Entangling two atoms via spontaneous emission

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Abstract. We discuss the creation of entanglement between two two-level atoms in the dissipative process of spontaneous emission. It is shown that spontaneous emission can lead to a transient entanglement between the atoms even if the atoms were prepared initially in an unentangled state. The amount of entanglement created in the system is quantified by using two different measures: concurrence and negativity. We find analytical formulas for the evolution of concurrence and negativity in the system. We also find the analytical relation between the two measures of entanglement. The system consists of two two-level atoms which are separated by an arbitrary distance $r_{12}$ and interact with each other via the dipole-dipole interaction, and the antisymmetric state of the system is included throughout, even for small inter-atomic separations, in contrast to the small sample model. It is shown that for sufficiently large values of the dipole-dipole interaction initially the entanglement exhibits oscillatory behaviour with considerable entanglement in the peaks. For longer times the amount of entanglement is directly related to the population of the slowly decaying antisymmetric state.


PACS numbers: 32.80.-t, 42.50.-p
1. Introduction

Entanglement is a property of quantum systems to exhibit correlations that cannot be accounted for classically. Entangled states of collective quantum systems, which are states that cannot be factorized into product states of the subsystems, are of fundamental interest in quantum mechanics. A number of methods for creating entanglement have been proposed involving trapped and cooled ions or neutral atoms [1, 2, 3, 4, 5, 6, 7, 8]. Of particular interest is generation of entangled states in two-atom systems, since they can represent two qubits, the building blocks of the quantum gates that are essential to implement quantum protocols in quantum information processing. It has been shown that entangled states in a two-atom system can be created by a continuous driving of the atoms with a coherent or chaotic thermal field [5, 9, 10], or by a pulse excitation followed by a continuous observation of radiative decay [11, 12, 13]. Moreover, the effect of spontaneous emission on initially prepared entangled state has also been discussed [4, 14, 15, 16, 17]. These studies, however, have been limited to the small sample (Dicke) model [18] or the situation involving noninteracting atoms strongly coupled to a cavity mode. The difficulty of the Dicke model is that it does not include the dipole-dipole interaction among the atoms and does not correspond to realistic experimental situations of atoms located (trapped) at different positions. In fact, the model corresponds to a very specific geometrical configuration of the atoms confined to a volume much smaller compared with the atomic resonant wavelength (the small-sample model). The present atom trapping and cooling techniques can trap two atoms at distances of order of a resonant wavelength [19, 20, 21], which makes questionable the applicability of the Dicke model to physical systems.

Recently, we have shown [22] that spontaneous emission from two spatially separated atoms can lead to a transient entanglement of initially unentangled atoms. This result contrasts the with the Dicke model where spontaneous emission cannot produce entanglement from initially unentangled atoms [10, 16]. We have numerically calculated the evolution of the concurrence and discussed the role of the maximally entangled collective states of the two-atom system: the rapidly decaying symmetric state and the slowly decaying antisymmetric state of the two-atom system.

In this paper we extend our study of spontaneously induced transient entanglement in a system of two atoms separated by an arbitrary distance $r_{12}$. We find analytical results for the two calculable measures of entanglement concurrence and negativity establishing the relation between the two. Our solutions are valid for a broad class of initial conditions including mixed states. It is shown that when the dipole-dipole interaction becomes larger than the atomic decay rate then the entanglement exhibits oscillatory behaviour, oscillating with twice the frequency describing the dipole-dipole interaction, which is the frequency separation between the symmetric and antisymmetric states of the two-atom system. Remarkable amounts of entanglement can be obtained at the maxima of the oscillations. For times longer that the decay rate of the superradiant symmetric state, when the population of the symmetric state is negligible, the oscillations disappear and the entanglement remaining in the system is related to the population of the slowly decaying antisymmetric state.

2. Measures of entanglement

To assess how much entanglement is stored in a given quantum system it is essential to have appropriate measures of entanglement. A number of measures have been proposed, which include entanglement of formation [23], entanglement of distillation [24], relative entropy of entanglement [25] and negativity [26, 27, 28, 29]. For pure states, the Bell states represent maximally entangled states, but for mixed states represented by a density matrix there are some difficulties with ordering the states according to various entanglement measures; different entanglement measures can give different orderings of pairs of mixed states and there is a problem of the definition of the maximally entangled mixed state [30, 31].

