Physica Scripta. Vol. T48, 53–60, 1993

Phase Distributions of Real Field States

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Received September 30, 1992; accepted in revised form November 5, 1992

Abstract

The phase distribution obtained within the Pegg–Barnett Hermitian phase formalism is compared to the phase distributions obtained from the s-parametrized quasiprobability distributions integrated over the "radial" variable for some real states of the field. Exact analytical formulas for the s-parametrized phase distributions of coherent states, squeezed states, and displaced number states are obtained. A general formula relating the s-parametrized phase distributions to the Pegg–Barnett distribution is derived. Numerical examples illustrating the similarities and differences are presented in a graphical form.

1. Introduction

The Hermitian phase formalism introduced recently by Pegg and Barnett [1–3] has successfully overcome the problems that existed earlier with the proper description of the phase in quantum mechanics [4, 5]. The Pegg–Barnett formalism has been applied for studying quantum phase properties of a number of real field states [6–33] revealing new features of optical fields. Simultaneously, the appearance of the Pegg–Barnett Hermitian phase formalism provoked a renewed interest in older attempts to introduce a Hermitian phase operator [34–38] and triggered a discussion resulting in some alternative ways that can be applied to describe phase properties of optical fields [39–46].

For real field states, or "physical states" according to the terminology used by Pegg and Barnett [3], the continuous phase distribution characterizing such states can be introduced. Such a distribution should be 2π-periodic and normalized to describe properly phase properties of the field. However, phase distributions that are 2π-periodic and normalized can also be obtained by integrating the Wigner function or the Q function over the "radial" variable. It is, thus, interesting to know to what extent such phase distributions reproduce the Pegg–Barnett phase distribution.

In this paper we give a general relation between the s-parametrized phase distribution (the phase distribution obtained by integrating the s-parametrized quasidistribution over the "radial" variable) and the Pegg–Barnett phase distribution. We have also obtained exact analytical formulas for the s-parametrized phase distributions for coherent states, squeezed states, displaced number states. The results obtained according to these formulas are compared to the corresponding results obtained from the Pegg–Barnett formalism.

We should mention here that the phase distribution associated with the Q function is significant from the experimental point of view, because it can be measured according to some realistic experimental schemes [46–49].

The earlier attempts to construct the Hermitian phase operator mentioned above [34–38] are rather unsatisfactory because they lead to the phase distributions that exhibit an asymmetry which is incompatible with the symmetry of the other phase distributions. The asymmetry appearing in the Garrison and Wong [34] and all equivalent to its phase distributions has been discussed by us elsewhere [33], and the details can be found there. The Garrison and Wong phase distribution, moreover, does not satisfy the 2π-periodicity demanded from the phase distribution. Thus, there are physical reasons for not using such phase distributions, and we refrain from discussing them here.

2. The Pegg–Barnett phase distribution

Pegg and Barnett [1–3] introduced the Hermitian phase formalism, which is based on the observation that in a finite-dimensional state space the states with the well-defined phase exist [50]. Thus they restrict the state space to a finite $(s+1)$-dimensional space Ψ spanned by the number states $|0\rangle, |1\rangle, \ldots, |s\rangle$. In this space they define a complete orthonormal set of phase states by

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle, \quad m = 0, 1, \ldots, s, \tag{1}$$

where the values of θ_m are given by

$$\theta_m = \theta_0 + \frac{2\pi m}{s+1}. \tag{2}$$

The value of θ_0 is arbitrary and defines a particular basis set of $(s+1)$ mutually orthogonal phase states. The Pegg–Barnett (PB) Hermitian phase operator is defined as

$$\hat{\Phi}_{\mathrm{PB}} = \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m|. \tag{3}$$

Of course, the phase states (1) are eigenstates of the phase operator (3) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and $\theta_0 + 2\pi$. The Pegg–Barnett prescription is to evaluate any observable of interest in the finite basis (1) and only after that take the limit $s \to \infty$.

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Since the phase states (1) are orthonormal, $\langle\theta_m|\theta_{m'}\rangle = \delta_{mm'}$, the kth power of the Pegg–Barnett phase operator (3) can be written as

$$\hat{\Phi}^k_{\mathrm{PB}} = \sum_{m=0}^{s} \theta_m^k |\theta_m\rangle\langle\theta_m|, \tag{4}$$

and the expectation value of the kth power of the phase operator can be calculated as

$$\langle f|\hat{\Phi}^k_{\mathrm{PB}}|f\rangle = \sum_{m=0}^{s} \theta_m^k |\langle\theta_m|f\rangle|^2, \tag{5}$$

where the quantity $|\langle\theta_m|f\rangle|^2$ gives a probability of being found in the phase state $|\theta_m\rangle$.

When "physical states", according to their definition by Pegg and Barnett [3], are considered, we can simplify the calculation of the sum in eq. (5) by replacing it by the integral in the limit as s tends to infinity. Since the density of states is $(s+1)/2\pi$, we can write eq. (5) as

$$\langle f|\hat{\Phi}^k_{\mathrm{PB}}|f\rangle = \int_{\theta_0}^{\theta_0+2\pi} \mathrm{d}\theta\theta^k P_{\mathrm{PB}}(\theta), \tag{6}$$

where the continuous-phase distribution $P_{\mathrm{PB}}(\theta)$ is introduced by

$$P_{\mathrm{PB}}(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} |\langle\theta_m|f\rangle|^2, \tag{7}$$

and θ_m has been replaced by the continuous-phase variable θ. If the state $|f\rangle$ has the number-state decomposition

$$|f\rangle = \sum_n b_n |n\rangle, \tag{8}$$

The Pegg–Barnett phase distribution is given by [3]

$$P_{\mathrm{PB}}(\theta) = \frac{1}{2\pi}\left\{1 + 2\,\mathrm{Re} \sum_{\substack{n,m\\ m>n}} b_m b_n^* \exp\left[-\mathrm{i}(m-n)\theta\right]\right\}. \tag{9}$$

In the case of fields being in mixed states described by the density matrix $\hat{\rho}$, formula (9) generalizes to

$$P_{\mathrm{PB}}(\theta) = \frac{1}{2\pi}\left\{1 + 2\,\mathrm{Re} \sum_{m>n} \rho_{mn} \exp\left[-\mathrm{i}(m-n)\theta\right]\right\}, \tag{10}$$

where, $\rho_{mn} = \langle m|\hat{\rho}|n\rangle$, are the density matrix elements in the number state basis. Formulas (9) and (10) can be used for calculations of the Pegg–Barnett phase distribution for any state with known amplitudes b_n or matrix elements ρ_{mn}. Formulas (9) and (10), although exact, can rarely be summed up into a closed form, and usually numerical summation must be performed to find the phase distribution. Such numerical summations have been widely applied in studying phase properties of optical fields [10–32] (see also [51]). The Pegg–Barnett phase distributions, eqs (9) or (10), is obviously 2π-periodic, and for all states with the density matrix diagonal in the number states the phase distribution is uniform over the 2π-wide phase window. These are non-diagonal elements of the density matrix that lead to the structure of the phase distribution. The Pegg–Barnett distribution is positive definite and normalized.

