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FINITE ENERGY STATES FOR PERIODICALLY KICKED NONLINEAR OSCILLATOR*

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We study a nonlinear oscillator interacting with a one-mode cavity field. We assume, that the cavity is periodically kicked by a series of ultra-short coherent pulses. We show that for a special choice of parameters the system evolution is restricted to a finite set of n-photon states. In consequence, the mean energy of the cavity remains finite despite the fact that the cavity is continuously pumped. We study the properties of the cavity field showing that the field exhibits nonclassical features.

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1 Introduction

Quantum engineering of the electromagnetic field states has been a subject of numerous papers during the past years ([1-11] and the references quoted therein). Among a large variety of systems discussed there, models involving a nonlinear oscillator are of great importance. This paper is devoted to a system of such kind. We assume here that the nonlinear oscillator is just a nonlinear Kerr-type medium located inside a one-mode, high-Q cavity. The cavity is irradiated by a series of ultra-short coherent pulses of classical light. The evolution of the system is governed by the following Hamiltonian (in units of $\hbar = 1$):

$$\hat{H} = \hat{H}_{NL} + \hat{H}_p \ , \tag{1}$$

where

$$\hat{H}_{NL} = \frac{\chi}{2}\hat{a}^{\dagger 2}\hat{a}^2 \tag{2}$$

and

$$\hat{H}_p = \epsilon(\hat{a}^\dagger + \hat{a})f(t) \ . \tag{3}$$

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The parameter χ appearing in (2) describes the nonlinearity of the medium, whereas ϵ corresponds to the strength of the pulses that pump the cavity. Since the external excitation is assumed to be in the form of a series of ultrashort pulses, we define the function $f(t)$ by the Dirac delta functions in the following form:

$$f(t) = \sum_{k=0}^{\infty} \delta(t - nT) \ . \tag{4}$$

The time T appearing here denotes the time between two subsequent pulses.

We neglect all damping processes in the model discussed in this paper . One could thus expect that due to periodic but permanent excitation by the external pulses the mean number of photons of the cavity field will grow infinitely, as it is the case for a cavity without nonlinear medium. However, as we shall show, for a special choice of the parameters describing the system its evolution can be practically restricted to a finite set of the number states $|n\rangle$.

In this paper we shall discuss two regimes. The first one concerns weak external excitations, *i.e.* we assume that $\epsilon \ll \chi$. Applying the standard perturbation theory, we shall obtain analytical formulas describing behaviour of the system. The second regime discussed here corresponds to a short time T between two subsequent external pulses – we assume that time $T \ll 1/\chi$. We shall study this case numerically and show that the dynamics of the system is restricted to a finite set of n-photon states.

2 Weak excitation limit

This section is devoted to the case when the external excitation is weak, *i.e.* we assume that $\epsilon \ll \chi$. In this case, we can apply the standard perturbation theory in the same way as in the paper [9].

One should note that the Hamiltonian $\hat{H}_{NL}$ has two degenerate eigenstates $|0\rangle$ and $|1\rangle$ with the eigenvalue equal to zero. However, the perturbation associated with the Hamiltonian $\hat{H}_p$ couples the two states removing the degeneracy. For weak external excitations the dynamics of the system can be restricted to the two degenerate states. Hence, the quantum state of the system can be written as:

$$|\Psi(t)\rangle = a_0(t)|0\rangle + a_1(t)|1\rangle \ . \tag{5}$$

Thus, the problem is reduced to an equivalent problem of a two-level atom interacting with the field having the envelope $f(t)$. Applying the procedure described in [12], we can write equations of motion for the probability amplitudes $a_0(t)$ and $a_1(t)$ as:

$$\begin{aligned} \frac{d}{dt}a_0(t) &= -i\epsilon f(t)a_1(t) \ , \\ \frac{d}{dt}a_1(t) &= -i\epsilon f(t)a_0(t) \ . \end{aligned} \tag{6}$$

Equations (6) lead to the following solutions [12]:

$$\begin{aligned} a_0(t) &= \cos\Theta(t) \ , \\ a_1(t) &= -i\sin\Theta(t) \ , \end{aligned} \tag{7}$$