Here we use two entanglement measures, i.e., concurrence and negativity to describe the amount of entanglement created in a two-atom system during spontaneous emission. The concurrence introduced by Wootters [23] is defined as

$$C = \max \left(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right),$$

(1)
where \( \{ \lambda_i \} \) are the eigenvalues of the matrix
\[
R = \rho \tilde{\rho}
\]
with \( \tilde{\rho} \) given by
\[
\tilde{\rho} = \sigma_x \otimes \sigma_x \rho^* \sigma_x \otimes \sigma_x ,
\]
\( \sigma_x \) is the Pauli matrix, and \( \rho \) is the density matrix representing the quantum state. The range of concurrence is from 0 to 1. For unentangled atoms \( C = 0 \) whereas \( C = 1 \) for the maximally entangled atoms.

Another measure of entanglement we use here is the negativity, which is based on the Peres-Horodecki [26, 27] criterion for entanglement and is defined by the formula
\[
N = \max \left( 0, -2 \sum \mu_i \right) ,
\]
where the sum is taken over the negative eigenvalues \( \mu_i \) of the partial transposition of the density matrix \( \rho \) of the system. The partial transposition means transposition with respect to the one atom only. For pure states, like in the case of concurrence, \( N = 1 \) for maximally entangled state and \( N = 0 \) for unentangled atoms.

The two entanglement measures, i.e., concurrence and negativity, give the same criteria for entanglement, but generally they give different values for a degree of entanglement [31]. We will give analytical expressions for both of them for the entanglement produced in spontaneous emission.

Introducing the computational basis for the two-atom system as product states of the individual atoms, as follows
\[
|1\rangle = |g_1\rangle \otimes |g_2\rangle ,
|2\rangle = |e_1\rangle \otimes |e_2\rangle ,
|3\rangle = |g_1\rangle \otimes |e_2\rangle ,
|4\rangle = |e_1\rangle \otimes |g_2\rangle ,
\]
where \( |g_i\rangle \) and \( |e_i\rangle \) (for \( i = 1, 2 \)) are the ground and excited states of the individual atoms, we can define the density matrix of the two-atom system as a \( 4 \times 4 \) matrix.

We assume that the density matrix of the system has the block form
\[
\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} & 0 & 0 \\
\rho_{21} & \rho_{22} & 0 & 0 \\
0 & 0 & \rho_{33} & \rho_{34} \\
0 & 0 & \rho_{43} & \rho_{44}
\end{pmatrix}
\]
(6)
with the condition \( \text{Tr} \rho = 1 \). We will show that, if the atoms initially start from a state described by the density matrix of the form (6), the evolution does not destroy this form in the sense that the blocks of zeros remain untouched. The other matrix elements evolve in time, and we find explicitly their time dependence.

The matrix \( \tilde{\rho} \), needed for calculation of the concurrence, has the form
\[
\tilde{\rho} = \begin{pmatrix}
\rho_{22} & \rho_{12} & 0 & 0 \\
\rho_{21} & \rho_{11} & 0 & 0 \\
0 & 0 & \rho_{44} & \rho_{34} \\
0 & 0 & \rho_{34} & \rho_{33}
\end{pmatrix}
\]
(7)
and the square roots of the eigenvalues of the matrix \( R \) given by (2) are the following
\[
\{ \sqrt{\lambda_i} \} = \{ \sqrt{\rho_{11}\rho_{22}} - |\rho_{12}|, \sqrt{\rho_{11}\rho_{22}} + |\rho_{12}|, \sqrt{\rho_{33}\rho_{44}} - |\rho_{34}|, \sqrt{\rho_{33}\rho_{44}} + |\rho_{34}| \} .
\]
(8)
Depending on the particular values of the matrix elements there are two possibilities for the largest eigenvalue, either the second term or the fourth term in (8). The concurrence is thus given by
\[
C = \max \{ 0, C_1, C_2 \} ,
\]
(9)
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with

\[ C_1 = 2 (|\rho_{12}| - \sqrt{\rho_{33}\rho_{44}}) , \]
\[ C_2 = 2 (|\rho_{34}| - \sqrt{\rho_{11}\rho_{22}}) , \]

and we have two alternative expressions for the concurrence depending on which of them is positive.

For calculation of the negativity we need the partially transposed density matrix. The transposition with respect to the indices of the first atom gives the matrix

\[ \rho_{T1} = \left( \begin{array}{cccc}
\rho_{11} & \rho_{14} & 0 & 0 \\
\rho_{34} & \rho_{22} & 0 & 0 \\
0 & 0 & \rho_{33} & \rho_{21} \\
0 & 0 & \rho_{12} & \rho_{44} \\
\end{array} \right) \]

which has the eigenvalues

\[ \{\nu_i\} = \left\{ \frac{1}{2} \left( \rho_{11} + \rho_{22} \pm \sqrt{(\rho_{11} + \rho_{22})^2 + 4 (|\rho_{12}|^2 - \rho_{11}\rho_{22})} \right) , \right. \]
\[ \left. \frac{1}{2} \left( \rho_{33} + \rho_{44} \pm \sqrt{(\rho_{33} + \rho_{44})^2 + 4 (|\rho_{34}|^2 - \rho_{33}\rho_{44})} \right) \right\} . \]