3. Quasiprobability distributions and phase distributions associated with them

According to Cahill and Glauber [52, 53] the s-parametrized quasidistribution function describing a field state can be derived from the formula

$$W(\alpha, s) = \frac{1}{\pi}\,\mathrm{Tr}\,\{\rho T(\alpha, s)\}, \tag{11}$$

where the operator $T(\alpha, s)$ is given by

$$T(\alpha, s) = \frac{1}{\pi}\int \exp\left(\alpha\xi^* - \alpha^*\xi\right) D(\xi, s)\,\mathrm{d}^2\xi, \tag{12}$$

and

$$D(\xi, s) = \mathrm{e}^{s|\xi|^2/2} D(\xi) \tag{13}$$

with $D(\xi)$ being the displacement operator; ρ is the density matrix of the field, and we have introduced the extra $1/\pi$ factor with respect to the original definition [53]. The operator $T(\alpha, s)$ can be rewritten in the form [52]

$$T(\alpha, s) = \frac{2}{1-s} \sum_{n=0}^{\infty} D(\alpha)|n\rangle \left(\frac{s+1}{s-1}\right)^n \langle n|D^+(\alpha), \tag{14}$$

which explicitly gives its s-dependence. So, in the number-state basis, we have

$$\begin{aligned} W(\alpha, s) &= \frac{1}{\pi}\sum_{m,n} \rho_{mn}\langle n|T(\alpha, s)|m\rangle \\ &= \frac{1}{\pi}\sum_{m,n} \rho_{mn}\left(\frac{n!}{m!}\right)^{1/2}\left(\frac{2}{1-s}\right)^{m-n+1}\left(\frac{s+1}{s-1}\right)^n \\ &\quad\times \mathrm{e}^{-\mathrm{i}(m-n)\theta}|\alpha|^{m-n}\exp\left(-\frac{2|\alpha|^2}{1-s}\right) \\ &\quad\times L_n^{m-n}\left(\frac{4|\alpha|^2}{1-s^2}\right), \end{aligned} \tag{15}$$

where we have used (14), and the fact that the matrix elements of the displacement operator are given by [52]

$$\langle m|D(\alpha)|n\rangle = \left(\frac{n!}{m!}\right)^{1/2} \alpha^{m-n}\mathrm{e}^{-|\alpha|^2/2} L_n^{m-n}(|\alpha|^2), \tag{16}$$

where $L_n^{m-n}(|\alpha|^2)$ is the associate Laguerre polynomial. In (15) we have also explicitly separated the phase of the complex number α by writing

$$\alpha = |\alpha|\mathrm{e}^{\mathrm{i}\theta} \tag{17}$$

The phase θ is later on treated as the quantity representing the field phase.

With the quasiprobability distributions $W(\alpha, s)$ the expectation values of the s-ordered products of the creation and annihilation operators can be obtained by proper integrations in the complex α plane. In particular, for $s = 1, 0, -1$, the s-ordered products are normal, symmetric, and anti-normal ordered products of the creation and annihilation operators, and the corresponding quasiprobability distribution are the P, Wigner, and Q functions.

When we integrate the quasiprobability distribution $W(\alpha, s)$ over the "radial" variable $|\alpha|$, we get the "phase distribution" associated with this quasiprobability distribution. The s-parametrized phase distribution is thus given by

$$P(\theta, s) = \int_0^{\infty} W(\alpha, s)|\alpha|\,\mathrm{d}|\alpha|, \tag{18}$$

which after inserting (15) gives

$$P(\theta, s) = \frac{1}{\pi} \sum_{m,n} \rho_{mn} \left(\frac{n!}{m!}\right)^{1/2} \left(\frac{2}{1-s}\right)^{m-n+1} \left(\frac{s+1}{s-1}\right)^{n} \times e^{-i(m-n)\theta} \int_0^\infty |\alpha|^{m-n} \exp\left(-\frac{2|\alpha|^2}{1-s}\right) \times L_n^{m-n}\left(\frac{4|\alpha|^2}{1-s^2}\right) |\alpha| \, d|\alpha|. \tag{19}$$

If the definition of the Laguerre polynomial is invoked, the integrations in (19) can be performed explicitly, and we get for the s-parametrized phase distribution the formula which is similar to the Pegg–Barnett phase distribution (10), and it reads

$$P(\theta, s) = \frac{1}{2\pi} \left\{1 + 2 \operatorname{Re} \sum_{m>n} \rho_{mn} e^{-i(m-n)\theta} G^{(s)}(m, n)\right\}. \tag{20}$$

The difference is in the coefficients $G^{(s)}(m, n)$ that appeared in (20), and which are given by

$$G^{(s)}(m, n) = \left(\frac{2}{1-s}\right)^{(m+n)/2} \sum_{l=0}^{\min(m,n)} (-1)^l \left(\frac{1+s}{2}\right)^l \times \sqrt{\binom{n}{l}\binom{m}{l}} \frac{\Gamma\left(\frac{m+n}{2} - l + 1\right)}{\sqrt{(m-l)!(n-1)!}}. \tag{21}$$

For $s = -1$, only the term with $l = 0$ survives in (21), and we get the coefficients obtained by us earlier [25] for the phase distribution associated with the Q function. For $s = 0$, we have the coefficients for the phase distribution associated with the Wigner function, which have been used in our studies of phase properties of the displaced number states [31]. It is seen from (21) that for $s = 1$ (the P function) the coefficients $G^{(s)}(m, n)$ become infinity, and the phase distribution is indeterminate. Our formula (20) allows for calculations of the s-parametrized phase distributions for any state with known ρ_{mn} and compare them to the Pegg–Barnett phase distribution, for which $G^{(s)}(m, n) = 1$.

