QUANTUM PHASE FLUCTUATIONS AND CORRELATIONS OF ELLIPTICALLY POLARIZED LIGHT PROPAGATING IN A KERR MEDIUM

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Abstract—The quantum theory of light propagation in a nonlinear Kerr medium is applied to calculate quantum phase fluctuations and correlations of elliptically polarized light propagating in the medium with dissipation. The Hermitian phase formalism of Pegg and Barnett is applied to describe the phase properties of the field. Exact analytical formulas that describe the mean phase, the phase variance, the inter-mode phase correlations and the phase-difference variance are derived. The results are illustrated graphically for different initial intensities of the field to show explicitly their intensity-dependence. The effect of dissipation on the nonlinear quantum effect of phase randomization is exactly accounted for.

I. INTRODUCTION

It is a well known experimental result [1] that when strong elliptically polarized light propagates through an isotropic nonlinear medium the medium becomes birefringent, which results in the self-induced rotation of the polarization ellipse. Nowadays, propagation of light in a nonlinear Kerr medium is a standard subject of textbooks on nonlinear optics [2,3]. To understand phenomena like optically induced birefrigence there is no need for field quantization. If, however, the quantum properties of light propagating through a Kerr medium are taken into account, some new effects like photon antibunching [4]–[6] and squeezing [7] can occur. Quantum description of elliptically polarized light propagating in a nonlinear Kerr medium requires, in general, a two-mode description of the field. When the light is circularly polarized, the problem can be reduced to the one-mode problem that is equivalent to the anharmonic oscillator model. This model, due to its simplicity allowing exact solutions, became very popular for studying various aspects of nonlinear quantum-field evolution [8]–[27]. To discuss effects associated with elliptical polarization the two-mode description is needed. Such descriptions have already been used in the early studies [4]–[7] of the quantum field effects that appear during the propagation. In those studies the Heisenberg equations of motion for the field operators were solved and their solutions were used to calculate the degree of photon antibunching or squeezing. Recently, Agarwal and Puri [28] have re-examined the problem of propagation of elliptically polarized light through a Kerr medium discussing not only the Heisenberg equations of motion for the field operators but also the evolution of the field states themselves. The polarization state of the field propagating in a Kerr medium can be described by the Stokes parameters which are the expectation values of the corresponding Stokes operators in the quantum description of the field. Quantum fluctuations in the Stokes parameters have recently been discussed by Tanaś and Kielich [29].

The effect of dissipation on the dynamics of the anharmonic oscillator, i.e., the one-mode propagation problem, has already been considered by Milburn and Holmes [10], and recently the exact solutions of the master equation for the system have been discussed [17,20,23,24]. For the two-mode case, the effect of losses and noise has been discussed by Horăk) and Peřina [30] whose approximate approach was based on the Heisenberg-Langevin equations of motion for the operators of the two coupled nonlinear oscillators. Quite recently, using the thermofield dynamics notation, Chaturvedi and Srinivasan [31,32] have found an elegant, exact solution of the master equation for a single nonlinear oscillator [31] as well as for coupled nonlinear oscillators [32].

In this paper we discuss phase properties of elliptically polarized light propagating through a Kerr medium with dissipation. To describe the phase properties of the field we use the Hermitian

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phase formalism introduced by Pegg and Barnett [33]–[35] which enables direct calculations of the expectation values and variances of the Hermitian phase operators for the two modes of the field as well as the correlations between the two phases. To include the dissipation into the system we adopt the master equation solution obtained by Chaturvedi and Srinivasan [32] to the propagation problem. Exact analytical formulae describing quantum phase fluctuations and correlations of the two-mode field propagating in a Kerr medium with dissipation are derived and illustrated graphically for different values of the mean initial numbers of photons to show explicitly the intensity-dependence of the phase properties of the field.

