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PHYSICAL OPTICS

Phase Distributions for a Shifted Fock State and Shifted Random State

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Abstract—Phase properties of a shifted Fock state and a shifted random state are studied with the use of two different approaches for the description of phase. The Pegg–Barnett phase distribution and the phase distribution associated with the Wigner function are shown to be close to each other for a shifted Fock state, whereas the phase distribution associated with the Q-function is of lower phase information content. In the case of a shifted random state, all the phase distributions are qualitatively similar and change to the uniform distribution in the limit of a large average number of photons. The results are clearly interpreted in terms of the concept of the overlapping domain in the phase state.

INTRODUCTION

Determination of the phase distribution for a quantum state is a nontrivial problem [1, 2]. This is associated with the fact that the Hermitian operator of phase is difficult to construct. Pegg and Barnett have recently shown [3–5] how such an operator can be constructed for a quantized electromagnetic field. The Pegg–Barnett formalism allows the phase distribution for a quantum state to be obtained using eigenstates of the Hermitian phase operator.

Another approach for the problem of phase description in quantum optics [6–8], exists in which quasiprobability distribution functions are used, such as the Glauber–Sudarshan $\mathcal{P}$-function, the Q-function, and the Wigner function. These are functions of a complex number $\alpha = |\alpha|\exp(i\theta)$, which is an eigenvalue of the annihilation operator. By integrating functions of quasiprobability distribution over the radial variable $|\alpha|$, functions periodic in the phase angle θ can be obtained. For most known states, these functions possess all the properties required for a phase distribution [6–8]. In our recent works, we compared the two aforementioned approaches for the description of phase for the states of light propagating in a Kerr medium [9] and generated in a multiphoton downconversion [10], as well as for a shifted Fock state [11] and for a squeezed Fock state and squeezed random state [12].

The purpose of this work is to compare these two approaches using a shifted Fock state (SFS) and a shifted random state (SRS) as an example. Fock states (random states) are determined only by the number of photons (the average number of photons) and have a uniformly distributed phase. As a result of the operation of translation, they acquire nonzero average values of the field amplitude and phase, after which the phase distributions become nonuniform. Thus, the study of phase properties of SFS and SRS appears to be helpful. Note that SFS possess interesting nonclassical properties, which were studied in detail in [13, 14].

RESULTS AND DISCUSSION

A SFS is determined by the action of the translation operator $D(\alpha)$ upon a Fock state $|N\rangle$

$$|\psi\rangle = |\alpha, N\rangle = D(\alpha)|N\rangle, \tag{1}$$

where

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a). \tag{2}$$

Expansion of SFS (1) in a basis of Fock states has the form

$$\begin{aligned}|\psi\rangle &= \sum_n |n\rangle\langle n|\psi\rangle = \sum_n |n\rangle\langle n|D(\alpha)|N\rangle \\ &= \sum_n b_n e^{i\varphi_n}|n\rangle,\end{aligned} \tag{3}$$

where

$$b_n = \left(\frac{N!}{n!}\right)^{1/2} |\alpha|^{n-N} e^{-|\alpha|^2/2} L_N^{n-N}(|\alpha|^2) \tag{4}$$

and

$$\varphi_n = (n-N)\varphi \tag{5}$$

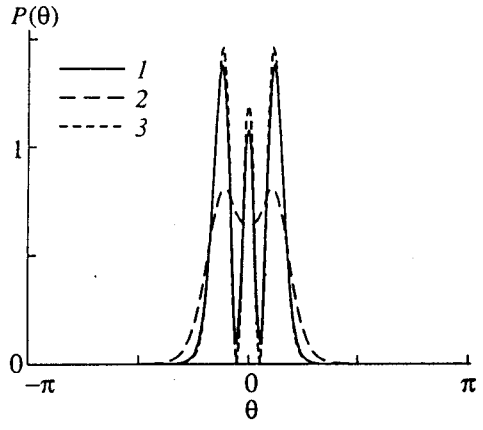

Fig. 1. Phase distributions (*1*) $P^{(PB)}(\theta)$, (*2*) $P^{(Q)}(\theta)$, and (*3*) $P^{(W)}(\theta)$ for shifted Fock states for $|\alpha| = 3$, and $N = 2$.