There are two candidates for being negative among the roots \[ \{\rho_{12}\} \], however, they cannot be negative simultaneously because the inequality \[ |\rho_{34}| - \sqrt{\rho_{11}\rho_{22}} > 0 \] implies that \[ |\rho_{12}| - \sqrt{\rho_{33}\rho_{44}} < 0 \], and vice versa. It is also easy to find the two alternative values for the concurrence \[ \{\rho\} \] inside the square roots. So, the negativity defined by \[ \{\nu\} \] has also two alternative forms

\[ N = \max \left\{ 0, \sqrt{4 (|\rho_{12}|^2 - \rho_{33}\rho_{44}) + (\rho_{33} + \rho_{44})^2 - (\rho_{33} + \rho_{44})} , \right. \]
\[ \left. \sqrt{4 (|\rho_{34}|^2 - \rho_{11}\rho_{22}) + (\rho_{11} + \rho_{22})^2 - (\rho_{11} + \rho_{22})} \right\} , \]

\[ = \max \left\{ 0, \sqrt{C_1 C_1^+ + (\rho_{33} + \rho_{44})^2 - (\rho_{33} + \rho_{44})} , \right. \]
\[ \left. \sqrt{C_2 C_2^+ + (\rho_{11} + \rho_{22})^2 - (\rho_{11} + \rho_{22})} \right\} , \]

where the appropriate expression from \[ \{\nu\} \] is to be substituted to the two alternative terms in \[ \{\rho\} \]. The quantities \[ C_1^+ \], and \[ C_2^+ \], which are always nonnegative, represent two alternative expressions associated with corresponding expressions \[ C_1 \] and \[ C_2 \] for the concurrence \[ \{\rho\} \], and they have the following form

\[ C_1^+ = 2 (|\rho_{12}| + \sqrt{\rho_{33}\rho_{44}}) , \]
\[ C_2^+ = 2 (|\rho_{34}| + \sqrt{\rho_{11}\rho_{22}}) . \]

For pure states \[ C_1^+ \] and \[ C_2^+ \] are equal to \[ C_1 \] and \[ C_2 \], respectively, and in this case the negativity is equal to the concurrence.

The equality \[ \{\nu\} \] establishes the relation between the negativity and the concurrence for the system described by the density matrix of the form \[ \{\rho\} \]. It is evident that both quantities give the same criterion for entanglement, that is, positive value of \[ C \] implies positive value of \[ N \], but the degree of entanglement indicated by the two quantities can be quite different. It is also clear from \[ \{\nu\} \] that the product \[ C_1 C_1^+ (C_2 C_2^+) \] can as a whole serve as a measure of entanglement: it is zero if \[ C = 0 \], and it is unity for maximally entangled pure state, but for mixed states it gives yet another value for the degree of entanglement.

It is interesting to express the results for concurrence and negativity in the Bell basis which is defined as follows

\[ |1\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle) , \]
\[ |2\rangle = |\Phi^-\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle) , \]
\[ |3\rangle = |\Psi^+\rangle = \frac{1}{\sqrt{2}} (|3\rangle + |4\rangle) , \]
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\[ |4'⟩ = |Ψ^−⟩ = -\frac{1}{\sqrt{2}} (|3⟩ - |4⟩) . \] (16)

The transformation of the density matrix \( \rho \) given by (15) from the original basis (15) to the Bell basis (10) is performed with the transformation matrix

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \] (17)

leading to the new density matrix

\[ \rho' = U \rho U^+ , \] (18)

which has the same block form as (15) but the new matrix elements have primed indices (\( \rho_{1'1'}, \rho_{1'2'}, \ldots \)). The matrix elements in the Bell basis are related to the original matrix elements as follows

\[
\begin{align*}
\rho_{1'1'} &= \frac{1}{2} [\rho_{11} + \rho_{22} + (\rho_{12} + \rho_{21})], \\
\rho_{2'2'} &= \frac{1}{2} [\rho_{11} - \rho_{22} + (\rho_{12} - \rho_{21})], \\
\rho_{1'2'} &= -\frac{1}{2} [\rho_{11} - \rho_{22} + (\rho_{12} + \rho_{21})], \\
\rho_{2'1'} &= -\frac{1}{2} [\rho_{11} + \rho_{22} - (\rho_{12} - \rho_{21})], \\
\rho_{3'3'} &= \frac{1}{2} [\rho_{33} + \rho_{44} + (\rho_{12} + \rho_{21})], \\
\rho_{4'4'} &= \frac{1}{2} [\rho_{33} + \rho_{44} - (\rho_{12} - \rho_{21})] .
\end{align*}
\] (19)

