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QUANTUM PHASE PROPERTIES OF NONLINEAR OPTICAL PHENOMENA[1]

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Quantum phase properties of optical fields generated in nonlinear optical processes are reviewed. Phase distributions obtained from the Hermitian phase formalism of Pegg and Barnett and from the s-parametrized quasidistributions are used to represent the phase of the field.

1. Introduction

In recent years, a significant progress has been achieved in clarifying the status of the quantum mechanical phase operator, describing phase properties of optical fields in terms of various phase distribution functions, and measuring phase dependent physical quantities. So, although the quantum phase is still a subject of some controversy, we can now say that, despite the existence of various different conceptions of phase, we are on the way to a unified view and understanding of the quantum-optical phase.

In this review we are not going to give detailed account of different descriptions of the quantum phase showing their similarities and differences. We shall rather concentrate on the description of quantum properties of real field states that are generated in various nonlinear optical processes. Nonlinear optical phenomena are sources of optical fields, statistical properties of which have been changed in a nontrivial way as a result of nonlinear transformation. Quantum phase properties are among those statistical properties that undergo nonlinear changes, and fields generated in different nonlinear processes have different phase properties. With the existing now phase formalisms the quantum phase properties of such fields can be studied in a systematic way, and quantitative comparisons between different quantum field states can be made. We shall use the Pegg-Barnett (PB) phase formalism and the phase formalism based on the s-parametrized quasidistribution functions to give several examples of quantum phase distributions and other phase characteristics associated with the particular one- and two-mode field states.

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2. The Pegg-Barnett phase distribution

Pegg and Barnett [1, 2] (see also [3]) introduced the Hermitian phase formalism, which is based on the observation that in a finite-dimensional state space the states with the well-defined phase exist [4]. Thus they restrict the state space to a finite $(\sigma+1)$-dimensional Hilbert space $\mathcal{H}^{(\sigma)}$ spanned by the number states $|0\rangle$, $|1\rangle$,..., $|\sigma\rangle$. In this space they define a complete orthonormal set of phase states by

$$|\theta_m\rangle = \frac{1}{\sqrt{\sigma+1}}\sum_{n=0}^{\sigma}\exp(in\theta_m)\,|n\rangle, \qquad m=0,1,\ldots,\sigma, \tag{1}$$

where the values of θ_m are given by

$$\theta_m = \theta_0 + \frac{2\pi m}{\sigma+1}. \tag{2}$$

The value of θ_0 is arbitrary and defines a particular basis set of $(\sigma+1)$ mutually orthogonal phase states.

The PB Hermitian phase operator is defined as

$$\hat{\Phi}_\theta \equiv \left(\hat{\Phi}_{\theta_0}^{(\sigma)}\right)_{\mathrm{PB}} = \sum_{m=0}^{\sigma}\theta_m\,|\theta_m\rangle\langle\theta_m|. \tag{3}$$

The phase states (1) are eigenstates of the phase operator (3) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and $\theta_0+2\pi\frac{\sigma}{\sigma+1}$. The PB prescription is to evaluate any observable of interest in the finite basis (1) and only after that take the limit $\sigma\to\infty$.

Since the phase states (1) are orthonormal, $\langle\theta_m|\theta_{m'}\rangle=\delta_{mm'}$, the kth power of the PB phase operator (3) can be written as

$$\hat{\Phi}_\theta^k = \sum_{m=0}^{\sigma}\theta_m^k\,|\theta_m\rangle\langle\theta_m|. \tag{4}$$

As the Hermitian phase operator is defined, one can calculate the expectation value and variance of this operator for a given state of the field $|f\rangle$. Moreover, the PB phase formalism allows to introduce the continuous phase probability distribution that is a representation of the quantum state of the field and describes the phase properties of the field in a very spectacular fashion.