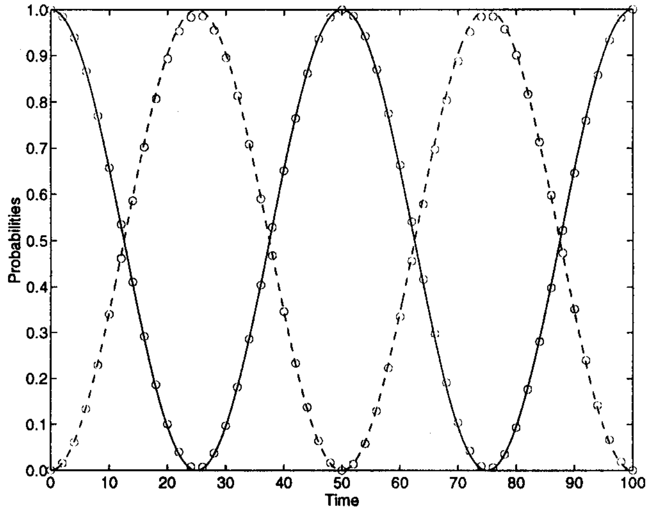

Fig. 1. Time evolution of the probability amplitudes a_0 (solid line) and a_1 (dashed line). The time $T = \pi$, and the pulse strength $\epsilon = \pi/50$ (we use units of $\chi = 1$). The circle marks correspond to the numerical results.

where

$$\Theta(t) = \epsilon \int_0^t f(t')dt' \tag{8}$$

contains the integral denoting the area under the excitation field envelope. Since the function $f(t)$ [Eq. (4)] consists of a series of the Dirac delta functions, the quantity $\Theta(t)$ can be easily calculated. It can be expressed as $\Theta = \epsilon k$, where k denotes the number of pulses. In consequence, the evolution of the system is restricted to the two-dimensional space spanned on the states $|0\rangle$ and $|1\rangle$. This fact can lead to the one-photon state generation phenomenon already discussed in [7,9]. Fig. 1 shows the probabilities $|a_0|^2$ and $|a_1|^2$ as a function of time and the results of direct computer simulations based on the evolution operator. The probabilities oscillate periodically between 0 and 1, and the value of $|a_0(t)|^2 + |a_1(t)|^2$ is practically equal to unity, which means that the dynamics is restricted to the two degenerate states. Moreover, we see that the lines corresponding to the analytical result given by (7) and (8) agree perfectly with the marks denoting the direct numerical solutions.

3 Short time limit

This section is devoted to the case of short time T between two subsequent external pulses, *i.e.* $T \ll 1/\chi$. In this case we perform numerical calculations to investigate the dynamics of the system. Our calculations are based on the method applied in the papers [7,9-11]. Assuming that the damping processes are absent, we are able to describe the evolution of the system using