In the Bell basis (16), the concurrence alternatives (10) and the negativity (14) take the following form

\[
\begin{align*}
C_1 &= \sqrt{(\rho_{1'1'} - \rho_{2'2'})^2 - (\rho_{1'2'} - \rho_{2'1'})^2} - \sqrt{(\rho_{3'3'} + \rho_{4'4'})^2 - (\rho_{3'4'} + \rho_{4'3'})^2} , \\
C_2 &= \sqrt{(\rho_{3'3'} - \rho_{4'4'})^2 - (\rho_{3'4'} - \rho_{4'3'})^2} - \sqrt{(\rho_{1'1'} + \rho_{2'2'})^2 - (\rho_{1'2'} + \rho_{2'1'})^2} .
\end{align*}
\] (20)

\[ \mathcal{N} = \max \left\{ 0, \sqrt{C_1^+ C_1^+} + (\rho_{3'3'} + \rho_{4'4'}) - (\rho_{3'4'} + \rho_{4'3'}) \right\} , \] (21)

where

\[
\begin{align*}
C_1^+ &= \sqrt{(\rho_{1'1'} - \rho_{2'2'})^2 - (\rho_{1'2'} - \rho_{2'1'})^2} + \sqrt{(\rho_{3'3'} + \rho_{4'4'})^2 - (\rho_{3'4'} + \rho_{4'3'})^2} , \\
C_2^+ &= \sqrt{(\rho_{3'3'} - \rho_{4'4'})^2 - (\rho_{3'4'} - \rho_{4'3'})^2} + \sqrt{(\rho_{1'1'} + \rho_{2'2'})^2 - (\rho_{1'2'} + \rho_{2'1'})^2} .
\end{align*}
\] (22)

From (20) and (21), it is evident that for any Bell state (16) the concurrence and negativity become unity. For mixed states the situation is much more complicated with the values of concurrence and negativity which are different in this case, and have values between zero and unity. Later on we apply the general formulas derived in this Section to find the evolution of entanglement in spontaneous emission from a system of two two-level atoms.

3. Atomic evolution

We consider a system of two non-overlapping two-level atoms with ground states \( |g_i⟩ \) and excited states \( |e_i⟩ \) (\( i = 1, 2 \)) connected by dipole transition moments \( \vec{μ}_i \). The atoms are located at fixed positions \( \vec{r}_1 \) and \( \vec{r}_2 \) and coupled to all modes of the electromagnetic field, which we assume are in the vacuum state. We consider spontaneous emission from identical as well as non-identical atoms prepared in different initial states. In the case of nonidentical atoms, we assume that atoms have equal dipole moments \( \mu_1 = \mu_2 = \mu \), but different transition frequencies \( \omega_1 \) and \( \omega_2 \), such that \( \omega_2 - \omega_1 \ll \omega_0 = (\omega_1 + \omega_2)/2 \), so that the rotating-wave approximation can be applied to calculate the dynamics of the system.

The time evolution of the system of atoms coupled through the vacuum field is given by the following master equation

\[
\frac{\partial \rho}{\partial t} = -i \sum_{i=1}^{2} \omega_i [S^+_i, \rho] - i \sum_{i \neq j}^{2} \Omega_{ij} [S^+_i S^-_j, \rho] - \frac{1}{2} \sum_{i,j=1}^{2} \Gamma_{ij} (\rho S^+_i S^-_j + S^+_i S^-_j \rho - 2 S^-_j S^+_i) ,
\] (23)
where \( S^+ (S^-) \) are the dipole raising (lowering) operators and \( S^z \) is the energy operator of the \( i \)th atom. In Eq. (23), \( \Gamma_{ij} (i = j) \) are the spontaneous emission rates of the atoms, equal to the Einstein A coefficient for spontaneous emission, whereas \( \Gamma_{ij} \) and \( \Omega_{ij} (i \neq j) \) describe the interatomic coupling [32, 33, 34], and are the collective damping and the dipole-dipole interaction potential defined, respectively, by

\[
\Gamma_{ij} = \Gamma_{ji} = \frac{3}{2} \Gamma \left\{ \left[ 1 - (\hat{\mu} \cdot \hat{r}_{ij})^2 \right] \frac{\sin(k_0 r_{ij})}{k_0 r_{ij}} + \left[ 1 - 3 (\hat{\mu} \cdot \hat{r}_{ij})^2 \right] \frac{\cos(k_0 r_{ij})}{(k_0 r_{ij})^3} \right\},
\]

