Quantum Opt. 6 (1994) 517–526. Printed in the UK

Phase properties of binomial and negative binomial states

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Received 14 June 1994

Abstract. Phase properties of binomial and negative binomial states are studied within the Pegg–Barnett Hermitian phase formalism

1. Introduction

Binomial and negative binomial states are the field states that are superpositions of the number states with the appropriately chosen coefficients (amplitudes). Quantum properties of the binomial states have been discussed by Stoler *et al* [1], who have shown that such states can display antibunching and sub-Poissonian photon statistics as well as squeezing. Some possibilities of producing binomial states in practice have also been discussed in the literature [2]. The effect of binomial field distribution on collapses and revivals in the Jaynes–Cummings model has been studied by Joshi and Puri [3]. Joshi and Lawande [4] have considered the effects of negative binomial field distribution on Rabi oscillations in a two-level atom. The properties of the negative binomial states of the field and a possibility of their production in practice have been discussed by Agarwal [5]. The binomial state is 'intermediate' between a pure number state and a pure coherent state, and the negative binomial state intermediate between a pure thermal state and a pure coherent state. Both binomial and negative binomial states can exhibit squeezing, a phase sensitive effect [1–4]. It can be, thus, interesting to study their quantum phase properties.

In this paper we study the phase properties of both binomial and negative binomial states [6] using the Pegg–Barnett phase formalism [7–9].

2. Binomial and negative binomial states

The binomial states are defined as [1]

$$|\psi\rangle_b = \sum_{n=0}^{N} c_n(p)|n\rangle \tag{1}$$

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0954-8998/94/060517+10 $19.50

where

$$c_n(p)=\left[\binom{N}{n}p^n(1-p)^{N-n}\right]^{1/2} \qquad n=0,1,2,\ldots,N, \quad 0\leqslant p\leqslant 1 \quad (2)$$

which means that the binomial state describes the state of the field having a binomial photon distribution $P_n=|c_n(p)|^2$ with the mean photon number $\bar{n}=Np$, and the variance $(\Delta n)^2=\bar{n^2}-\bar{n}^2=Np(1-p)$. Since the variance is always less than the mean, the Mandel parameter $Q=(\Delta n)^2/\bar{n}-1$, which signifies deviations from the Poissonian distribution, is always negative. Thus photon statistics in the binomial states is always sub-Poissonian. For $p=1$, $c_n(p)=\delta_{nN}$ and the binomial state becomes the number state $|N\rangle$. In the opposite extreme, for $p\rightarrow 0$ and $N\rightarrow\infty$ but with $Np=\bar{n}=$ const, the binomial state becomes a coherent state with the mean number of photons $\bar{n}$.

The negative binomial states are defined as [6]

$$|\psi\rangle_{\mathrm{nb}}=\sum_{n=0}^{\infty}c_n(p,w)|n\rangle \quad (3)$$

where

$$c_n(p,w)=\left[\binom{n+w}{n}p^{w+1}(1-p)^n\right]^{1/2} \quad (4)$$

with $w\geqslant 0$, $0<p<1$, $n=0,1,2,\ldots,\infty$. The probability of finding n photons in the state (4) is given by the negative binomial distribution

$$P_n=|c_n(p,w)|^2=\binom{n+w}{n}p^{w+1}(1-p)^n \quad (5)$$

with the mean number of photons and the variance given by

$$\bar{n}=(1+w)\frac{1-p}{p} \qquad (\Delta n)^2=(1+w)\frac{1-p}{p^2}. \quad (6)$$

The Mandel Q parameter for the negative binomial states equals

$$Q\equiv\frac{(\Delta n)^2}{\bar{n}}-1=\frac{1}{p}-1 \quad (7)$$

and is always positive since p lies between zero and one. This means that photon statistics in the negative binomial states is always super-Poissonian. For $w\rightarrow 0$ the photon number distribution (5) reduces to the Bose–Einstein distribution with the mean number of photons $\bar{n}-(1-p)/p$. In the opposite limit, for $w\rightarrow\infty$, $p\rightarrow 1$, but with $\bar{n}=(1+w)(1-p)/p=$ const it reduces to the Poissonian distribution.