for $n \geq N$. Here, φ is the phase of the complex number $\alpha = |\alpha|\exp(i\varphi)$, and $L_N^{n-N}(|\alpha|^2)$ is the associated Laguerre polynomial. For $n < N$, we have

$$b_n = \left(\frac{n!}{N!}\right)^{1/2}(-1)^{N-n}|\alpha|^{N-n}e^{-|\alpha|^2/2}L_n^{N-n}(|\alpha|^2), \quad (6)$$

and the phase φ_n remains the same as in (5).

A SRS is determined by the action of the translation operator $D(\alpha)$ upon a random state

$$\rho_{d.th} = D(\alpha)\rho_{th}D^{\dagger}(\alpha) = \frac{1}{1+\bar{n}}\sum_{N=0}^{\infty}\left(\frac{\bar{n}}{1+\bar{n}}\right)^N |\alpha, N\rangle\langle\alpha, N|, \quad (7)$$

where $\bar{n}$ is the average number of photons in the initial random state.

As shown in [9–11], the Pegg–Barnett phase distribution and distributions obtained by integrating the Q-function and the Wigner function can be combined into one common formula

$$P^{(s)}(\theta) = \frac{1}{2\pi}\left\{1 + 2\sum_{\substack{n,k=0\\ n>k}}^{\infty} b_n b_k \times \cos[(n-k)\theta]G^{(s)}(n,k)\right\}, \quad (8)$$

where the coefficients $G^{(s)}(n, k)$ are determined as follows:

(a) for the Pegg–Barnett distribution $P^{(PB)}(\theta)$

$$G^{(PB)}(n, k) \equiv 1; \quad (9)$$

(b) for the distribution $P^{(Q)}(\theta) = \int_0^{\infty} Q(\beta)|\beta|d|\beta|$ [9, 10]

$$G^{(Q)}(n, k) = \frac{\Gamma((n+k)/2+1)}{\sqrt{n!k!}}; \quad (10)$$

(c) for the distribution $P^{(W)}(\theta) = \int_0^{\infty} W(\beta)|\beta|d|\beta|$ [11]

$$G^{(W)}(n, k) = \sum_{m=0}^{p}(-1)^{p-m}2^{(|n-k|+2m)/2} \times \sqrt{\binom{p}{m}\binom{q}{p-m}}G^{(Q)}(m, |n-k|+m), \quad (11)$$

where

$$p = \min(n, k), \quad q = \max(n, k). \quad (12)$$

All the matrices $G^{(s)}(n, k)$ are symmetric, and their diagonal elements are equal to unity:

$$G^{(s)}(n, k) = G^{(s)}(k, n), \quad G^{(s)}(n, n) = 1. \quad (13)$$

Let us examine the phase distribution $P^{(Q)}(\theta)$ associated with the Q-function. Because we have $G^{(Q)}(n, k) < 1$ for $n \neq k$, $P^{(Q)}(\theta)$ is always broader than the Pegg–Barnett distribution $P^{(PB)}(\theta)$ [10] and thus carries a smaller amount of phase information in comparison with the latter. Braunstein and Caves [6] called $P^{(Q)}(\theta)$ a "classical" distribution function, because the Q-function is associated with simultaneous measurement of two noncommuting observables, a process inevitably accompanied by additional noise. In the case of the phase distribution $P^{(W)}(\theta)$, associated with the Wigner function, the off-diagonal matrix elements $G^{(W)}(n, k)$ $(n \neq k)$ take values greater, as well as smaller, than unity, and their effect upon the distribution form is not as trivial as in the case of $P^{(Q)}(\theta)$. The $P^{(PB)}(\theta)$, $P^{(Q)}(\theta)$, and $P^{(W)}(\theta)$ distributions for the SFS are presented in Fig. 1 for $|\alpha| = 3$ and $N = 2$. The $P^{(PB)}(\theta)$ and $P^{(W)}(\theta)$ distributions are seen to be very close to each other and have $N + 1$ maxima, whereas $P^{(Q)}(\theta)$ is somewhat broader and has only two maxima. Note that this is true for any $N \geq 1$. Let us consider the Q-function and the Wigner function for the SFS to explain such a form of the phase distributions. The quasi-probability distribution functions for the SFS are obtained by mere translation of the corresponding distribution functions for Fock states and have the form