The phase distributions associated with particular quasiprobability distributions have been used in literature to describe phase properties of field states. For example, the integrated Wigner function ($s = 0$) has been applied by Schleich, Horowicz and Varro [39, 40] in their description of the phase probability distribution for a highly squeezed states. The integrated Q function ($s = -1$) has been used by Braunstein and Caves [49] to describe phase properties of the generalized squeezed states. Eiselt and Risken [54] have used the s-parametrized quasiprobability distributions to study properties of the Jaynes–Cummings model with cavity damping. In their approach, Eiselt and Risken have used the series expansions of the quasiprobability distribution functions, and they have found an expression relating the Pegg–Barnett phase distribution to the quasiprobability distributions in a form of the integral relation and applied it to the Jaynes–Cummings model. Their formulas, however, do not work for $s = -1$, i.e. for the Q function.

For some field states the quasiprobability distribution functions $W(\alpha, s)$ can be found in a closed form via direct integrations according to the definitions (11)–(13), and sometimes the next integration leading to the s-parametrized phase distributions can also be performed according to the definition (18). We have found the exact analytical formulas for the s-parametrized phase distributions for coherent states, squeezed states, and displaced number states.

3.1. *Coherent states*

For a coherent state $|\alpha_0\rangle$ we have

$$|\alpha_0\rangle = D(\alpha_0)|0\rangle, \tag{22}$$

and the s-parametrized quasiprobability distribution function can be calculated from (11)–(13) as

$$\begin{aligned} W_{\text{coh}}(\alpha, s) &= \frac{1}{\pi^2} \int \exp(\alpha\xi^* - \alpha^*\xi + s|\xi|^2/2) \times \langle 0|D^+(\alpha_0)D(\xi)D(\alpha_0)|0\rangle \, d^2\xi \\ &= \frac{1}{\pi^2} \int \exp[(\alpha - \alpha_0)\xi^* - (\alpha^* - \alpha_0^*)\xi + s|\xi|^2/2)] \times \langle 0|D(\xi)|0\rangle \, d^2\xi \\ &= \frac{1}{\pi^2} \int \exp[(\alpha - \alpha_0)\xi^* - (\alpha^* - \alpha_0^*)\xi + s|\xi|^2/2) - |\alpha_0|^2/2] \, d^2\xi \\ &= \frac{1}{\pi} \frac{2}{1-s} \exp\left\{-\frac{2}{1-s}|\alpha - \alpha_0|^2\right\}. \end{aligned} \tag{23}$$

The corresponding phase distribution is

$$\begin{aligned} P_{\text{coh}}(\theta, s) &= \int_0^\infty W_{\text{coh}}(\alpha, s)|\alpha| \, d|\alpha| \\ &= \frac{1}{2\pi} \exp[-(X_0^2 - X^2)]\{\exp(-X^2) + \sqrt{\pi} X[1 + \operatorname{erf}(X)]\}, \end{aligned} \tag{24}$$

where

$$X = X(\theta, s) = \sqrt{\frac{2}{1-s}} |\alpha_0| \cos(\theta - \theta_0'), \tag{25}$$

and $X_0 = X(\theta_0', s)$, θ_0' is the phase of α_0.

Our formula (24) is exact, it is 2π-periodic, positive definite and normalized, so it satisfies all requirements for the phase distribution. Moreover, formula (24) has quite simple and transparent structure. For small $|\alpha_0|$, the first term in braces plays an essential role, and for $|\alpha_0| \to 0$ we get uniform phase distribution. For large $|\alpha_0|$, the second term in the braces predominates, and if we replace $\operatorname{erf}(X)$ by the unity, we obtain the approximate asymptotic formula given by Schleich *et al.* [55] (for $s = 0$)

$$P_{\text{coh}}(\theta, s) \approx \sqrt{\frac{2}{\pi}} |\alpha_0| \cos(\theta - \theta_0') \times \exp[-2|\alpha_0|^2 \sin^2(\theta - \theta_0')], \tag{26}$$

which however, can be applied only for $-\pi/2 \leqslant \overset{-\theta_0'}{\theta} \leqslant \pi/2$. After linearization of (26) with respect to θ, the approximate formula for coherent states with large mean number of photons obtained by Barnett and Pegg [2] is recovered. The presence of the error function in (24) handles properly the phase behaviour in the total range of phase values $-\pi \leqslant \overset{-\theta_0'}{\theta} \leqslant \pi$.

3.2. *Squeezed states*

Similar calculations can be performed for squeezed states defined by [56]

$$|\alpha_0, \zeta\rangle = D(\alpha_0)S(\zeta)|0\rangle, \tag{27}$$

where $S(\zeta)$ is the squeezing operator [56]

$$S(\zeta) = \exp\left(\tfrac{1}{2}\zeta^* a^2 - \tfrac{1}{2}\zeta a^{+2}\right), \tag{28}$$

and ζ is the complex squeeze parameter

$$\zeta = |\zeta| e^{2i\eta}, \quad |\zeta| = r. \tag{29}$$

The direct integrations lead to the s-parametrized quasiprobability distribution (for $\eta = 0$)

$$W_{sq}(\alpha, s) = \frac{2}{\sqrt{(\mu - s)(\mu^{-1} - s)}} \exp\left\{-\frac{2}{\mu - s}[\mathrm{Im}\,(\alpha - \alpha_0)]^2 - \frac{2}{\mu^{-1} - s}[\mathrm{Re}\,(\alpha - \alpha_0)]^2\right\}, \tag{30}$$

where we have used the notation

$$\mu = e^{2r}. \tag{31}$$

After the integration over $|\alpha|$, assuming that α_0 is real, we arrive at the formula

$$P_{sq}(\theta, s) = \frac{1}{2\pi}\frac{\sqrt{(\mu - s)(\mu^{-1} - s)}}{(\mu - s)\cos^2\theta + (\mu^{-1} - s)\sin^2\theta} \times \exp[-(X_0^2 - X^2)]\{\exp(-X^2) + \sqrt{\pi}X[1 + \mathrm{erf}\,(X)]\}, \tag{32}$$

where

$$X = X(\theta, s) = \sqrt{\frac{2}{\mu^{-1} - s}} \times \frac{\alpha_0\sqrt{\mu - s}\cos\theta}{\sqrt{(\mu - s)\cos^2\theta + (\mu^{-1} - s)\sin^2\theta}}. \tag{33}$$

Although the variable X is slightly different, the main structure of the phase distribution is preserved. Formula (32) is valid for both small and large α_0. For $\alpha_0 = 0$ we have the result for squeezed vacuum. After appropriate approximations one can easily obtain the formula obtained by Schleich *et al.* [39] for a highly squeezed state. Again, our formula is exact and works for all phase values.

