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Kicked Nonlinear Kerr Medium and Fock States Generation

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Abstract—We discuss a cavity filled with the "Kerr" medium with the $(2q-1)$th nonlinearity $\chi^{(2q-1)}$, periodically kicked by a series of ultrashort laser pulses. Applying the Floquet state and perturbation methods, we find the analytic formulas for the probabilities of the n-photon states. We show that our system can produce pure Fock states. Moreover, we perform numerical calculations to validate our analytical results.

1. INTRODUCTION

During the past few years, much attention has been paid to generation of various nonclassical states of electromagnetic field. Nonetheless, the experimental realization of those states is not a trivial problem. Among many papers concerning methods of generation of various quantum states one can mention the Hong and Mandel paper [1], where they have shown that a one-photon Fock state is produced in the parametric down converter. Stoler and Yurke have studied theoretically the possibility of generation of antibunched light [2]. Another method of producing Fock states has been proposed by Brune *et al.* [3, 4], who have proposed the quantum nondemolition experiment in which detection of the atomic phase by the Ramsey method plays the role of a QND probe giving information on the cavity field energy. After a sequence of atomic measurements, the cavity field collapses into a Fock state with an unpredictable number of photons. A micromaser system in which two-level atoms are injected into a cavity gives also a possibility to generate highly excited Fock states. This model has been discussed by Filipowicz *et al.* [5]. A method of generation of various quantum states based on the interaction with a cavity electromagnetic field of N two-level atoms injected to a single-mode resonator has been presented by Vogel *et al.* [6]. Quite recently, Kozierowski and Chumakov [7] have shown that in the spontaneous emission of the partially inverted Dicke model Fock states can also be generated.

The system discussed in this paper contains a cavity filled with a passive nonlinear "Kerr" medium, which is characterized by the $(2q-1)$th susceptibility $\chi^{(2q-1)}$. The cavity is periodically kicked by a series of ultrashort laser pulses. Moreover, we assume that the field inside the cavity is initially in the vacuum state $|0\rangle$. We will show that for a sufficiently weak external excitation, resonance effects start to play a significant role and lead, in effect, to Fock states. The effectiveness of the Fock state preparation is, of course, considerably diminished by the cavity losses. Nevertheless, it seems important to us that a cavity with a nonlinear Kerr medium and a field initially in the vacuum state, kicked periodically by a train of classical pulses, can be, to high accuracy, a source of Fock states. In our previous paper [8] the problem of generation of the one-photon state has been discussed for such a model with the use of numerical approach and approximate representation for the unitary evolution operator. Here, we will derive approximate analytical formulas for the evolution of the probabilities corresponding to the particular Fock states in which we are interested. We will apply the Floquet states method [9–11] and the perturbation theory to get analytical results. For comparison, we will perform calculations in which we solve the dynamics of our system numerically.

2. THE MODEL AND FLOQUET STATES

The system we in which are interested is governed by the following Hamiltonian:

$$\hat{H}(t) = \hat{H}_{NL} + \hat{H}_1(t), \tag{1}$$

where

$$\hat{H}_{NL} = \frac{\hbar\chi^{(2q-1)}}{q}(\hat{a}^\dagger)^q\hat{a}^q \tag{2}$$

describes the cavity field interaction mediated by the nonlinear medium, whereas

$$\hat{H}_1(t) = \hbar\epsilon(\hat{a}^\dagger + \hat{a})\sum_{n=0}^{\infty}\delta(t-nT) \tag{3}$$

is a time-dependent Hamiltonian corresponding to the driving of the cavity by the external classical field, which is a series of ultrashort pulses modeled by the Dirac delta functions. Moreover, we assume that this interaction is weak, i.e., $\epsilon \ll \chi^{(2q-1)}$. Although the Hamiltonian (2) describes any order of the nonlinearity, in this paper we will restrict our considerations to the case $q = 2$ only. Since the Hamiltonian (1) is a periodic function of time t with the period T, it generates the unitary evolution operator $\hat{U}$ that transforms the initial state of the system $|\Phi(0)\rangle$ (the state for the time $t = 0$)