3. Phase properties

To describe quantum phase properties of the binomial and negative binomial states we apply the Hermitian phase formalism introduced by Pegg and Barnett [7–9]. This formalism is based on introducing a finite $(s+1)$-dimensional space Ψ spanned by the number states $|0\rangle$, $|1\rangle,\ldots,|s\rangle$, for a given mode of the field. The Hermitian phase operator operates on this finite space and, after all necessary expectation values have been calculated in Ψ, the value of s is allowed to tend to infinity. A complete orthonormal basis of $(s+1)$ states is defined on Ψ as

$$|\theta_m\rangle \equiv \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle \tag{8}$$

where

$$\theta_n \equiv \theta_0 + \frac{2\pi m}{s+1} \qquad m=0, 1, \ldots, s. \tag{9}$$

The value of θ_0 is arbitrary and defines a particular basis set of $(s+1)$ mutually orthogonal phase states. The Hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m| \tag{10}$$

where the subscript θ indicates the dependence on the choice of θ_0. The phase states (8) are eigenstates of the phase operator (10) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and $\theta_0 + 2\pi$. The unitary phase operator $\exp(i\hat{\phi}_\theta)$ is defined as the exponential function of the Hermitian operator $\hat{\phi}_\theta$ and has the form [7–9]

$$\exp(i\hat{\phi}_\theta) \equiv \sum_{n=0}^{s-1} |n\rangle\langle n+1| + \exp[i(s+1)\theta_0]|s\rangle\langle 0|. \tag{11}$$

The last term in (11) ensures the unitarity of this operator. The first sum reproduces the Susskind–Glogower [10] phase operator in the limit $s \to \infty$.

The expectation value of the phase operator (10) in a state $|\psi\rangle$ is given by

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \sum_{m=0}^{s} \theta_m |\langle\theta_m|\psi\rangle|^2 \tag{12}$$

where $|\langle\theta_m|\psi\rangle|^2$ gives the probability of being found in the phase state $|\theta_m\rangle$. The density of phase states is $(s+1)/2\pi$, so in the continuum limit as s tends to infinity, we can write equation (12) as

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta P(\theta)\,d\theta \tag{13}$$

where the continuum phase distribution $P(\theta)$ is introduced by

$$P(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} |\langle\theta|\psi\rangle|^2 \tag{14}$$

with θ_m being replaced by the continuous phase variable θ. Once the phase distribution function $P(\theta)$ is known, all the quantum mechanical phase expectation values can be calculated with this function in a classical-like manner by integrating over θ. The choice of θ_0 defines a particular window of phase values.

Equation (14) defining the phase distribution function can be generalized for any physically realizable field state described by the density matrix ρ, and it takes the form

$$P(\theta)=\frac{1}{2\pi}\sum_{n,n'=0}^{\infty}\rho_{n,n'}\,\mathrm{e}^{-\mathrm{i}(n-n')\theta} \tag{15}$$

where $\rho_{n,n'}$ are the matrix elements of the density operator in the number state basis.

Of particular interest in description of the phase properties of the field is the phase variance that can be calculated according to the formula

$$\langle(\Delta\hat{\phi})^2\rangle=\int\theta^2P(\theta)\,\mathrm{d}\theta-\left(\int\theta P(\theta)\,\mathrm{d}\theta\right)^2. \tag{16}$$

Since the number and phase are conjugate quantities, they obey the uncertainty relation [9]

$$\Delta n\Delta\phi\geqslant\tfrac{1}{2}|\langle[\hat{n},\hat{\phi}]\rangle|. \tag{17}$$

Knowing the variances for the number of photons and phase one can calculate the uncertainty product

$$\Delta n\Delta\phi=[\langle(\Delta n)^2\rangle\langle(\Delta\hat{\phi})^2\rangle]^{1/2} \tag{18}$$

and, on the other hand, the number-phase commutator appearing on the right-hand side of equation (17) can be easily evaluated, for any physical state, from the relation [9]

$$\langle[\hat{n},\hat{\phi}]\rangle=\mathrm{i}[1-2\pi P(\theta_0)]. \tag{19}$$

So, both sides of the uncertainty relation (17) can be calculated independently for a given state of the field, and the uncertainty relation itself can be tested for finding, for example, the minimum uncertainty states.