3.3. *Displaced number states*

Other states that are interesting from the point of view of their phase properties are the displaced number states [57], for which corresponding formulas are given by

$$W_{dn}(\alpha, s) = \frac{1}{\pi}\frac{2}{1 - s}(-1)^n\left(\frac{1 + s}{1 - s}\right)^n \times \exp\left\{-\frac{2}{1 - s}|\alpha - \alpha_0|^2\right\}L_n\left(\frac{4|\alpha - \alpha_0|^2}{1 - s^2}\right), \tag{34}$$

and

$$P_{dn}(\theta, s) = \left(\frac{2}{1 - s}\right)^n \sum_{k=0}^{n}\frac{(-1)^{n-k}}{k!}\left(\frac{1 + s}{2}\right)^{n-k}\binom{n}{k} \times \sum_{l=0}^{k}\binom{k}{l}\frac{N_{k-l}(2k - 2l)!}{2^{2k-2l}(k - l)!}(X_0^2 - X^2)^l P_{k-l}(X), \tag{35}$$

here

$$P_n(X) = \frac{N_n^{-1}}{2\pi}\exp[-(X_0^2 - X^2)] \times \exp(-X^2)Q_n(X) + \sqrt{\pi}X[1 + \mathrm{erf}\,(X)]\}, \tag{36}$$

$$Q_n(X) = \frac{2^{2n}(n!)^2}{(2n)!}\sum_{k=0}^{n}\frac{1}{k!}X^{2k} - \sum_{k=1}^{n}\frac{2^{2k}k!}{(2k)!}X^{2k}, \tag{37}$$

and the normalization constant is equal to

$$N_n = 1 + \exp(-X_0^2)\frac{1}{2\pi}\int_{-\pi}^{\pi}\{Q_n[X(\theta)] - 1\}\,d\theta = 1 + \exp(-X_0^2)\left\{-1 + \frac{2^{2n}(n!)^2}{(2n)!} \times \sum_{k=0}^{n}\frac{(2k)!}{2^{2k}(k!)^3}X_0^{2k} - \sum_{k=1}^{n}\frac{1}{k!}X_0^{2k}\right\}. \tag{38}$$

The X variable in this case is

$$X = X(\theta, s) = \sqrt{\frac{2}{1 - s}}\,\alpha_0\cos\theta, \tag{39}$$

and we have assumed α_0 being real. Despite its more complex structure, formula (35) contains phase distributions $P(X)$ that exhibit main features of the previous phase distributions.

4. Quasiprobability versus Pegg–Barnett distributions

4.1. *General relation*

Now, we are going to illustrate the differences between the Pegg–Barnett phase distribution and s-parametrized phase distributions obtained by integrating the s-parametrized quasiprobability distribution functions. For any field with known number state matrix elements ρ_{mn} of the density matrix the s-parametrized phase distribution can be calculated according to formula (20) with the coefficients $G^{(s)}(m, n)$ given by (21). The distribution of the coefficients $G^{(s)}(m, n)$, for $s = 0, -1$, is illustrated in Fig. 1. It is seen that for $s = -1$ (the Q function) the coefficients monotonically decrease as we go far away from the diagonal. This means that all nondiagonal elements ρ_{mn} are weighted with numbers that are less than unity, and the phase distribution for $s = -1$ is always broader than the Pegg–Barnett phase distribution [for which $G^{(s)}(m, n) = 1$]. For $s = 0$ the situation is not that simple, because the coefficients $G^{(0)}(m, n)$ show oscillations with values that are both smaller and larger than unity. This leads to the phase structure that is sharper than the Pegg–Barnett distribution. Moreover, since the Wigner function ($s = 0$) can take on negative values, the positive definiteness of the $P(\theta, s)$ is not guaranteed, although there is no problem even for displaced number states, where the Wigner function oscillates between positive and negative values. From the form of the coefficients $G^{(s)}(m, n)$ it is evident that there is no s such that $G^{(s)}(m, n) = 1$ for all m, n. This means that there is no "phase ordering" of the field operators, that is, the ordering, for which $P(\theta, s)$ would be equal to $P_{PB}(\theta)$. However, for a given state of the field one can find s such that the two distributions are "almost identical". Formula (20) is quite general, and it has been used in