to a state corresponding to the time $t = T$. Hence, the state of the system corresponding to the arbitrary number of pulses k is determined by

$$|\Phi(t = kT)\rangle = \hat{U}^k|\Phi(t = 0)\rangle, \tag{4}$$

where k denotes the number of kicks. Owing to the periodicity of our system, we can introduce the Floquet states $|E\rangle$ of the Hamiltonian $\hat{H}$:

$$\hat{U}|E\rangle = e^{-i\frac{ET}{\hbar}}|E\rangle. \tag{5}$$

These states belong to the Hilbert space $\mathcal{H}_1$ spanned by the Fock number states $|n\rangle$, $n = 0, 1, 2, \ldots$. The Hamiltonian $\hat{H}$ is time-dependent and obeys the time-dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle. \tag{6}$$

Due to periodicity, it is possible to extend the Hilbert space and reduce the solution of (6) to the eigenvalue problem [9–11] and next to apply the time-independent perturbation theory. To perform calculations using this method we introduce the momentum-like and the position-like operators $\hat{h}$ and $\hat{t}$, which obey the well-known commutation relation $[\hat{h}, \hat{t}] = -i\hbar$. The operator $\hat{h}$ acts on the Hilbert space $\mathcal{H}_2$ generated by the eigenstates $|m\rangle$:

$$\hat{h}|m\rangle = m\hbar\Omega|m\rangle, \tag{7}$$

where $m = 0, \pm1, \pm2, \ldots, \pm\infty$ and the characteristic frequency Ω [12, 13] is related to the periodicity of the system

$$\Omega = \frac{2\pi}{T}. \tag{8}$$

In the next step, we define the following quasi-Hamiltonian $\hat{K}$:

$$\hat{K} = \hat{h} + \hat{H}_{NL} + \hat{H}_1, \tag{9}$$

in extended Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. This operator satisfies the eigenvalue problem

$$\hat{K}|\xi\rangle = \xi|\xi\rangle. \tag{10}$$

The state $|\xi\rangle$ appearing in (10) is related to the Floquet state $|E\rangle$. Since the latter is defined in the Hilbert space $\mathcal{H}_1$, it can be obtained by the following product:

$$|E\rangle = \langle t = 0|\xi\rangle, \tag{11}$$

where

$$|t\rangle = \sum_{m=-\infty}^{+\infty} e^{-i\Omega mt}|m\rangle. \tag{12}$$

The state $|t\rangle$ is defined in the Hilbert space $\mathcal{H}_2$ and obeys the eigenvalue equation $\hat{t}\,|t\rangle = t|t\rangle$, and the Floquet state $|E\rangle$, given by (11), has the energy $E = \xi(\mathrm{mod}\,\hbar\Omega)$.

Thus, to find the Floquet states for the system is equivalent to diagonalizing $\hat{K}$ in the extended Hilbert space. We start with the "free" evolution first, i.e., the parameter ϵ is assumed to be equal to zero. For this case, the quasi-Hamiltonian has the form

$$\hat{K}_0 = \hat{h} + \hat{H}_{NL}. \tag{13}$$

Since the following eigenvalue equations are satisfied,

$$\hat{H}_{NL}|n\rangle = \frac{\hbar\chi}{2}n(n-1)|n\rangle, \qquad \hat{h}|m\rangle = \hbar\Omega m|m\rangle, \tag{14}$$

the eigenstates $|n, m\rangle = |n\rangle \otimes |m\rangle$ of the operator $\hat{K}_0$ are labeled by the following quantum numbers: $n = (0, 1, 2, \ldots)$ and $m = (0, \pm1, \pm2, \ldots)$. Obviously, eigenenergies of the quasi-Hamiltonian $\hat{K}_0$ are determined by these two numbers and are equal to