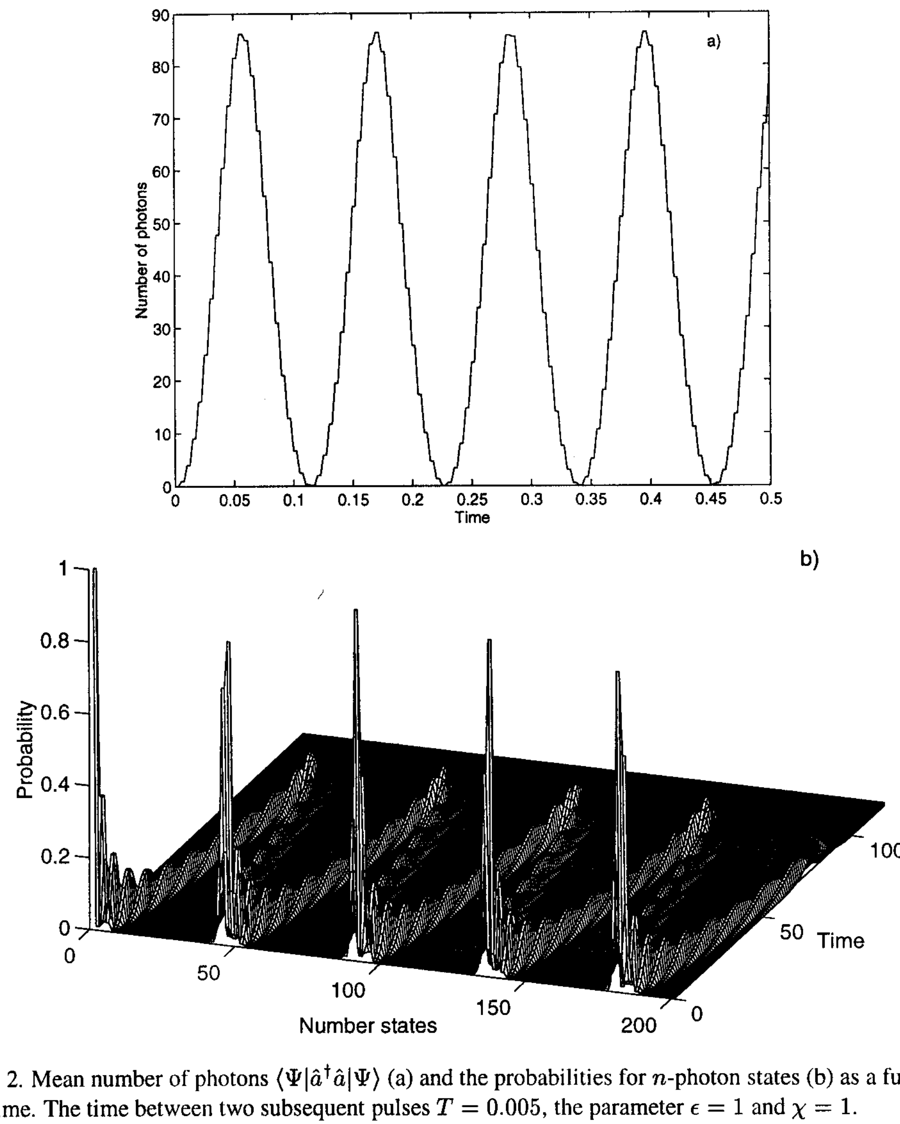

Fig. 2. Mean number of photons $\langle\Psi|\hat{a}^{\dagger}\hat{a}|\Psi\rangle$ (a) and the probabilities for n-photon states (b) as a function of time. The time between two subsequent pulses $T = 0.005$, the parameter $\epsilon = 1$ and $\chi = 1$.

the following quantum state decomposition in the n-photon basis:

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} a_n(t)|n\rangle \ . \tag{9}$$

The dynamics of the system can be determined by the unitary evolution operators. They are defined by the Hamiltonians $\hat{H}_{NL}$ and $\hat{H}_p$. One can easily see that the evolution of the system can be divided into two types. The first one is the evolution during the times between two