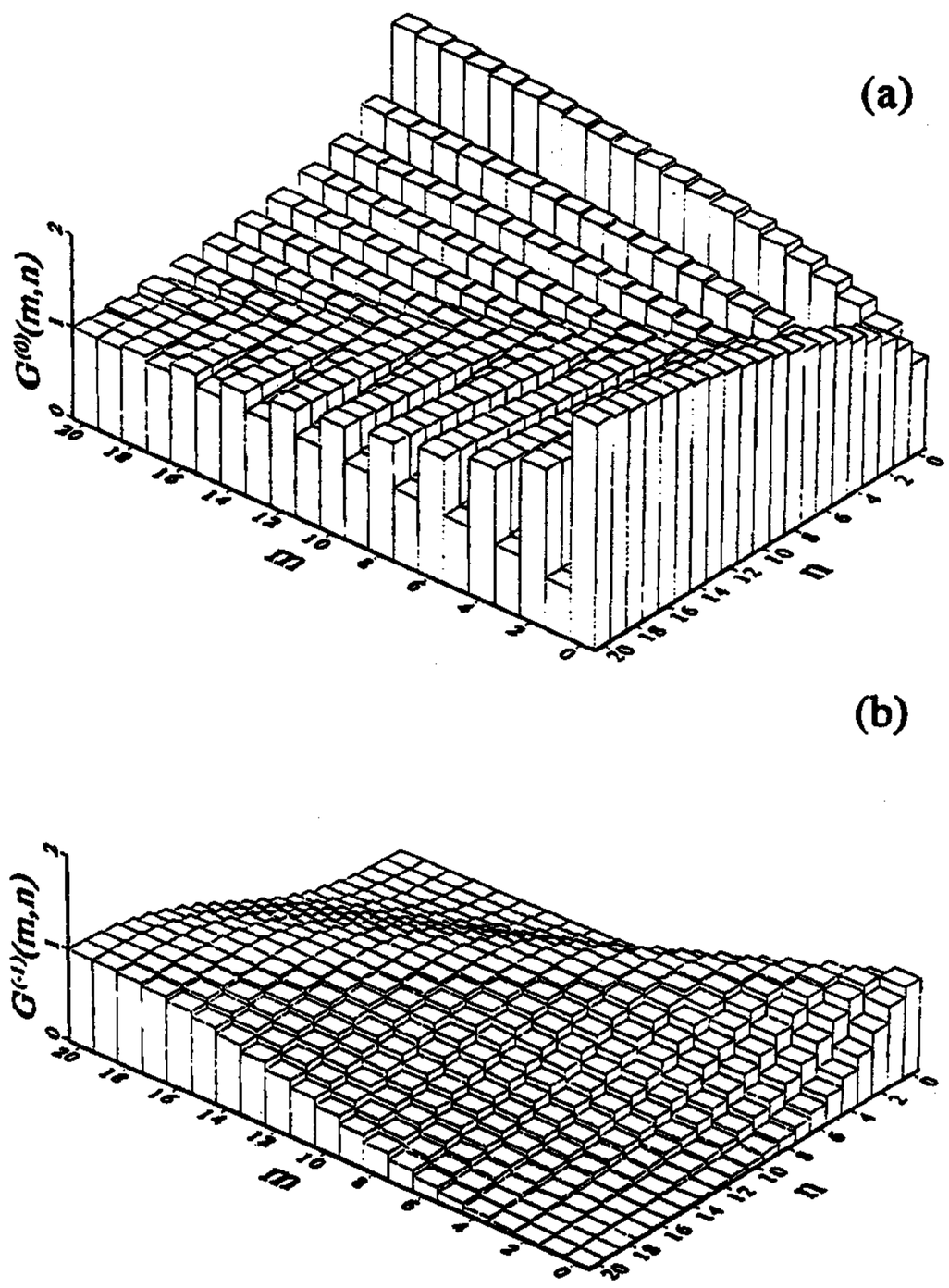

Fig. 1. Distributions of the coefficients $G^{(s)}(m, n)$ for (a) $s = 0$, and (b) $s = -1$.

our earlier studies of phase properties of the anharmonic oscillator [25], parametric down-conversion [58] and displaced number states [31]. A disadvantage of the formula (20) is the fact that the numerical summations can be time consuming and even difficult to perform for the field states with slowly converging number state expansions. This, for example, is the case for highly squeezed states. In some cases, instead of using the number state expansions we can find analytical formulas for $P(\theta, s)$ via direct integrations, as shown in Section 3. In many cases such formulas can be treated as good approximations to the Pegg–Barnett phase distribution.

4.2. *Coherent states*

The exact formula for the s-parametrized phase distributions for coherent states is given by (24) and (25). In Fig. 2 we show the phase distributions $P_{PB}(\theta)$, $P(\theta, 0)$, and $P(\theta, -1)$ for coherent states with the mean number of photons $|\alpha_0|^2 = 2$ (a), and $|\alpha_0|^2 = 0.01$ (b). It is seen that the Pegg–Barnett phase distribution is located somewhere between the phase distribution associated with the Wigner function and that associated with the Q function. It becomes closer to $P(\theta, 0)$ for $|\alpha_0|^2 \gg 1$, and closer to $P(\theta, -1)$ for $|\alpha_0|^2 \ll 1$. For $|\alpha_0|^2 \to \infty$, the Pegg–Barnett distribution tends to the distribution associated with the Wigner function [2, 39], and for $|\alpha_0|^2 \to 0$ all the distributions tend to the uniform distribution, but the Pegg–Barnett distribution in this region tends to the distribution associated with the Q function. This means that for coherent states with large mean numbers of photons $P(\theta, 0)$ is a good approximation to the Pegg–Barnett phase distribution, while for small numbers of photons $P(\theta, -1)$ becomes a good approximation to the PB distribution.

4.3. *Squeezed states*

The exact analytical formula for the s-parametrized phase distribution for squeezed states is given by (32) and (33). For the squeezed vacuum, we have

$$P_{sq}(\theta, s) = \frac{1}{2\pi} \frac{\sqrt{(\mu - s)(\mu^{-1} - s)}}{(\mu - s)\cos^2\theta + (\mu^{-1} - s)\sin^2\theta}, \tag{40}$$

where $\mu = \exp(2r)$. This formula exhibits a two-peak structure with the peaks for $\theta = \pm\pi/2$ for $r > 0$). It is easy to find that the peaks heights are

$$P_{sq}(\pi/2, s) = \frac{1}{2\pi}\sqrt{\frac{\mu - s}{\mu^{-1} - s}}, \tag{41}$$

which means that for $s = 0$ the peak height goes as μ. In Fig. 3 we have illustrated the dependence of the peak heights on the squeeze parameter r. It is seen that the Pegg–Barnett result lies between the $s = 0$ and $s = -1$ curves, but the three curves are divergent for large r. Qualitatively all three distributions give the same two-peak phase distributions, but quantitatively they differ: the sharpest peaks are those of $P(\theta, 0)$, and the broadest those of $P(\theta, -1)$.

For squeezed states with different from zero displacement α_0, an additional factor of the form identical to that for coherent states, except for the different meaning of $X(\theta)$, appears in the phase distribution $P_{sq}(\theta, s)$. Since this extra factor shows a peak at $\theta = 0$, a competition arises between the two-peak structure of the squeezed vacuum and the one peak structure of the coherent component. This competition leads to the bifurcation in the phase distribution discussed by Schleich, Horowicz and Varro [39, 40]. In Fig. 4 we show the pictures of such a bifurcation for $\alpha_0 = 1$, exhibited by all three distributions plotted in the same scale to visualize the differences. Qualitatively the pictures are quite similar, and the differences are only in the widths of the peaks. To calculate the Pegg–Barnett phase distribution we have applied formula (9) with b_n given by [56]

$$b_n = \langle n | \alpha_0, \zeta \rangle = \frac{1}{\sqrt{n!\cosh r}} [\tfrac{1}{2} e^{2i\eta} \tanh r]^{n/2} \times H_n\left[\frac{\alpha_0 + \alpha_0^* e^{2i\eta}\tanh r}{\sqrt{2 e^{2i\eta}\tanh r}}\right] \times \exp\{-\tfrac{1}{2}[|\alpha_0|^2 + \alpha_0^{*2} e^{2i\eta}\tanh r]\}, \tag{42}$$

assuming $\eta = 0$ (results for $\eta = \pi/2$ can be obtained from our formulas replacing r by $-r$).