(24)

and

\[
\Omega_{ij} = \frac{3}{4} \Gamma \left\{ - \left[ 1 - (\hat{\mu} \cdot \hat{r}_{ij})^2 \right] \frac{\cos(k_0 r_{ij})}{k_0 r_{ij}} + \left[ 1 - 3 (\hat{\mu} \cdot \hat{r}_{ij})^2 \right] \frac{\sin(k_0 r_{ij})}{(k_0 r_{ij})^3} \right\},
\]

(25)

where \( k_0 = \omega_0/c \), \( r_{ij} = |r_j - r_i| \) is the distance between the atoms, \( \hat{\mu} \) is unit vector along the atomic transition dipole moments, that we assume are parallel to each other, and \( \hat{r}_{ij} \) is the unit vector along the interatomic axis.

The master equation (23) has been used for many years to study a wide variety of problems involving the interaction of collective atomic systems with the radiation field [35]. Using the master equation (23), we can write down the equations of motion for the components of the density matrix of the two-atom system in the basis \( 5 \) of the product states of the individual atoms. However, the problem simplifies by working in the basis of the collective states of the system which contains symmetric and antisymmetric combinations of the product states. For identical atoms (\( \omega_1 = \omega_2 \)) the collective states are \( 18, 32 \)

\[
|g\rangle = |1\rangle, \\
|e\rangle = |2\rangle, \\
|s\rangle = \frac{1}{\sqrt{2}} (|3\rangle + |4\rangle), \\
|a\rangle = \frac{1}{\sqrt{2}} (|4\rangle - |3\rangle),
\]

(26)

where we used the basis \( 6 \).
In the collective state representation, the two-atom system behaves as a single four-level system, illustrated in Fig. 1, with the ground state |g⟩, the upper state |e⟩, and two intermediate states: the symmetric |s⟩ and antisymmetric |a⟩ states. The most important property of the collective states is that the symmetric and antisymmetric states are maximally entangled states. The states are linear superpositions of the product states which cannot be separated into product states of the individual atoms. They are in fact two of the Bell states introduced in (16): |s⟩ = |3′⟩ and |a⟩ = |4′⟩. The symmetric and antisymmetric states are eigenstates of the system of two identical atoms with the dipole-dipole interaction included. The basis of atomic states (26) can be considered as an effect of partial transformation to the Bell basis (16) in which the transformation has been performed in the lower block only. Such basis is convenient for finding the solution to the master equation (23) describing spontaneous emission in the system.

Assuming that initially the state of the system has been prepared in the block form (26), from the master equation (23), we get the following set of differential equations describing the evolution of the system in the basis of collective atomic states (26)

\[
\begin{align*}
\dot{\rho}_{ee} &= -2\Gamma \rho_{ee}, \\
\dot{\rho}_{eg} &= -(\Gamma + 2\Omega_0) \rho_{eg}, \\
\dot{\rho}_{as} &= -(\Gamma + \Gamma_{12}) (\rho_{as} - \rho_{ee}) + i\Delta (\rho_{as} - \rho_{sa}), \\
\dot{\rho}_{aa} &= -(\Gamma - \Gamma_{12}) (\rho_{aa} - \rho_{ee}) - i\Delta (\rho_{as} - \rho_{sa}), \\
\dot{\rho}_{as} &= -(\Gamma + 2\Omega_{12}) \rho_{as} + i\Delta (\rho_{ss} - \rho_{aa})
\end{align*}
\]

(27)

with the condition \( \rho_{eg} + \rho_{as} + \rho_{ss} + \rho_{aa} = 1 \), and with \( \Delta = (\omega_2 - \omega_1)/2 \). All other matrix elements related to the blocks of zeros in (26) remain zeros, if the evolution is govern by the master equation (23).

Equations (27) show that all transitions rates to and from the symmetric state are equal to \( (\Gamma + \Gamma_{12}) \). On the other hand, all transitions rates to and from the antisymmetric state are equal to \( (\Gamma - \Gamma_{12}) \). Thus, the symmetric state decays with an enhanced (superradiant) rate, while the antisymmetric state decays with a reduced (subradiant) state. Hence, the population of the antisymmetric state experiences a variation on a time scale of order \( (\Gamma - \Gamma_{12})^{-1} \), which can lead to interesting effects not observed in the Dicke model. These effects result from the fact that the set of equations (27) has two different solutions depending on whether \( \Gamma_{12} = \Gamma \) or \( \Gamma_{12} \neq \Gamma \). The case of \( \Gamma_{12} = \Gamma \) corresponds to the small sample (Dicke) model, whereas the case of \( \Gamma_{12} \neq \Gamma \) corresponds to spatially extended atomic systems. The existence of two different solutions of Eq. (27) is connected with conservation of the total spin \( S^2 \), that \( S^2 \) is a constant of motion for the Dicke model and \( S^2 \) not being a constant of motion for a spatially extended system of atoms. We can explain it by expressing the square of the total spin of the two-atom system in terms of the density matrix elements of the collective system as

\[
S^2(t) = 2 - 2\rho_{aa}(t).
\]