To describe the relative quantum fluctuation, with respect to the minimum uncertainty, it is convenient to introduce the notion of the number and phase squeezing defined by [11, 12]

$$S_n=\frac{(\Delta n)^2}{\frac{1}{2}|\langle[\hat{n},\hat{\phi}]\rangle|}-1 \tag{20}$$

$$S_\phi=\frac{(\Delta\phi)^2}{\frac{1}{2}|\langle[\hat{n},\hat{\phi}]\rangle|}-1. \tag{21}$$

The value of -1 of these equations means perfect squeezing of the photon number or the phase.

We use the field characteristics defined above to study quantum properties of the binomial and negative binomial states of the field mode.

3.1. Binomial states

The binomial states given by (1) and (2) are finite superpositions of the number states. In the two limits they become either the number state $|N\rangle$ or a coherent state. One can thus expect that their phase properties will reflect this feature and will interpolate between the completely random phase of the number state and the phase properties of the coherent state. The build-up of the phase peak in the phase distribution $P(\theta)$ given by (15), with the density matrix elements being the products of the amplitudes (2), is

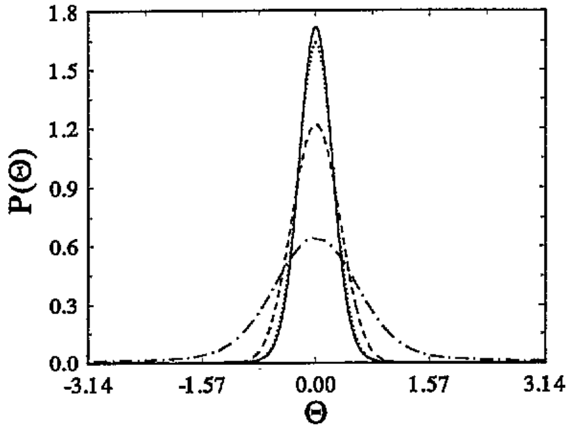


Figure 1. Phase distribution $P(\theta)$ for the binomial states with different N, for the mean number of photons $\bar{n}=5$: $N=6$ (dotted curve), $N=10$ (chain curve), and $N=50$ (broken curve). The full curve corresponds to the phase distribution of the coherent state with $\bar{n}=5$.

shown in figure 1, for the mean number of photons $\bar{n}=5$ and various N. It is seen that for $N \gg \bar{n}$ the phase distribution $P(\theta)$ for the binomial state becomes indistinguishable from the corresponding distribution for the coherent state. As one could expect the phase distribution for the binomial states with small N is much broader than the phase distribution for the coherent state with the same mean number of photons, and it becomes narrower and closer to the coherent state phase distribution as N increases. The binomial states have their phase properties that are intermediate between the number states with completely random phase and the coherent states. In figure 2 we plot the phase variances with respect to p for given N (figure 2(*a*)) and with respect to N for given p (figure 2(*b*)). It is seen that for given N the phase variance approaches a minimum for $p=0.5$. This means that the state with $p=0.5$ has the best defined phase for given N. For given p, the phase variance decreases as N increases and asymptotically tends to zero as $N \to \infty$. This could be expected, since the mean number of photons $\bar{n}=Np$ tends to infinity in this limit, and the phase of the field becomes well defined.

The other interesting quantities characterizing the states are the number and phase squeezing defined by (20) and (21). It is seen from figure 3 that these two quantities when plotted with respect to p show opposite behaviour in the sense that if one of them goes down the other one goes up. This behaviour confirms the fact that the number of photons and the phase are conjugate quantities. Moreover, it is interesting to notice that for $N>1$ the number-phase uncertainty product has a maximum in the vicinity of the point where the number and phase squeezing curves cross. At the maximum the number–phase uncertainty product differs significantly from the half of the modulus of the number–phase commutator (RHS of equation (17)), which sets the level of quantum noise. For $\bar{n}=Np \gg 1$, the binomial states become intelligent states [11], that is the states for which inequality (17) becomes equality (see figure 3(*c*)). The maximum of the uncertainty product appears for the mean photon numbers $\bar{n} \sim 1$, which means $p \sim 1/N$. This is also the region where the curves for the phase and number squeezing cross. However, the crossing point does not correspond exactly to the maximum uncertainty product, but they become closer as N increases. The