4.4. *Displaced number states*

Both for coherent states and squeezed states there was no qualitative difference between various phase distributions. So, one could say that, at least qualitatively, all the phase distributions carried the same phase information. Here, we give an example of states for which such statement is no longer true. These are displaced number states. Phase properties of such states have been discussed earlier [31] with the use of our general formula (20). It has been shown that there is qualitative difference between the phase distribution associated with the Q function on the one side,

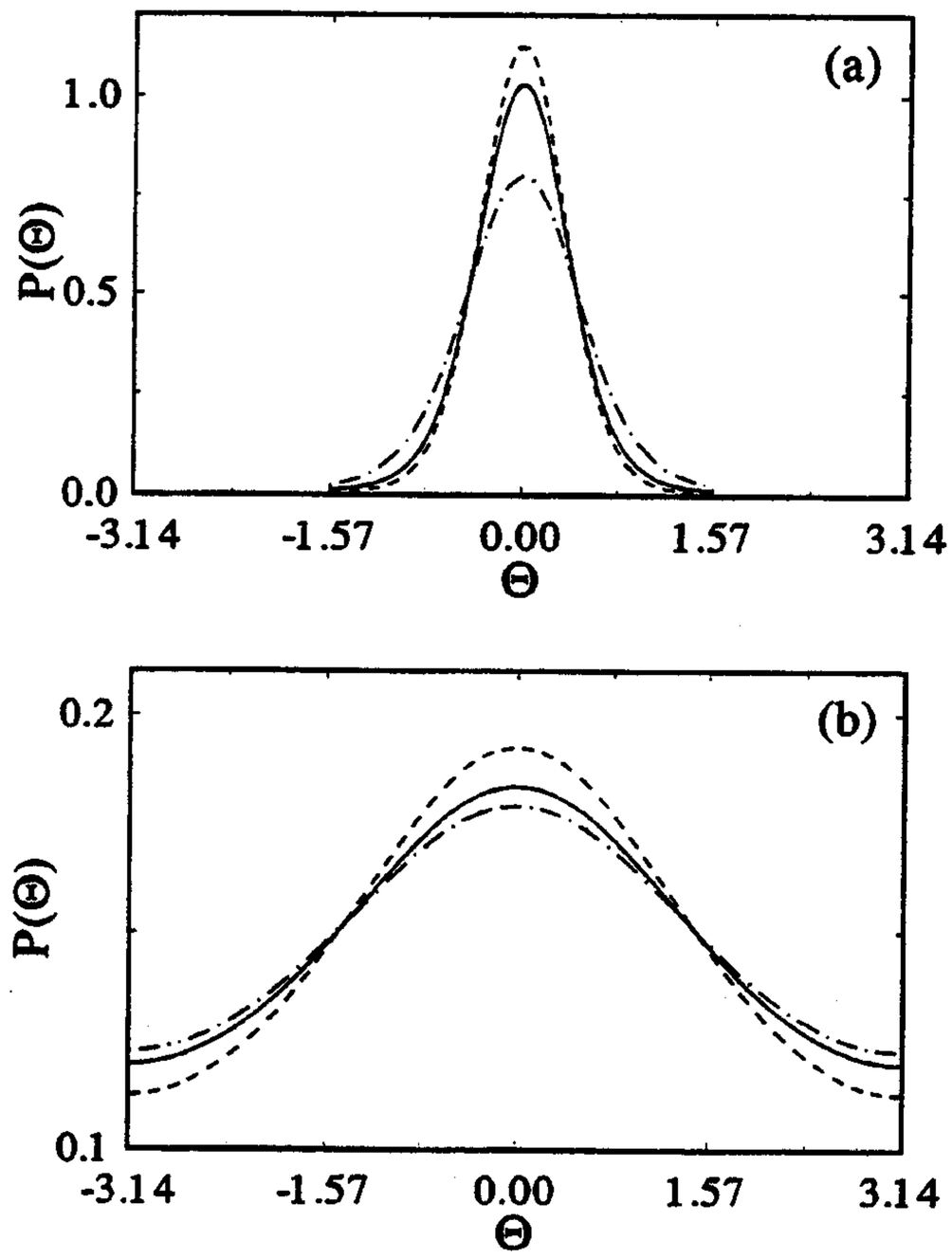

Fig. 2. Phase distributions for the coherent states with the mean number of photons: (a) $|\alpha_0|^2 = 2$, and $|\alpha_0|^2 = 0.01$; the Pegg–Barnett distribution – solid line, $P(\theta, 0)$ – dashed line, and $P(\theta, -1)$ – dotted-dashed line.

and the PB phase distribution and the phase distribution associated with the Wigner function on the other side. There is an essential loss of information in the case of the phase distribution associated with the Q function. The differences can be easily interpreted [31] when the concept of the area of overlap in phase space introduced by Schleich and Wheeler [59] is invoked. A possibility of deeper insight into the structure of the s-parametrized phase distributions gives us formula (35). The phase distribution $P_{dn}(\theta, s)$ is a result of competition between the functions $P_n(X)$, which are peaked at $\theta = 0$, and the functions $(X_0^2 - X^2)^l$, which have peaks for $\theta = \pm\pi/2$. For $s = -1$ only the term with $n - k = 0$ survives, and there is no modulation due to $(-1)^{n-k}$ factor. This is the reason for which the phase distribution associated with the Q function can have only two peaks, no

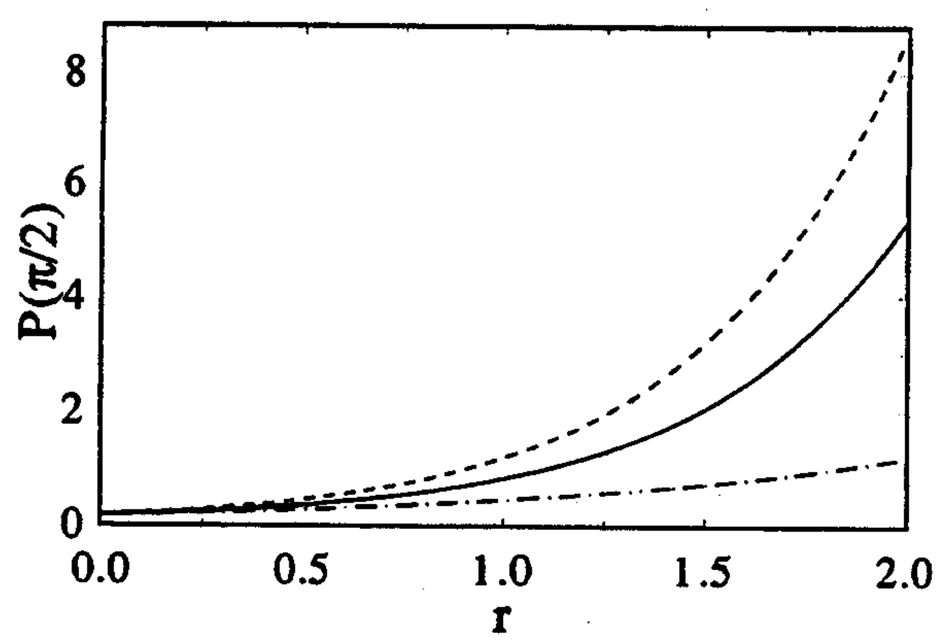

Fig. 3. Height of the peak *vs.* the squeeze parameter r for the squeezed vacuum. Meaning of the lines is the same as in Fig. 2.