$$Q_{dN}(\beta) = Q_N(\beta - \alpha) = \frac{1}{\pi}e^{-|\beta-\alpha|^2}\frac{|\beta-\alpha|^{2N}}{N!}, \quad (14)$$

$$W_{dN}(\beta) = W_N(\beta - \alpha) = \frac{2}{\pi}\exp(-2|\beta-\alpha|^2)(-1)^N L_N(4|\beta-\alpha|^2), \quad (15)$$

where $L_N(x)$ is the Laguerre polynomial of the Nth order. These functions are plotted in Fig. 2 for $|\alpha| = 3$ and $N = 2$. It is seen from (14) that the Q-function van-

ishes for $|\beta - \alpha| = 0$ for $N \geq 1$ (Fig. 2a). Consequently, the distribution $P^{(Q)}(\theta)$ has two maxima corresponding to two positions of the azimuth half-plane when the area of its intersection with the Q-function takes the largest values. This is just the concept of the overlapping domain in the phase space [7] applied to the Q-function. Because the Wigner function oscillates as a function of N (Fig. 2b), the application of this idea to it explains the presence of $N + 1$ peaks in the distribution $P^{(W)}(\theta)$. Thus, there is a considerable difference in the phase information contained in the $P^{(Q)}(\theta)$ and $P^{(W)}(\theta)$ distributions. A certain part of the phase information is lost in $P^{(Q)}(\theta)$ because of the averaging with the weight $G^{(Q)}_{(n,k)}$ [10]. The $P^{(\mathrm{PB})}(\theta)$ and $P^{(W)}(\theta)$ distributions almost coincide and carry basically identical phase information, at least in the case of the SFS. This similarity agrees with arguments of the concept of the overlapping domain in the phase space. However, the positive definiteness of the function $P^{(W)}(\theta)$ is not guaranteed, because the Wigner function may take negative values, while there is no such problems for the Pegg–Barnett phase distribution. Completing the consideration of the phase distributions for the SFS, we note that no well-defined Glauber–Sudarshan $\mathcal{P}$-functions exist for these states, and, consequently, the corresponding phase distribution $P^{(\mathcal{P})}(\theta)$ cannot be defined. Now, we will briefly discuss the phase properties of the SRS. According to definition (7), the phase distribution for this state can be obtained by summing the corresponding phase distributions for the SFS over the number of photons N with a weight of $\bar{n}^N/(1 + \bar{n})^{N+1}$. The $\mathcal{P}$-function for the SRS is obtained by mere translation of the $\mathcal{P}$-function for the random state

$$\mathcal{P}_{\mathrm{d.th}}(\beta) = \mathcal{P}_{\mathrm{th.}}(\beta - \alpha) = \frac{1}{\pi\bar{n}}\exp\left(-\frac{|\beta - \alpha|^2}{\bar{n}}\right). \quad (16)$$