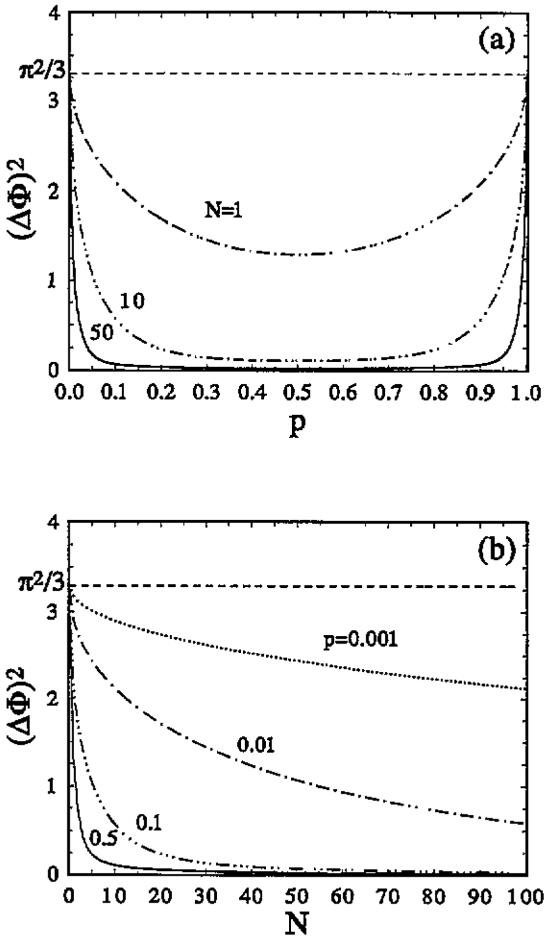
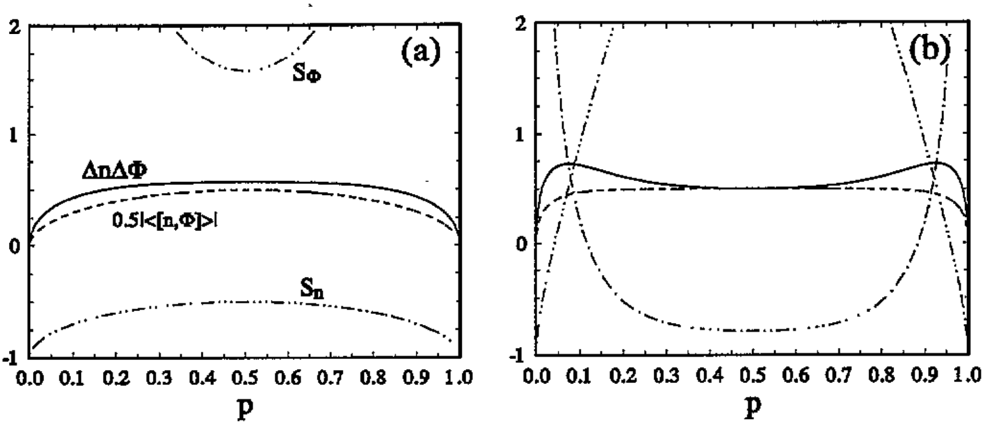


Figure 2. Phase variances for the binomial states.

crossing point means the state with the quantum noise for the phase variable being equal to the noise for the photon number, $\langle(\Delta n)^2\rangle = \langle(\Delta\phi)^2\rangle$. Our considerations show that the binomial states have really, as expected, phase properties that interpolate between the number and coherent states and can be used to model fields with such phase properties.

3.2. Negative binomial states

The negative binomial states defined by equations (3) and (4) are, in contrast to the binomial states, infinite superpositions of the number states with the extra index $w \geqslant 0$ defining particular states. Their phase properties can be studied in the same way as has been done for the binomial states. The results are presented in figures 4–6. In figure 4 the phase distribution calculated according to the formula (15) is shown for several negative binomial states ($w = 0, 1, 3$) for the mean number of photons $\bar{n} = 5$. For comparison, the phase distribution for the coherent state with the same mean number of photons is also plotted. It is interesting to see that the negative binomial states have a phase peak narrower than the coherent state, so the phase peak for such states



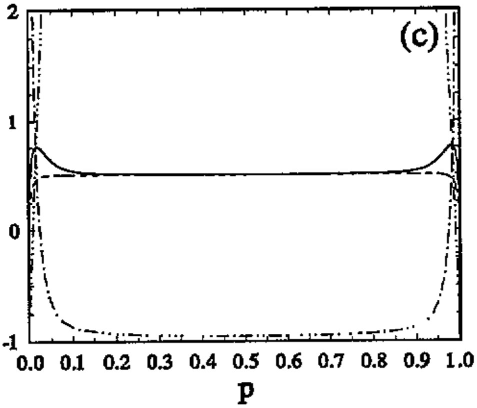


Figure 3. Plots of the number and phase squeezings and the uncertainty product for the binomial states: (*a*) $N=1$, (*b*) $N=10$, (*c*) $N=50$.