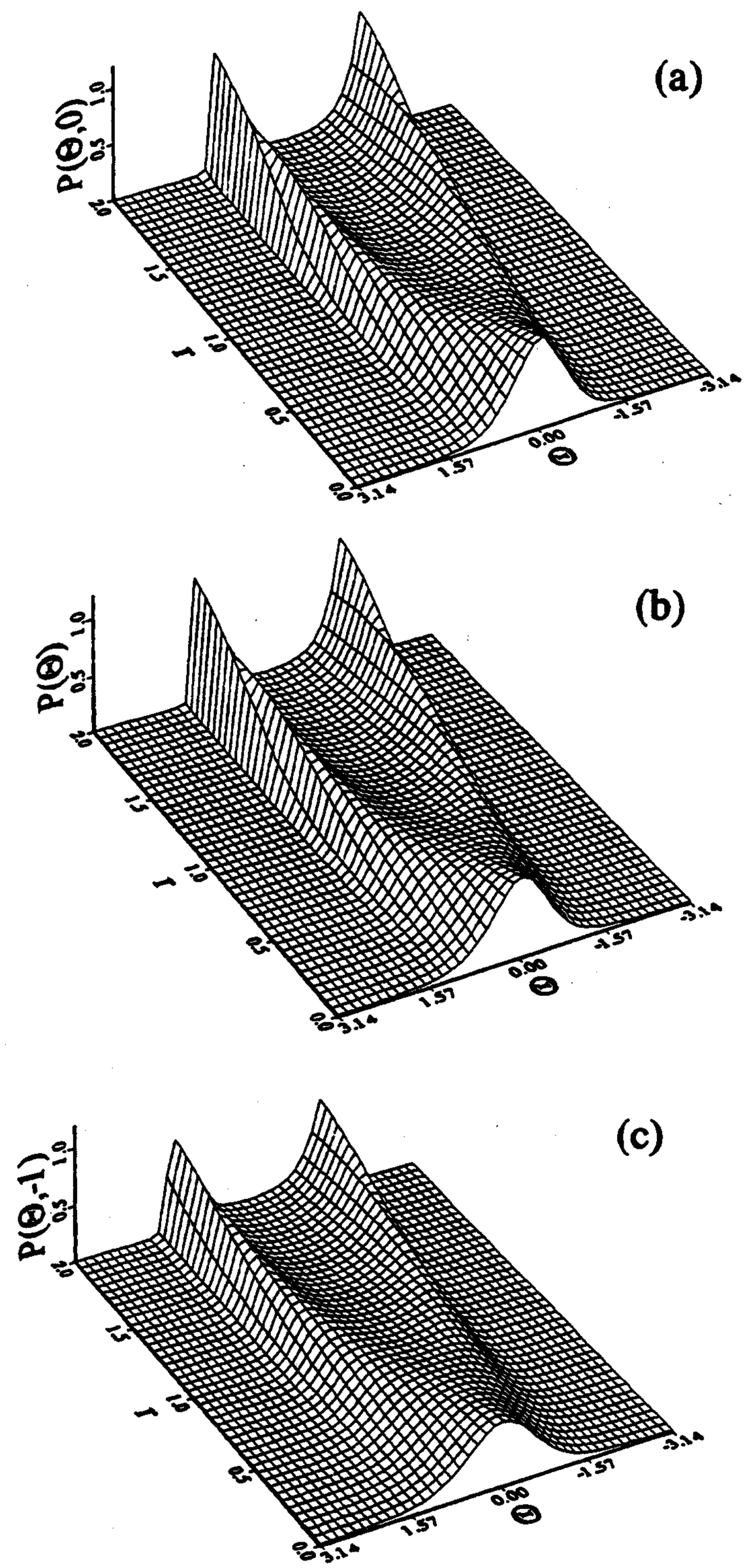

Fig. 4. Pictures of the phase bifurcation for the squeezed state with the mean number of photons $|\alpha_0|^2 = 1$. The distributions are: (a) $P(\theta, 0)$, (b) Pegg–Barnett, and (c) $P(\theta, -1)$.

matter how large is n. Both for the PB phase distribution and $P(\theta, 0)$ there are $n + 1$ peaks. It is also worth mentioning that despite the fact that the Wigner function (34) oscillates between positive and negative values, the phase distribution (35) is positive definite. An illustration of the differences between the phase distributions for the displaced number states with $n = 2$ and $|\alpha_0|^2 = 9$ is shown in Fig. 5. It is seen that the PB phase distribution is very close to the $P_{dn}(\theta, 0)$, and they carry basically the same phase information, while there is an essential loss of phase information carried by $P_{dn}(\theta, -1)$. This is even more convincingly shown

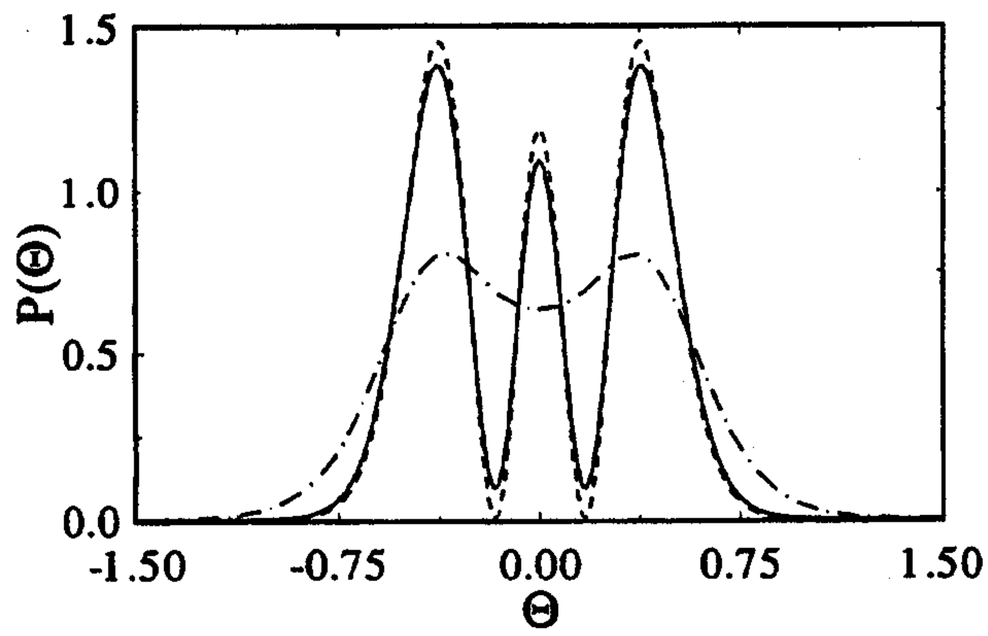

Fig. 5. Phase distributions for the displaced number state with $n = 2$ and $\alpha_0 = 3$. Meaning of the lines is the same as in Fig. 2.