The mean value of the phase operator and its variance can be calculated according to the formulas

$$\langle f|\hat{\Phi}_\theta|f\rangle = \sum_m \theta_m\left|\langle\theta_m|f\rangle\right|^2, \tag{5}$$

$$\langle(\Delta\hat{\Phi}_\theta)^2\rangle = \sum_m(\theta_m-\langle\hat{\Phi}_\theta\rangle)^2\left|\langle\theta_m|f\rangle\right|^2, \tag{6}$$

where $\left|\langle\theta_m|f\rangle\right|^2$ denotes the probability of being found in the phase state $|\theta_m\rangle$ The mean and variance of $\hat{\Phi}_\theta$ will depend on the chosen value of θ_0. If the field state $|f\rangle$ is a partial phase state, i. e., the amplitude c_n of its decomposition in the Fock state basis can be written as

$$c_n = b_n \mathrm{e}^{\mathrm{i}n\varphi}, \tag{7}$$

the most convenient and physically transparent way of choosing θ_0 is to symmetrize the phase window with respect to the phase φ. This means the choice

$$\theta_0 = \varphi - \frac{\pi\sigma}{\sigma+1}, \tag{8}$$

and after introducing new phase label

$$\mu = m - \frac{\sigma}{2}, \tag{9}$$

the phase probability distribution becomes

$$\left|\langle\theta_\mu|f\rangle\right|^2 = \frac{1}{\sigma+1} + \frac{2}{\sigma+1}\sum_{n>k} b_n b_k \cos\left[(n-k)\frac{2\pi\mu}{\sigma+1}\right], \tag{10}$$

with μ which goes in integer steps from $-\sigma/2$ to $\sigma/2$. Since the distribution (10) is symmetrical in μ, we immediately get, according to (8)–(10),

$$\langle f|\hat{\Phi}_\theta|f\rangle = \varphi. \tag{11}$$

This result means that for a partial phase state with phase φ, the choice of θ_0 as in (8) relates directly the expectation value of the phase operator with the phase φ.

We should make a remark here that it is not quite correct to take the amplitudes c_n, eq. (7), obtained from the decomposition of the state $|f\rangle$ in the infinite-dimensional Fock basis and apply them in the $(\sigma+1)$-dimensional space. In the finite-dimensional space the amplitudes c_n should be redefined as to make the state $|f\rangle$ normalized in this space. Coherent states in a finite-dimensional basis have been discussed by Bužek et al. [5] and Miranowicz at al. [6]. However, if σ is taken so large that the probabilities $|c_n|^2$ for n approaching σ are negligible, the use of the infinite basis expansion coefficients leads to a negligible error and is justified.

All field states generated in real experiments belong to the so-called physical states [2]. They are defined as the states of finite-energy (finite mean photon number and its higher moments). For such states the continuous phase distribution can be introduced. Since the density of states is $(\sigma+1)/2\pi$, we can write the expectation value of the kth power of the phase operator as

$$\langle f|\hat{\Phi}_\theta^k|f\rangle = \int_{\theta_0}^{\theta_0+2\pi} \mathrm{d}\theta\, \theta^k P(\theta), \tag{12}$$

where the continuous-phase distribution $P(\theta) = P_{\rm PB}(\theta)$ is introduced by

$$P(\theta) = \lim_{\sigma\to\infty} \frac{\sigma+1}{2\pi} |\langle\theta_m|f\rangle|^2, \tag{13}$$

and θ_m has been replaced by the continuous-phase variable θ. If the state $|f\rangle$ has the number-state decomposition with the amplitudes c_n then the PB phase distribution is given by [2]

$$P(\theta) = \frac{1}{2\pi}\left\{1+2{\rm Re}\sum_{\substack{n,m\\ m>n}} c_m c_n^* \exp[-{\rm i}(m-n)\theta]\right\}, \tag{14}$$

and for fields being in mixed states described by the density matrix $\hat\rho$, formula (14) generalizes to

$$P(\theta) = \frac{1}{2\pi}\left\{1+2{\rm Re}\sum_{m>n} \rho_{mn} \exp[-{\rm i}(m-n)\theta]\right\}, \tag{15}$$

where, $\rho_{mn} = \langle m|\hat\rho|n\rangle$, are the density matrix elements in the number state basis. Formulas (14) or (15) can be used for calculations of the PB phase distribution for any state with known amplitudes c_n or matrix elements ρ_{mn}, but despite the fact that the formulas are exact, they can rarely be summed up into a closed form, and usually numerical summation must be performed to find the phase distribution. Such numerical summations have been widely applied in studying phase properties of optical fields. The PB phase distribution, Eqs. (14) or (15), is obviously 2π-periodic, and for all states with the density matrix diagonal in the number states the phase distribution is uniform over the 2π-wide phase window. These are nondiagonal elements of the density matrix that lead to the structure of the phase distribution. The PB distribution is positive definite and normalized, and it is a good representation of the quantum state of the field.