$$E_{n,m} = \hbar\left(\frac{\chi}{2}n(n-1) + m\Omega\right). \tag{15}$$

It is clear from (15) that the states $|0, m\rangle$ and $|1, m\rangle$ are degenerate and their energies are equal to $E_{0,m} = E_{1,m} = m\hbar\Omega$. In our further calculations we will only consider characteristic frequencies Ω that guarantee that any other states degenerate with $|0, m\rangle$ and $|1, m\rangle$ are $|n', m'\rangle$, where $n' \geq 3$. Since the interaction $\hat{H}_1$ is linear in the cavity field operators, to diagonalize $\hat{K}$, it is sufficient to perform perturbation calculations on the states $|0, m\rangle$ and $|1, m\rangle$ to the first order in ϵ omitting other degenerate states. When we include the interaction with the external field ($\epsilon \neq 0$), the quasi-Hamiltonian becomes

$$\hat{K} = \hat{K}_0 + \epsilon\hat{K}_1, \tag{16}$$

where

$$\hat{K}_1 = \frac{\hbar}{T}(\hat{a}^\dagger + \hat{a})\sum_{l=-\infty}^{+\infty} c_l e^{i\Omega l\hat{t}}. \tag{17}$$

The sum appearing in (17) is a Fourier representation of the series of ultrashort kicks, and the coefficients c_l are equal to

$$c_l = e^{-i\Omega l\tau}\Big|_{\tau\to 0}, \tag{18}$$

where the small interval of time τ has been introduced to ensure that a single kick is located inside the period T. We want to find the evolution of the states $|0, m\rangle$ and $|1, m\rangle$. Expanding the state $|\xi\rangle$ and the value ξ in series of ϵ and inserting into them (10) we get a set of equations for finding the eigenstates and eigenvalues for a given order of perturbation. In zeroth order the state is given by

$$|\xi^{(0)}\rangle = a|0, m\rangle + b|1, m\rangle. \tag{19}$$

To find the first-order contributions we have to solve the eigenvalue equation

$$\frac{\hbar}{T}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \xi^{(1)}\begin{pmatrix} a \\ b \end{pmatrix}, \tag{20}$$

which has the following solution:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \quad (21)$$

The quantity $\epsilon\xi^{(1)}$ is the first-order correction to the energy, and we have $\xi^{(1)} = \pm k/T$. The resulting states labeled by $|\xi_{\pm}^{(0)}\rangle$ correspond to the Floquet states $|E_{\pm}^{(0)}\rangle$. According to (11), the latter are given by

$$|E_{\pm}^{(0)}\rangle = \langle t = 0|\xi_{\pm}^{(0)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle). \quad (22)$$

After continuing the perturbation procedure, we get the first-order solutions for the states in extended Hilbert space and, in consequence, the first-order Floquet states and their energies. These states are of the form

$$|E_{\pm}^{(1)}\rangle = (1 \pm \epsilon A)|E_{\pm}^{(0)}\rangle \pm \epsilon B|2\rangle + \mathcal{O}(\epsilon^2), \quad (23)$$

where

$$A = \frac{1}{2\pi}\sum_{\substack{l \neq 0 \\ l = -\infty}}^{+\infty} \frac{c_l}{l}, \quad B = \frac{1}{2\pi}\sum_{l=-\infty}^{+\infty} \frac{c_l}{l + \chi/\Omega}, \quad (24)$$

and the energies $E_{\pm}^{(1)}$ are

$$E_{\pm}^{(1)} = \pm\frac{\epsilon\hbar}{T} + \mathcal{O}(\epsilon^2). \quad (25)$$

Since we are interested in the generation of the Fock states from the system with the cavity field in the vacuum state, we can express $|0\rangle$ as a function of the first-order solution for the Floquet state (23). Then, after straightforward algebra, we obtain the probability amplitudes for the Fock states. Thus, the probabilities $P_n = |\langle n|U^m|0\rangle|^2$ ($n = 0, 1, 2, \ldots$) for the times t just after mth kick can be written as

$$\begin{aligned} P_0(k) &= \cos^2(k\epsilon) + \mathcal{O}(\epsilon^2), \\ P_1(k) &= \sin^2(k\epsilon) + \mathcal{O}(\epsilon^2), \\ P_2(k) &= \frac{2\epsilon^2}{\sin^2(\chi T/2)}\sin^2(k\epsilon) + \mathcal{O}(\epsilon^4). \end{aligned} \quad (26)$$