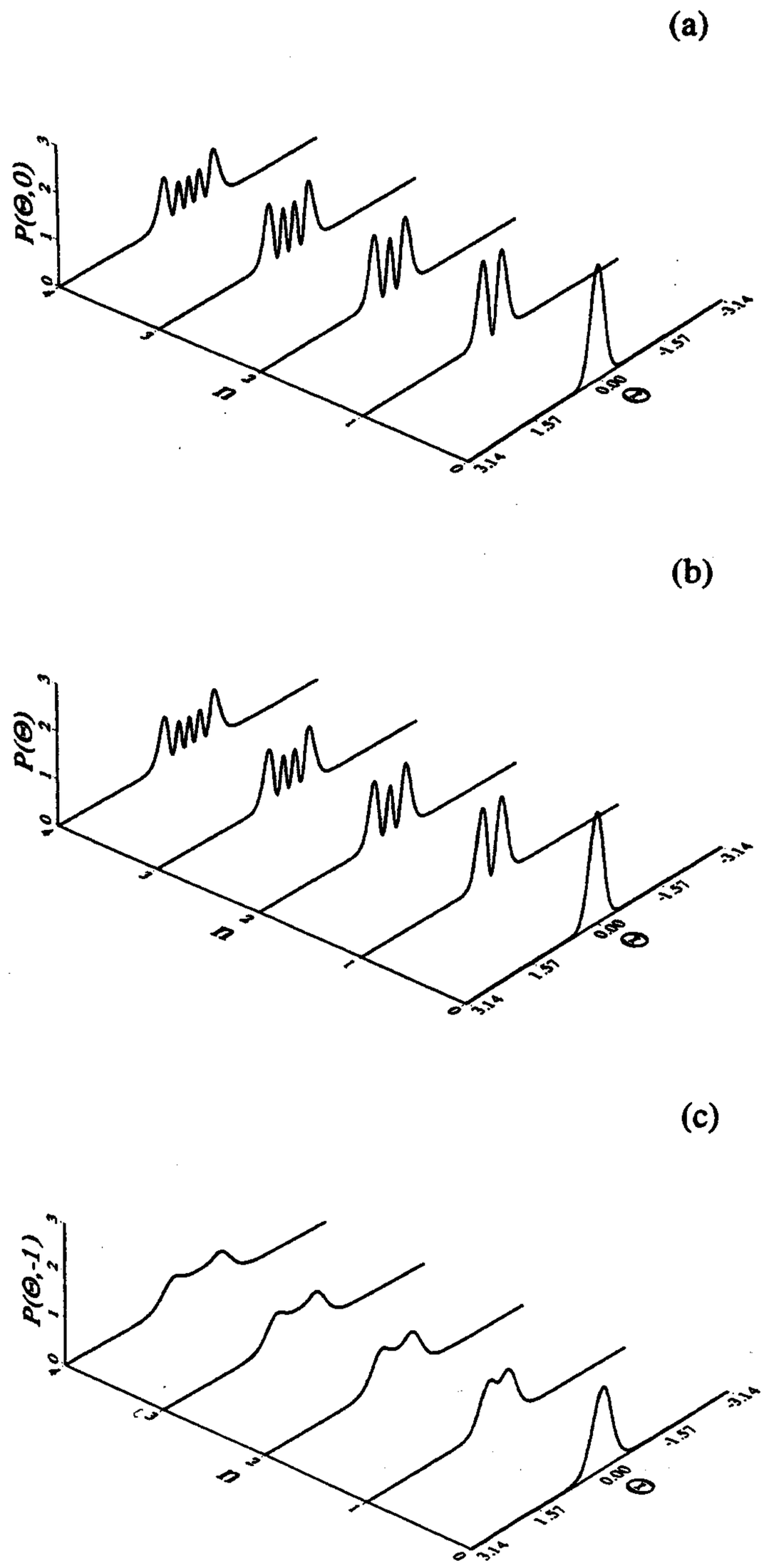

Fig. 6. Phase distributions for the displaced number state with the numbers $n = 0, \ldots, 4$ and $\alpha_0 = 3$. The distributions are: (a) $P(\theta, 0)$, (b) Pegg–Barnett, and (c) $P(\theta, -1)$.

in Fig. 6, where we present phase distributions for the displaced number states with numbers $n = 0, \ldots, 4$. The Pegg–Barnett and $P(\theta, 0)$ are very similar for given n, while $P(\theta, -1)$ has only two peaks that become broader as n increases.

5. Conclusions

In this paper we have made a comparison of various phase distributions for several real field states. The general formula relating the phase distributions obtained by the integration of the s-parametrized quasiprobability distribution functions to the Pegg–Barnett phase distribution has been derived. It has been shown that for any state of the field with known number state matrix elements the s-parametrized phase distribution is obtained by multiplying the nondiagonal elements of the density matrix by the coefficients $G^{(s)}(m, n)$ given by eq. (21). We have also derived exact analytical formulas in closed form for the s-parametrized phase distributions of coherent states, squeezed states, and displaced number states. In many cases such formulas can be treated as a good approximation to the Pegg–Barnett phase distribution. Numerical examples illustrating the similarities and differences between various distributions are given. From these examples, it is clear that qualitatively the Pegg–Barnett phase distribution is well represented by the phase distribution associated with the Wigner function, which is sharper than the Pegg–Barnett distribution, but contains all the details of the latter. However, since the Wigner function can take on negative values it can lead to troubles with positive definiteness of $P(\theta, 0)$, whereas, of course, no such troubles appear for the PB distribution.

Acknowledgement

This work was partially supported by the Polish Government Grant KBN 201509101.

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