Integrating $\mathcal{P}_{\mathrm{d.th}}(\beta)$ over the radial variable $|\beta|$, we obtain the phase distribution $P^{(\mathcal{P})}(\theta)$. The $P^{(\mathrm{PB})}(\theta)$, $P^{(Q)}(\theta)$, $P^{(W)}(\theta)$, and $P^{(\mathcal{P})}(\theta)$ phase distributions for the SRS are presented in Fig. 3 for $|\alpha| = 3$ and $\bar{n} = 0.3$. The $P^{(\mathcal{P})}(\theta)$ and $P^{(Q)}(\theta)$ distributions are seen to be the narrowest and broadest (as in the case of the SFS) distributions, respectively, whereas the $P^{(W)}(\theta)$ and $P^{(\mathrm{PB})}(\theta)$ distributions virtually coincide. However, despite the features indicated above, all of them carry qualitatively identical phase information. If we take into account that the Q-function and the Wigner function for the SRS can be obtained by replacing $\bar{n}$ by $(\bar{n} + 1)$ and $(\bar{n} + 1/2)$, respectively, in formula (16), such a form of the phase distributions can be easily interpreted with the help of the concept of the overlapping domain in the phase space, because the $\mathcal{P}$-function, the Q-function, and the Wigner function in this space have identical shapes of a Gaussian bell, translated by α from the origin of coordinates. The bell of the $\mathcal{P}$-function is the narrowest and highest, and the bell of the Q-function is the broadest and lowest. By intersecting these bells with an azimuth half-plane at an angle of θ and measuring the intersection areas, the corresponding phase distributions can be obtained. Note that in the limit of large $\bar{n}$, all the distributions considered become uniform, i.e., $P(\theta) = 1/2\pi$.

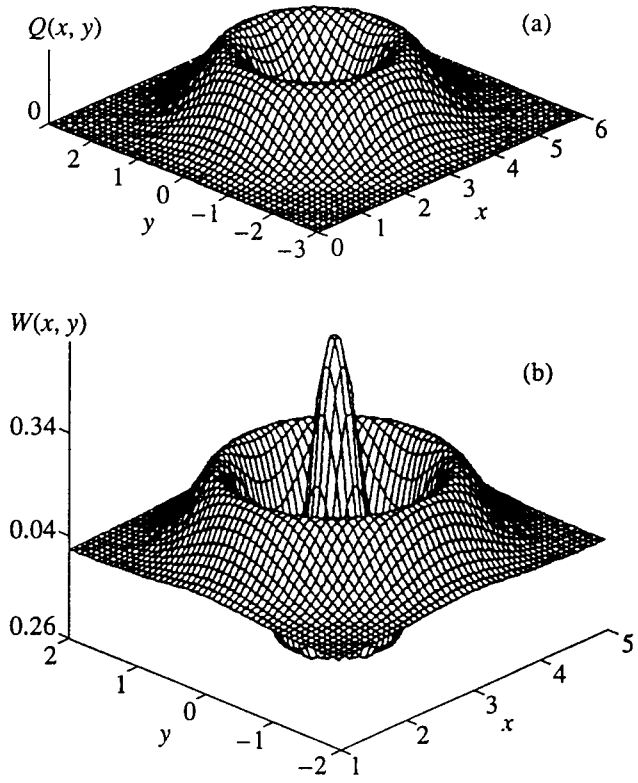

Fig. 2. (a) Q-functions and (b) Wigner functions for shifted Fock states; $x = \mathrm{Re}(\beta - \alpha)$, $y = \mathrm{Im}(\beta - \alpha)$, $\varphi = 0$, $|\alpha| = 3$, and $N = 2$.

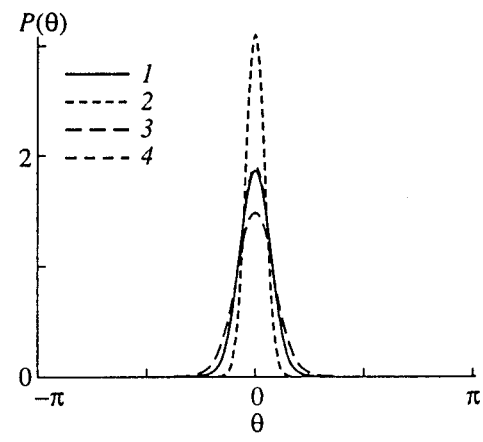

Fig. 3. Phase distributions (*1*) $P^{(\mathrm{PB})}(\theta)$, (*2*) $P^{(\mathcal{P})}(\theta)$, (*3*) $P^{(Q)}(\theta)$, and (*4*) $P^{(W)}(\theta)$ for shifted random states; $|\alpha| = 3$, and $\bar{n} = 0.3$.

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