It is seen that the system starts to evolve from the vacuum state $|0\rangle$. Then, after subsequent kicks, the probability $P_0(k)$ decreases, whereas the probability corresponding to the one-photon state $P_1(k)$ increases. In consequence, after $k = \pi\epsilon$ pulses, the system evolves to the pure one-photon state. The influence of the higher n-photon states on the system is proportional to ϵ^2 and for weak external excitations can be neglected. For instance, the probability for the two-photon state $P_2(k)$ oscillates with the amplitude equal to $\epsilon^2/(2\sin^2(\chi T/2))$. This approach enables us to get close form of analytical results describing the evolution of the system.

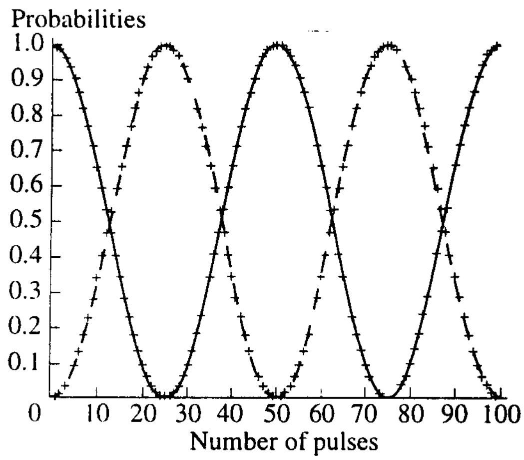

Fig. 1. Time evolution of the probabilities for the vacuum $|0\rangle$ (solid line) and one-photon $|1\rangle$ states. The time $T = \pi$, and the kick strength is $\epsilon = \pi/50$. The cross marks correspond to the numerical results.

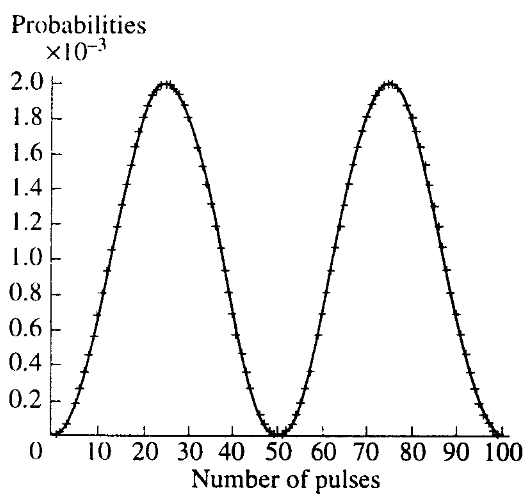

Fig. 2. The same as for Fig. 1, but for the probability P_2 corresponding to the two-photon state $|2\rangle$. Cross marks correspond to the numerical results again.

3. NUMERICAL APPROACH

To verify our analytical results, we perform numerical calculations and compare their results with those based on formulas (26). This will be done on the basis of the unitary evolution operator $\hat{U}$ (4), similarly to [8]. Owing to the fact that the ultrashort pulses are modeled by the Dirac delta functions, the time-evolution of the system can be divided into two different stages. The first is "free" evolution determined by the Hamiltonian $\hat{H}_{HL}$, during the time T between two subsequent pulses. This evolution is described by the following unitary operator:

$$\hat{U}_0 = \exp\left(-i\frac{\chi T}{2}(\hat{a}^{\dagger})^2\hat{a}^2\right). \quad (27)$$