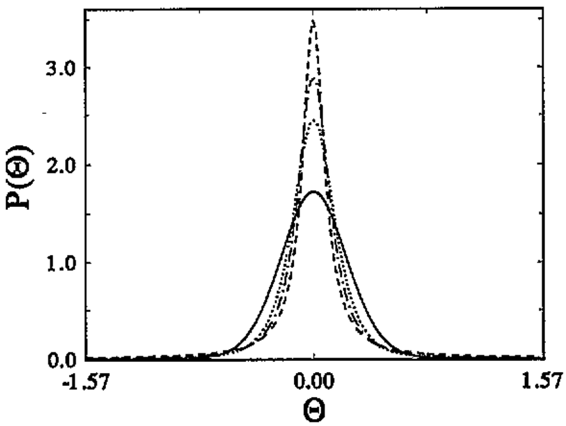


Figure 4. Phase distribution $P(\theta)$ for the negative binomial states with different w, for the mean number of photons $\bar{n}=5$: $w=0$ (dotted curve), $w=1$ (chain curve), and $w=3$ (broken curve). The full curve corresponds to the phase distribution of the coherent state with $\bar{n}=5$.

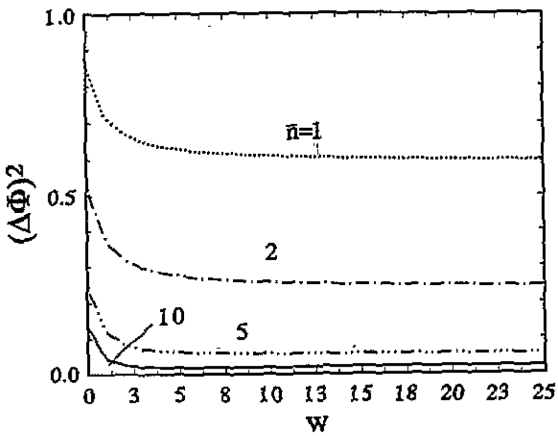


Figure 5. Phase variances for the negative binomial states plotted against w for several values of the mean photon number.