(28)

It is clear from Eq. (28) that \( S^2 \) is conserved only in the Dicke model, in which the antisymmetric state is ignored. For a spatially extended system the antisymmetric state participates fully in the dynamics and \( S^2 \) is not conserved. The Dicke model evolves between the triplet states |e⟩, |s⟩, and |g⟩, while the spatially extended two-atom system evolves between the triplet and the antisymmetric states.

The problem of spontaneous emission from two atoms can be solved analytically even for general case of nonidentical atoms (\( \Delta \neq 0 \)), but the general solutions are rather lengthy, and we will give here the solutions for the simpler case of identical atoms only. It is seen from (27) that the first two equations, belonging to the upper block of (26), are decoupled from the other equations belonging to the lower block of (26), and they have simple exponential solutions

\[
\begin{align*}
\rho_{ee}(t) &= \rho_{ee}(0) e^{-2\Gamma t}, \\
\rho_{eg}(t) &= \rho_{eg}(0) e^{-(\Gamma + 2\Omega_0)t}.
\end{align*}
\]

(29)

For identical atoms, \( \Delta = 0 \), the remaining equations simplify considerably and their solutions are as follows

\[
\rho_{ss}(t) = \rho_{ss}(0) e^{-(\Gamma + \Gamma_{12})t} + \rho_{ee}(0) \frac{\Gamma + \Gamma_{12}}{\Gamma - \Gamma_{12}} \left( e^{-(\Gamma + \Gamma_{12})t} - e^{-2\Gamma t} \right).
\]
The evolution within the two blocks runs independently except for the fact that all the states decay to the ground state $|g\rangle$, and the population of this state is

$$\rho_{gg}(t) = 1 - \rho_{ce}(t) - \rho_{ss}(t) - \rho_{aa}(t),$$

i.e., eventually total atomic population accumulates in the ground state.

The solutions (29) and (30) are particularly simple in the basis (26), but it is quite easy to transform them into the original basis (5) or into the Bell basis (16) using the relations (19). The same transformation can be used to transform the initial conditions. In this way we obtain analytical results for the matrix elements of the density matrix in either the original basis or the Bell basis as the linear combinations of the solutions (29) and (30), for any initial conditions that preserve the block form (6) of the density matrix.

4. Entanglement in the two-atom system

The solutions obtained in the previous Section can be used in formulas (9), (10) or (20) for the concurrence and in formulas (13) or (21) for the negativity giving the analytical expressions for the quantities describing degree of entanglement in the system. For example, if $\rho_{gg}(0) = \rho_{21}(0) = 0$, i.e., there is no two-photon coherence in the system initially, then $C_1$ cannot be positive, so it cannot contribute to the concurrence $C$, and the concurrence is equal to $C_2$, if it is positive. We have

$$C(t) = \max \{0, C_2(t)\},$$

and

$$C_2(t) = \sqrt{[\rho_{sa}(t) - \rho_{aa}(t)]^2 - [\rho_{sa}(t) - \rho_{as}(t)]^2 - 2\rho_{ce}(t)\rho_{gg}(t)},$$

with the solutions (29) and (30). The solution (30) still covers a broad range of initial conditions, i.e., such that the upper block in (4) is diagonal but the lower block is arbitrary.

It is immediately seen from (30) and the solutions (29) that, for two identical atoms prepared initially in one of the maximally entangled states $|s\rangle$ or $|a\rangle$, the concurrence for any time is equal to the population of the corresponding state $\rho_{ss}(t)$ or $\rho_{aa}(t)$: it is unity at time $t = 0$ and decays in time at rate $\Gamma + \Gamma_{12}$ for the symmetric state and at rate $\Gamma - \Gamma_{12}$ for the antisymmetric state.