The second stage of the time evolution of the system is caused by its interaction with the infinitely short pulse. This part of the evolution is described by the Hamiltonian $\hat{H}_1$ (3). Thus, the evolution operator corresponding to the interaction during one pulse can be written as

$$\hat{U}_1 = \exp(-i\epsilon(\hat{a}^{\dagger} + \hat{a})). \tag{28}$$

In consequence, the total evolution of the state of the system can be described as subsequent action of the above operators ($\hat{U}_0$ and $\hat{U}_1$) on the initial state. Assuming that for the time $t = 0$ the system was in the vacuum state, we express the state $|\Phi_k\rangle$ just after kth kick as

$$|\Phi_k\rangle = (\hat{U}_0\hat{U}_1)^k|0\rangle. \tag{29}$$

Formula (29) is a starting point for our numerical calculations.

Figure 1 shows the probabilities $P_0(k)$ and $P_1(k)$ corresponding to the vacuum $|0\rangle$ and one-photon $|1\rangle$ states, respectively. For the plots, we use units such that $\chi = 1$. The probabilities are plotted as functions of the number of pulses k. The lines illustrate the evolution of the vacuum state and the one-photon state according to analytical formulas (26). The system starts its evolution from the vacuum state $|0\rangle$. Then the probability for the one-photon state increases, and, in consequence, for the pulse number $k = \pi/\epsilon$, the system is almost ideally in the pure one-photon Fock state (up to terms $\sim\epsilon^2$). With crosses we have marked the results of our numerical calculations, and it is quite evident that they agree very well with those obtained from the analytical formulas. This is because $\epsilon \ll 1$ and the influence of the higher n-photon states (for $n > 1$) on P_0 and P_1 is negligible. For reference we have plotted in Fig. 2 the probability P_2 for the two-photon state $|2\rangle$. We see that it oscillates as a sine squared function, but the amplitude of these oscillations is much smaller than those in Fig. 1. This amplitude is proportional to $\epsilon^2 \sim 0.002$, and it is negligible in the scale of Fig. 1. The results of the numerical calculations are also marked, and, as in Fig. 1, they agree perfectly with those obtained from the analytical formula.

Obviously, for real physical systems we cannot avoid dissipation. In consequence, we cannot take ϵ too small in order to avoid complete damping of the field during the evolution between two subsequent pumping kicks. Moreover, the dissipation in the system leads to a mixture of the quantum states. Hence, the pure state picture of the field evolution presented above will be obscured. However, for the cases when the damping is weak, it is still possible to get the field in a cavity being very close to the one-photon state. For the case of the nonlinear medium with damping the master equation method should be applied. This problem has been solved exactly by Milburn and Holmes [14, 15] who have solved the master equation in their discussion of quantum and classical dynamics of a pulsed parametric oscillator with a Kerr nonlinearity. We simply take advantage of this solution and apply it here to take into account the dissipation in the system. The results are shown in Fig. 3, where the probabilities of the vacuum and the one-photon states are plotted for $\epsilon = \pi/50$, $T = \pi$, and various values of damping parameter γ. It is seen that for $\gamma = 0.01$ it is still over 75% of the population that is found in the state $|1\rangle$, while for $\gamma = 0.1$ it is already less than 15%. Thus, the dissipation in the system drastically lowers the effectiveness of producing the one-photon state. Nonetheless, our system seems to be interesting enough to be worthy of further investigation.

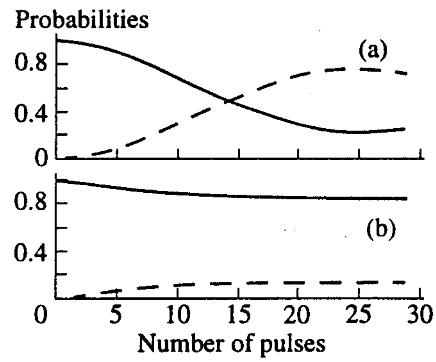

Fig. 3. The probabilities for the vacuum $|0\rangle$ (solid lines) and one-photon $|1\rangle$ states (dashed lines). The damping constant is γ = (a) 0.01 and (b) 0.1. The kick strength is $\epsilon = \pi/50$ and the time is $T = \pi$.

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