3. Phase distributions associated with the quasiprobability distributions

The quasidistribution functions such as Glauber-Sudarshan P function, Wigner function, or the Husimi Q function are special examples of the more general s-parametrized quasidistributions introduced by Cahill and Glauber [7, 8]. Such distributions are representations of the quantum states in the complex plain, and the parameter s is related to the particular ordering of the annihilation and creation operators. If such quasidistributions are integrated over the "radial" variable, the normalized, 2π-periodic phase distributions are obtained [9]. In the Fock basis the resulting formula for the phase distribution is very similar to the PB phase distribution and is given by [9]

$$P^{(s)}(\theta) = \frac{1}{2\pi}\left\{1+2{\rm Re}\sum_{m>n} \rho_{mn} {\rm e}^{-{\rm i}(m-n)\theta} G^{(s)}(m,n)\right\}. \tag{16}$$

The difference is in the coefficients $G^{(s)}(m,n)$ that appeared in (16), and which are given by [9]

$$\begin{aligned} G^{(s)}(m,n) &= \left(\frac{2}{1-s}\right)^{\frac{m+n}{2}} \sum_{l=0}^{\min(m,n)} (-1)^l \left(\frac{1+s}{2}\right)^l \\ &\times \sqrt{\binom{n}{l}\binom{m}{l}} \frac{\Gamma\left(\frac{m+n}{2}-l+1\right)}{\sqrt{(m-l)!(n-l)!}}. \end{aligned} \tag{17}$$

Formulas (16)- (17) allow for calculations of the s-parametrized phase distributions for any state with known ρ_{mn} and compare them to the PB phase distribution, for which $G^{(s)}(m,n)=1$. The phase distributions associated with particular quasiprobability distributions have been used in literature to describe phase properties of field states. For example, the integrated Wigner function ($s=0$) has been applied by Schleich, Horowicz and Varro [10] in their description of the phase probability distribution for a highly squeezed states. The integrated Q-function ($s=-1$) has been used by Braunstein and Caves [11] to describe phase properties of the generalized squeezed states. Eiselt and Risken [12] have used the s-parametrized quasiprobability distributions to study properties of the Jaynes-Cummings model with cavity damping. In their approach, Eiselt and Risken [12] have used the series expansions of the quasiprobability distribution functions, and they have found an expression relating the PB phase distribution to the quasiprobability distributions in a form of the integral relation and applied it to the Jaynes-Cummings model . Their formulas, however, do not work for $s=-1$, i.e. for the Q-function. For some field states the phase distributions $P^{(s)}(\theta)$ can be found in a closed form via direct integrations.

The s-parametrized phase distributions are different from the PB phase distribution, but in some cases the phase information carried by such distributions is basically the same as that of the PB phase distribution. The coefficients $G^{(s)}(m,n)$ that multiply the nondiagonal elements of the field density matrix ρ_{mn}, for $s\le -1$ have values smaller than unity, and the resulting phase distribution is broader than the PB distribution. Such distributions can be associated with the noisy measurements of the phase distribution [13]. We will show several examples of the different phase distributions that are found for fields generated in nonlinear optical processes.

4. Phase properties of field states

Optical fields produced as a result of nonlinear transformation of the incoming field in the nonlinear optical processes have their statistical properties changed with respect to the original field. Quantum phase properties of the resulting field belong to this class. Each quantum state of the field is characterized by its own phase properties which are represented by the phase distribution of the state and/or by the values of mean phase, variance, phase correlation, etc. Many different states of the field have been studied from the point of view of their quantum phase properties (see the special issue of Physica Scripta, vol. T48, 1993 and references therein). Here, we are going to give only few examples illustrating the use of the PB and s-parametrized phase distributions to describe phase properties of optical fields.