The quantity $C_2^0$, defined by (16), associated with $C_2$ is then given by

$$C_2^0(t) = \sqrt{[\rho_{sa}(t) - \rho_{aa}(t)]^2 - [\rho_{sa}(t) - \rho_{as}(t)]^2 + 2\rho_{ce}(t)\rho_{gg}(t)},$$

and the negativity $N$, given by (14), evolves in time according to the formula

$$N(t) = \max \{0, N_2(t)\},$$

where

$$N_2(t) = \sqrt{C_2(t)C_2^+(t) + \rho_{gg}(t) + \rho_{ce}(t)]^2 - [\rho_{gg}(t) + \rho_{ce}(t)]}.$$

Equations (30) and (36) are exact analytical formulas describing the time evolution of entanglement created in the system of two identical atoms via the process of spontaneous emission.

Let us now consider two special cases of the initial conditions: (i) initially only one atom excited, $\rho_{44}(0) = 1$, (ii) both atoms initially excited, $\rho_{ce} = 1$.

For case (i), we have $\rho_{ss}(0) = \rho_{aa}(0) = \rho_{as}(0) = \rho_{sa}(0) = 1/2$, $\rho_{ce}(0) = 0$, and equation (30) takes the form

$$C_2(t) = \frac{1}{2} \sqrt{\left[e^{-(\Gamma + \Gamma_{12})t} - e^{-(\Gamma - \Gamma_{12})t}\right]^2 + e^{-2\Gamma t} \sin^2(2\Omega_{12}t)}.$$

From (37) it is seen that $C_2(0) = 0$, there is no entanglement at $t = 0$, as it should be since the initial state is a product state. However, for $t > 0$, $C_2(t)$ becomes positive, which means that the two atoms
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become entangled, and the degree of entanglement measured by the concurrence is given by \( C(t) \). For long times, all terms in \( C(t) \) decay to zero, and the concurrence goes to zero. One more interesting feature of the evolution is seen from \( C(t) \), it is the oscillatory behaviour of the concurrence which can be observed at times shorter than \( (2\Gamma)^{-1} \), when the oscillatory term contributes significantly to the evolution.

The oscillations are with the frequency \( 2\Omega_{12} \), which is equal to the separation of the symmetric and antisymmetric states, and the oscillations become visible when the dipole-dipole interaction is sufficiently strong, i.e., for \( \Omega_{12} \gg \Gamma \). For times longer than \( (2\Gamma)^{-1} \) the only term that survives is the term that decays with the rate \( \Gamma - \Gamma_{12} \), which comes from the evolution of the slowly decaying antisymmetric state.

and the concurrence becomes equal to the population of this state. We have numerically studied this behaviour, but for the interatomic distances not so short as to reveal the oscillations in the concurrence.

In Fig. 2 we present the oscillatory behaviour of the concurrence \( C(t) \) for the case of initially one atom excited and the interatomic distance \( r_{12} = \lambda/12 \), which gives the values \( \Gamma_{12} = 0.95 \Gamma \) and \( \Omega_{12} = 9.30 \Gamma \).

\[ C(t) = \frac{1}{4} \left\{ e^{-(\Gamma+\Gamma_{12})t} - e^{-(\Gamma-\Gamma_{12})t} \right\}^2 + e^{-2\Gamma t} \sin^2(2\Omega_{12} t) + \rho_{gg}(t) - \rho_{gg}(t), \]

where

\[ \rho_{gg}(t) = 1 - \frac{1}{2} \left[ e^{-(\Gamma+\Gamma_{12})t} + e^{-(\Gamma-\Gamma_{12})t} \right]. \]

In Fig. 3 we compare the time evolution of the two measures of entanglement: concurrence and negativity for the same values of the parameters as in Fig. 2. Generally, the negativity takes smaller values than the
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Concurrence

Figure 3. Comparison of concurrence (solid) and negativity (dashed) for the same parameters as in Fig. 2.

Concurrence, except for the initial value which is zero for both of them, and the value for $t \to \infty$ which is also zero.

For case (ii), we have $\rho_{ee}(0) = 1$ and the concurrence takes the form

$$C_2(t) = \frac{\Gamma + \Gamma_{12}}{\Gamma - \Gamma_{12}} \left( e^{-(\Gamma + \Gamma_{12})t} - e^{-2\Gamma t} \right) - \frac{\Gamma - \Gamma_{12}}{\Gamma + \Gamma_{12}} \left( e^{-(\Gamma - \Gamma_{12})t} - e^{-2\Gamma t} \right) - 2e^{-\Gamma t} \sqrt{\rho_{gg}}$$

with

$$\rho_{gg}(t) = 1 - \left[ \frac{\Gamma + \Gamma_{12}}{\Gamma - \Gamma_{12}} \left( e^{-(\Gamma + \Gamma_{12})t} - e^{-2\Gamma t} \right) + \frac{\Gamma - \Gamma_{12}}{\Gamma + \Gamma_{12}} \left( e^{-(\Gamma - \Gamma_{12})t} - e^{-2\Gamma t} \right) + e^{-2\Gamma t} \right].$$