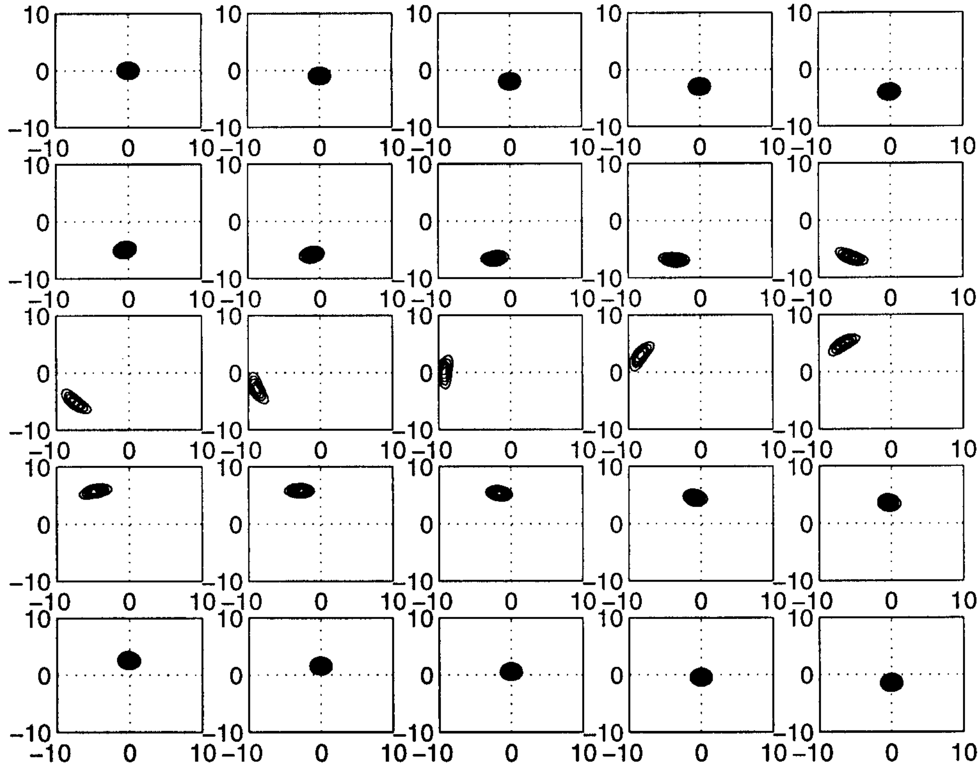

Fig. 3. The Q function evolution for the first 25 pulses. All parameters are the same as in Fig. 2.

subsequent pulses. For these times the the evolution is governed by the operator:

$$\hat{U}_{NL} = e^{-i\frac{\chi T}{2}(\hat{a}^\dagger)^2\hat{a}^2} \ . \tag{10}$$

The second type is the evolution during the pulse. Owing to the fact that the external excitation is modeled by a series of the Dirac delta functions, the unitary evolution operator acting on the quantum state during the pulse takes the following form:

$$\hat{U}_p = e^{-i\epsilon(\hat{a}^\dagger+\hat{a})} \ . \tag{11}$$

In consequence, the evolution of the system from the time just after jth pulse to the time just after $(j+1)$th pulse is determined by the operator $\hat{U} = \hat{U}_p\hat{U}_{NL}$. Hence, assuming that the system was initially in the vacuum state $|0\rangle$, the state corresponding to the time just after kth pulse can be written as:

$$|\Psi_k\rangle = \sum_{n=0}^{\infty} a_n(k)|n\rangle = (\hat{U}_p\hat{U}_{NL})^k|0\rangle \ . \tag{12}$$

Applying this formula we can determine the evolution of the state of the system and, hence, the time dependence of the mean values of the field operators. In Fig. 2(a) we plot the mean number of photons $\langle n\rangle = \langle\Psi|\hat{a}^\dagger\hat{a}|\Psi\rangle$ as a function of time. We see that $\langle n\rangle$ oscillates in a regular way between 0 and about 85. Fig. 2(b) shows the evolution of the probabilities $|a_n(t)|^2$ corresponding to the time t just after the first 200 subsequent pulses. It is seen that the evolution starts from the vacuum state $|0\rangle$, then the wave packet, in the space of n-photon states, is formed. This packet moves toward the higher and higher Fock states up to $n \sim 95$. After that the packet motion is reversed, and it comes back to the lower values of n and, for $t \sim 0.12$, it restores its initial form corresponding to the vacuum state. Next, the situation repeats periodically.

This behaviour can be explained by looking at the evolution of the Husimi Q function of the field state. Figure 3 depicts the contour plots of this function corresponding to the times just after the first 25 subsequent pulses. We see that for the time $t = 0$ the Q function corresponds to the vacuum state and it has a peak located at the origin of the coordinate system. Then, after about 5 pulses the peak of this function moves over the complex α plane toward negative values of $\mathrm{Im}(\alpha)$. As the mean number of photons $|\alpha^2| = \langle n\rangle$ reaches sufficiently high values the Q function changes its character from that reminding a coherent state to the crescent shape characteristic for the Kerr states. At these times the unitary operator $\hat{U}_{NL}$ describing nonlinear interaction starts to play more significant role in the whole evolution operator $\hat{U}$ because U_{NL} contains terms proportional to n^2 which dominate over the terms proportional to $\sqrt{n}$ and n. In consequence, the nonlinear evolution prevails and the Q function is rotated around the center of the coordinate system. Due to this rotation the peak moves around the point $(0,0)$. Then, subsequent pulses shift the peak down toward the point of the initial position. Moreover, during the last stage the Q function recovers its initial form corresponding to the vacuum state. This mechanism leads to the evolution of the system within a finite set of the n-photon states and, in consequence, to the generation of the states with finite energy. Moreover, as it is seen from Fig. 2 the evolution of the system has periodic character.

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