First, we remark that for coherent states the PB phase distribution is given by (14) with the coefficients c_n being Poissonian weight factors, and the phase distribution is obtained by performing the summations numerically. In contrast to this formula, the s-parametrized phase distribution for a coherent state can be obtained in a closed form [9]

$$\begin{aligned} P^{(s)}(\theta) &= \int_0^\infty \mathcal{W}^{(s)}(\alpha)|\alpha|\,\mathrm{d}|\alpha| \\ &= \frac{1}{2\pi}\exp[-(X_0^2 - X^2)]\left\{\exp(-X^2) + \sqrt{\pi}X\,(1+\mathrm{erf}(X))\right\}, \end{aligned} \tag{18}$$

where

$$X = X^{(s)}(\theta) = \sqrt{\frac{2}{1-s}}|\alpha_0|\cos(\theta - \vartheta_0), \tag{19}$$

and $X_0 = X^{(s)}(\vartheta_0)$, ϑ_0 is the phase of α_0. Formula (18) is exact, it is 2π-periodic, positive definite and normalized, so it satisfies all requirements for the phase distribution. Moreover, formula (18) has quite simple and transparent structure. For small $|\alpha_0|$, the first term in braces plays an essential role, and for $|\alpha_0| \to 0$ we get uniform phase distribution. For large $|\alpha_0|$, the second term in the braces predominates, and if we replace erf(X) by the unity, we obtain the approximate asymptotic formula given by Schleich, Dowling, Horowicz and Varro [14] (for $s = 0$)

$$P^{(0)}(\theta) \approx \sqrt{\frac{2}{\pi}}|\alpha_0|\cos(\theta - \vartheta_0)\exp[-2|\alpha_0|^2\sin^2(\theta - \vartheta_0)], \tag{20}$$

which however, can be applied only for $-\pi/2 \leq (\theta - \vartheta_0) \leq \pi/2$. After linearization of (20) with respect to θ, the approximate formula for coherent states with large mean number of photons obtained by Barnett and Pegg [3] is recovered. The presence of the error function in (18) handles properly the phase behavior in the total range of phase values $-\pi \leq (\theta - \vartheta_0) < \pi$. This example shows clear advantage of the s-parametrized phase distribution over the PB phase distribution from the point of view of calculation simplicity as well as interpretation insight into the form of the distribution. It was shown [9] that for large number of photons $P^{(0)}(\theta)$ is very close to the PB phase distribution, but for small number of photons the $P^{(-1)}(\theta)$ is closer to the PB distribution. So, in case of large photon numbers formula (18) is a very good and simple approximation to the PB formula (14).

Probably even more striking contrast between the analytical forms of the s-parametrized and PB phase distributions is seen for squeezed states, for which the s-parametrized phase distribution has the form [9]

$$\begin{aligned} P^{(s)}(\theta) &= \frac{1}{2\pi}\frac{\sqrt{(\mu - s)(\mu^{-1} - s)}}{(\mu - s)\cos^2\theta + (\mu^{-1} - s)\sin^2\theta} \\ &\times\ \exp[-(X_0^2 - X^2)]\left\{\exp(-X^2) + \sqrt{\pi}X\,(1+\mathrm{erf}(X))\right\}, \end{aligned} \tag{21}$$

where

$$X = X^{(s)}(\theta) = \sqrt{\frac{2}{\mu^{-1}-s}}\,\frac{\alpha_0\sqrt{\mu-s}\cos\theta}{\sqrt{(\mu-s)\cos^2\theta+(\mu^{-1}-s)\sin^2\theta}}, \tag{22}$$

and

$$\mu = e^{2r} \tag{23}$$

with r being the squeezing parameter and α_0 (assumed real) being the amplitude of the coherent component. Although the variable X is slightly different, the main structure of the phase distribution is the same as for the coherent state. Formula (21) is valid for both small and large α_0. For $\alpha_0 = 0$ we have the result for squeezed vacuum. After appropriate approximations one can easily reproduce the formula obtained by Schleich, Horowicz and Varro [10] for a highly squeezed state. In contrast, the PB phase distribution require summations in (14) with rather complicated coefficients c_n. Numerical calculations show [9] that, again, $P^{(0)}(\theta)$ is sharper than the PB distribution, but both have very similar shape.