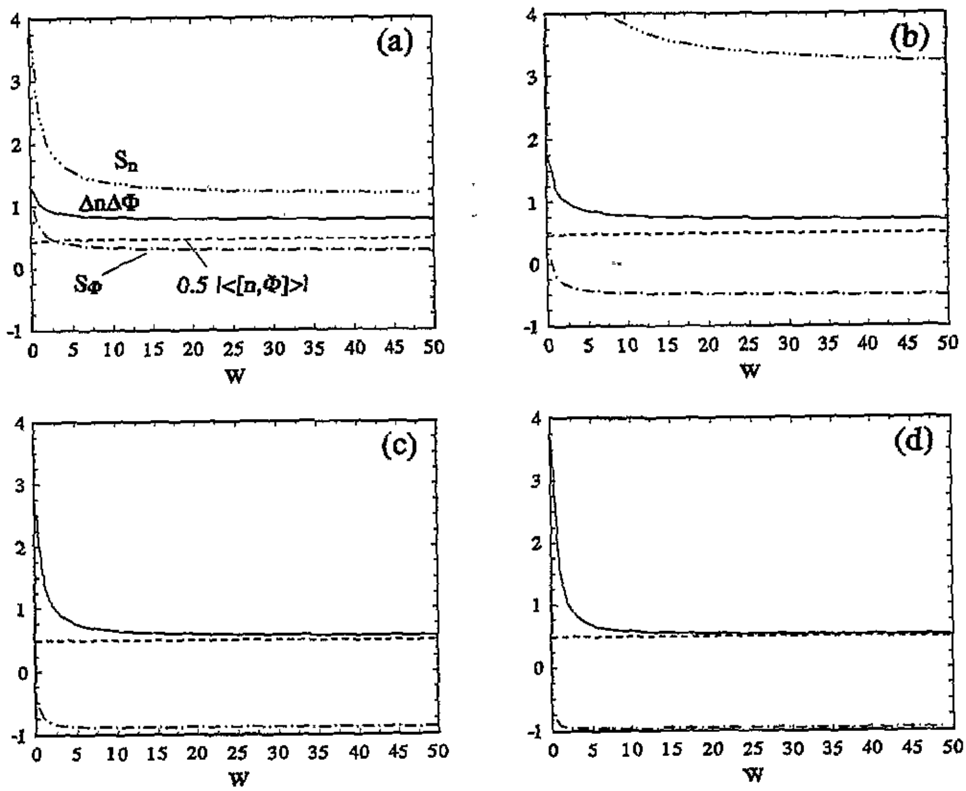


Figure 6. Plots of the number and phase squeezings and the uncertainty product for the negative binomial states: (a) $\bar{n}=1$, (b) $\bar{n}=2$, (c) $\bar{n}=5$ and (d) $\bar{n}=10$.

should be easier to detect. However, they have broader wings as compared to the coherent state. The state with sharpest and highest peak has the value of $w=0$, but it also has the most pronounced wings, which cause the phase variance for this state to be greater than the corresponding variances for the other negative binomial states with higher w. This is clearly seen from figure 5 where the phase variance is plotted against w for several values of the mean photon numbers. Next we have calculated the phase–number uncertainty product as well as the phase and number squeezings for the negative binomial states which are shown in figure 6. Here we see that for $\bar{n}\gg 1$, the quantum noise level (RHS in equation (17)) is practically equal to 0.5. The number–phase uncertainty product has its maximum for the states with $w=0$, but as w increases its value monotonically decreases to the value represented by a coherent state with the same mean number of photons. Unlike the binomial states, the negative binomial states always have their photon statistics being super-Poissonian (the Q parameter is always positive), so the number squeezing monotonically decreases as w increases. Since the phase squeezing also decreases, the two squeezings never cross. The negative binomial state with $w=0$ has the Bose–Einstein photon number distribution with the mean number of photons $\bar{n}=(1-p)/p$, which is characteristic of thermal states. However, the states (3) cannot be identified with the pure thermal states because the off-diagonal elements of the density operator $\rho=|\psi\rangle_{\rm nb\,nb}\langle\psi|$ in the Fock state basis are not zeros. This explains the sharp phase peak for the state with $w=0$. Pure thermal states with zero non-diagonal elements have flat phase distribution. For $w\rightarrow\infty$ and $1-p\rightarrow 0$, but $\bar{n}=(1+w)(1-p)/p$ being finite, the negative binomial states reduce to a coherent state with mean number of photons $\bar{n}$. In this limit, the phase properties of the negative binomial states correspond to the properties known for coherent states.

4. Conclusions

In this paper we have studied quantum phase properties of the binomial and negative binomial states. Using the Pegg–Barnett Hermitian phase formalism we have found the phase distributions as well as phase variances for such states. We have explicitly shown that the phase properties of the binomial states interpolate, as one could expect, between the number states and the coherent states. We have also shown that the photon number fluctuations and phase fluctuations for such states exhibit opposite behaviour, and there is a possibility to find the binomial state for which the photon number variance is equal to the phase variance. This behaviour is convincingly seen when plotting the number and phase squeezing curves, which cross when the two fluctuations are equal. This happens in the vicinity of the maximum of the number–phase uncertainty product.

For the negative binomial states we have found an interesting feature of the phase distribution for the state with $w=0$, for which the phase peak is sharper than for the coherent state, but because the wings of the distribution are more pronounced, the phase variance is still greater than that for a coherent state with the same number of photons. We have also studied the number–phase uncertainty product and the number and phase squeezing for such states.

In conclusion, we believe that the work discussed above adds some new facts to the already existing knowledge on the quantum properties of the binomial and negative binomial states.

Acknowledgments

Two of us (TG and AJ) would like to thank Professor H Walther for hospitality at the Max-Planck-Institute for Quantum Optics and TsG would also like to acknowledge the Alexander von Humboldt Foundation for the award of a Humboldt Research Fellowship.

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