Again, for $t = 0$ the concurrence is zero, but now it is not easy to see if $C_2(t)$ can be positive, and numerical evaluation is needed to check the positivity. What is clear from (40), however, is the fact that there are no oscillations in this case. One can also expect that for times for which the populations of the excited state and the symmetric state, which decay much faster than the antisymmetric state, are already close to zero, it is still some population in the antisymmetric state and $C_2(t)$ becomes positive. Numerical evaluation of (40) confirm that it is really true. Corresponding formula for the negativity can be obtained from (34), (36) and (40), but this are just simple substitutions, so we do not write it explicitly. In Fig. 4 we plot the time evolution of the concurrence $C(t)$, the negativity $N(t)$, and the population $\rho_{aa}(t)$ of the antisymmetric state for the initial state of both atoms excited ($\rho_{ee}(0) = 1$) and the interatomic distance $r_{12} = \lambda/12$, which gives the collective damping $\Gamma_{12} = 0.95 \Gamma$ and the dipole-dipole interaction frequency $\Omega_{12} = 9.30 \Gamma$. As expected, there is no entanglement before the populations of the exited state and the symmetric state depopulate, but some entanglement appears for longer times, and the concurrence again becomes equal to the population of the antisymmetric state. The values of the negativity in this case are much smaller than the values of the concurrence, which itself is very small. Exciting two atoms initially is thus very ineffective in producing entanglement.

For nonidentical atoms, although the analytical solution is possible, the formulas are rather lengthy and we will not adduce them here. Instead, we plot in Fig. 5 an example of the evolution for the...
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Concurrence for the case of atom “1” excited ($\rho_{44}(0) = 1$) with $\Delta = (\omega_2 - \omega_1)/2 = 10 \Gamma$ and $r_{12} = \lambda/12$ ($\Gamma_{12} = 0.95 \Gamma$, $\Omega_{12} = 9.30 \Gamma$). This means that we have $\Delta \sim \Omega_{12}$. As it is evident from the equations of motion [27], for two nonidentical atoms, i.e., for $\Delta \neq 0$, there is a coupling between the populations of the symmetric state and the asymmetric state, which leads to a coherent transfer of population from one state to the other. This introduces oscillations into the populations of both states. Since the oscillations of the two populations, for the parameters of Fig. 3, are opposite in phase, they add up in $\rho_{aa}(t) - \rho_{ss}(t)$ and subtract in $\rho_{aa}(t) + \rho_{ss}(t)$, as clearly seen from the figure. The concurrence is oscillating, similarly to the situation shown in Fig. 2 between the lower bound $(\rho_{aa}(t) - \rho_{ss}(t))$ and the upper bound $\rho_{aa}(t) + \rho_{ss}(t)$, but this time the lower bound itself undergoes oscillations, which results in increasing the concurrence at the maxima. The value at the first maximum is 0.88, which is higher than the corresponding value for identical atoms equal to 0.86. It is thus possible to enhance the transient entanglement in the two-atom system when the two atoms are nonidentical.

5. Conclusion

In this paper we have studied entanglement created in a system of two two-level atoms via the spontaneous emission. We have found analytical formulas for the concurrence and the negativity, the two different measures of entanglement usually used to quantify the amount of entanglement. Our formulas are valid for a broad class of initial conditions which are represented by the block form of the density matrix.

We have shown that for short times, when initially only one atom is excited, the amount of entanglement exhibits oscillatory behaviour until the time at which the population of the symmetric states becomes zero. For long times the concurrence becomes equal to the population $\rho_{aa}(t)$ of the slowly decaying antisymmetric state. For both atoms initially excited there are no oscillations, and the entanglement appears only for long times, when only the antisymmetric state contributes to the entanglement.
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\[ \Gamma t \]

Figure 5. Time evolution of the concurrence \( C(t) \) (solid line), \( \rho_{aa}(t) - \rho_{ss}(t) \) (dashed line), and \( \rho_{aa}(t) + \rho_{ss}(t) \) (dashed-dotted line) for two nonidentical atoms with \( \Delta = (\omega_2 - \omega_1)/2 = 10 \Gamma \); initially atom 1 is excited (\( \rho_{44}(0) = 1 \)), \( \hat{\mu} \perp \hat{r}_{12} \), and \( r_{12} = \lambda/12 \) (\( \Gamma_{12} = 0.95 \Gamma \), \( \Omega_{12} = 9.30 \Gamma \)).

References

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