Single-mode squeezed states differ essentially from the two-mode squeezed states discussed extensively by Caves and Schumaker [15] . The PB phase formalism has been applied by Barnett and Pegg [16], and Gantsog and Tanaś [17] to study the phase properties of the two-mode squeezed vacuum states.

The two-mode squeezed vacuum state is defined by applying the two-mode squeeze operator $S(r,\varphi)$ on the two-mode vacuum, and is given by [18]

$$\begin{aligned} |0,0\rangle_{(r,\varphi)} &= \hat{S}(r,\varphi)|0,0\rangle \\ &= (\cosh r)^{-1}\exp\left(e^{2i\varphi}\tanh r\,\hat{a}_1^\dagger\hat{a}_2^\dagger\right)|0,0\rangle \\ &= (\cosh r)^{-1}\sum_{n=0}^{\infty}\left(e^{2i\varphi}\tanh r\right)^n|n,n\rangle, \end{aligned} \tag{24}$$

where $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ are the creation operators for the two modes, r $(0 \le r < \infty)$ is the strength of squeezing, and φ $(-\pi/2 < \varphi \le \pi/2)$ is the phase (note the shift in phase by $\pi/2$ with respect to the usual choice of φ).

The state (24), after generalizing the PB formalism to the two-mode case, leads to the joint probability distribution for the phases θ_1 and θ_2 of the two modes in the form [16]

$$P(\theta_1,\theta_2) = (4\pi^2\cosh^2 r)^{-1}\left(1+\tanh^2 r-2\tanh r\cos(\theta_1+\theta_2)\right)^{-1}. \tag{25}$$

One important property of the two-mode squeezed vacuum, which is seen from (25), is that $P(\theta_1,\theta_2)$ depends on the sum of the two phases only. Integrating $P(\theta_1,\theta_2)$ over one of the phases gives the marginal phase distribution $P(\theta_1)$ or $P(\theta_2)$ for the phase θ_1 or θ_2

$$P(\theta_1) = \int_{-\pi}^{\pi} P(\theta_1,\theta_2)\,d\theta_2 = P(\theta_2) = \frac{1}{2\pi}, \tag{26}$$

which means that the phases θ_1 and θ_2 of the individual modes are uniformly distributed. This gives

$$\langle\hat{\Phi}_{\theta_1}\rangle = \varphi + \int_{-\pi}^{\pi} \theta_1 P(\theta_1)\,\mathrm{d}\theta_1 = \langle\hat{\Phi}_{\theta_2}\rangle = \varphi, \tag{27}$$

and

$$\langle\hat{\Phi}_{\theta_1} + \hat{\Phi}_{\theta_2}\rangle = 2\varphi, \qquad \langle\hat{\Phi}_{\theta_1} - \hat{\Phi}_{\theta_2}\rangle = 0. \tag{28}$$

So, the phase-sum operator is related to the phase 2φ defining the two-mode squeezed vacuum state (24).

The two-mode squeezed vacuum has very specific phase properties: individual phases as well as the phase difference are random, and the only non-random phase is the phase sum.

The variance of the phase-sum operator can be calculated according to the general formula

$$\langle[\Delta(\hat{\Phi}_{\theta_1} + \hat{\Phi}_{\theta_2})]^2\rangle = \langle(\Delta\hat{\Phi}_{\theta_1})^2\rangle + \langle(\Delta\hat{\Phi}_{\theta_2})^2\rangle + 2C_{12} \tag{29}$$

in terms of the individual phase variances $\langle(\Delta\hat{\Phi}_{\theta_{1,2}})^2\rangle$ and the phase correlation function (correlation coefficient)

$$\begin{aligned} C_{12} &\equiv \langle\hat{\Phi}_{\theta_1}\hat{\Phi}_{\theta_2}\rangle - \langle\hat{\Phi}_{\theta_1}\rangle\langle\hat{\Phi}_{\theta_2}\rangle. \\ &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \theta_1\theta_2 P(\theta_1,\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \end{aligned} \tag{30}$$

The variances $\langle(\Delta\hat{\Phi}_{\theta_{1,2}})^2\rangle$ are simply $\pi^2/3$ (because of (26)), and the phase correlation function C_{12} is equal to

$$C_{12} = -2(\cosh\, r)^{-2}\sum_{n>k}\frac{(\tanh\, r)^{n+k}}{(n-k)^2} = -2\,\mathrm{dilog}(1-\tanh\, r). \tag{31}$$

This correlation function describes the correlation between the phases of the two modes of the two-mode squeezed vacuum. The correlation is negative and asymptotically, as r tends to infinity, approaches $-\pi^2/3$. The strong negative correlation between the two phases lowers the variance (29) of the phase-sum operator. Asymptotically, for $r\to\infty$, this variance tends to zero, which means that for very large squeezing the sum of the two phases becomes well defined (the two phases are locked). The phase sum variance finally takes the form

$$\langle[\Delta(\hat{\Phi}_{\theta_1} + \hat{\Phi}_{\theta_2})]^2\rangle = 2\frac{\pi^2}{3} - 4\,\mathrm{dilog}(1-\tanh\, r). \tag{32}$$

The dependence of the joint phase distribution (25) on the sum of the individual phases suggests that, after the appropriate change of variables to the sum and the difference of the individual phases, we get the phase distribution for a single phase variable, the phase sum, with the properties of a single phase distribution, e.g., the 2π-periodicity. This transition, however, requires some care with handling properly the ranges of the phase values, because the phase sum and difference have the 4π-wide ranges of values. Barnett and Pegg [16] proposed a casting procedure to cast the new phases into the 2π-wide window. As a result of this procedure we get the mod(2π) phase distribution for the phase sum θ_+ in the form [16]

$$P_{2\pi}(\theta_+) = (2\pi\ \cosh^2 r)^{-1} \left(1 + \tanh^2 r - 2\tanh r \cos\theta_+\right)^{-1}. \tag{33}$$

where θ_+ is in the range $-\pi$ and π. The phase sum variance is now given by [16]

$$\langle[\Delta(\hat{\Phi}_{\theta_1} + \hat{\Phi}_{\theta_2})]^2\rangle_{2\pi} = \frac{\pi^2}{3} + 4\,\mathrm{dilog}(1 + \tanh\ r) \tag{34}$$

It is evident that the two variances, eqs. (32) and (34), have generally different values, although the asymptotic values are the same. This means that the two descriptions, which are equally well justified, must be interpreted with care. The original approach is better, for example, in showing explicitly the intermode phase correlations. The mod(2π) phase distribution for the phase sum, on the other hand, is simpler in calculations of the phase sum properties. The intermode correlations in this case are hidden in the value of the phase sum variance and are not seen directly.

5. Conclusion

The above examples of the fields, which are pretty simple, were chosen to illustrate some essential points of the quantum phase description of optical fields. We have shown the relation between the PB phase description and the description based on the s-parametrized phase distributions. There is a general relation given by (16) which says that the s-parametrized phase distribution can be obtained from the PB phase distribution (15) by multiplying the field density matrix elements ρ_{mn} by the coefficients $G^{(s)}(m,n)$ given by eq. (17). The calculation of the phase distribution in this way requires, however, numerical summations. In some cases, the s-parametrized phase distributions can be obtained by direct integrations in a closed form, which allows for better insight into the structure of the distribution.

There is a large variety of the nonlinear optical phenomena that can produce states with different phase properties. We just mention a few of them, the phase properties of which are known, like Jaynes-Cummings model [19, 20, 21, 22, 23], anharmonic oscillator model [24, 25, 26, 27, 28], harmonics generation [29, 30, 31], or down-conversion [32, 33, 34, 35, 36]. The space available for this article does not allow for presentation their phase properties, but more information can be found in our review article [37].

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