2. QUANTUM DESCRIPTION OF ELLIPTICALLY POLARIZED LIGHT

In the quantum description of the electromagnetic field it is convenient to write the field as a sum of the positive and negative frequency parts

$$E_i(r,t) = E_i^{(+)}(r,t) + E_i^{(-)}(r,t), \tag{1}$$

where i denotes a polarization component of the field. A mode decomposition of the field can be performed next, which for the plane-wave decomposition of the free field propagating in a medium with (linear) refractive index n(ω) gives

$$E_i^{(+)}(r,t) = \sum_{k,\lambda} i\left[\frac{2\pi\hbar\omega_k}{n^2(\omega)V}\right]^{1/2} e_{ki}^{(\lambda)} \exp\left[-i(\omega_k t - k\cdot r)\right] \tag{2}$$

where $e_{ki}^{(\lambda)}$ is the $i =$ th component of the polarization state λ and the propagation vector k, and V is the quantization volume. The operators $\alpha_{k\lambda}$ and $\alpha_{k\lambda}^{\dagger}$ are the annihilation and creation operators of photons with the propagation vector k and polarization λ satisfying the commutation relations

$$\left[\alpha_{k\lambda}, \alpha_{k'\lambda'}^{\dagger}\right] = \delta_{k,k'}\delta_{\lambda,\lambda'}. \tag{3}$$

The polarization vectors satisfy the orthogonality conditions

$$\left.\begin{aligned} &\sum_i e_{ki}^{(\lambda)*} e_{ki}^{(\lambda')} = \delta_{\lambda,\lambda'} \\ &\sum_i e_{ki}^{(\lambda)} k_i = 0. \end{aligned}\right\} \tag{4}$$

For a monochromatic field of frequency ω propagating along the z-axis of the laboratory reference frame, we can drop the index k in our notation and write

$$E_i^{(+)}(z,t) = i\left[\frac{2\pi\hbar\omega}{n^2(\omega)V}\right]^{1/2} \exp\left[-i(\omega t - kz)\right] \sum_{\lambda=1,2} e_i^{\lambda}\alpha_{\lambda} \tag{5}$$

with $k = n(\omega)\omega/c$. Since the summation over the two mutually orthogonal polarizations still remains in Equation (5), we have a two-mode description of the field. If the field is a superposition of these two modes, the two-mode description can be replaced by one mode of the elliptically polarized field

$$e_i a = e_i^{(1)} a_1 + e_i^{(2)} a_2, \tag{6}$$

where $e_i^{(1)}$ and $e_i^{(2)}$ are the i-th components of the orthogonal unit polarization vectors $\hat{e}^{(1)}$ and $\hat{e}^{(2)}$ of the modes a_1 and a_2, and e_i is the i-th component of the polarization vector $\hat{e}$ of the mode a. Relation (6) can also be considered in the reverse sense as a decomposition of initially elliptically polarized light into two orthogonal modes. Applying the orthogonality condition (4) for the polarization vectors, we obtain the formula

$$a = e_1^{*} a_1 + e_2^{*} a_2,$$

where

$$e_1^{*} = \hat{e}^{*}\hat{e}^{(1)}, e_2^{*} = \hat{e}^{*}\hat{e}^{(2)}.$$

So far the decomposition (6) (or, equivalently, (7)) is quite general and can be further specified either for two modes with mutually perpendicular linear polarizations or for right and left-circularly polarized modes.

If a Cartesian basis is chosen, the unit polarization vectors are $\hat{e}^{(1)} = \hat{x}, \hat{e}^{(2)} = \hat{y}^{*}$, whereas in a circular basis we have

$$\hat{e}^{(1)} = \hat{e}^{(+)} = (\hat{x} + i\hat{y})/\sqrt{2}, \hat{e}^{(2)} = \hat{e}^{(-)} - (\hat{x} - i\hat{y})/\sqrt{2},$$

with $\hat{x}$ and $\hat{y}$ being the unit vectors along the x and y axes, respectively. The unit vector $\hat{e}$ of the elliptically polarized light can be written in either a Cartesian or a circular basis as

$$\hat{e} = e_x^* \hat{X} + e_y \hat{y} = e_+ \hat{e}^{(+)} + e_- \hat{e}^{(-)} \tag{8}$$

with e_x and e_y given by [36]

$$\left.\begin{aligned} e_x &= \cos\eta\cos\theta - i\sin\eta\sin\theta, \\ e_y &= \cos\eta\cos\theta + i\sin\eta\sin\theta, \end{aligned}\right\} \tag{9}$$

and

$$e_\pm = \frac{1}{\sqrt{2}}\left(e_x \mp ie_y\right) = \frac{1}{\sqrt{2}}(\cos\eta \pm \sin\eta)e^{\mp i\theta}. \tag{10}$$

The parameters θ and η define the polarization ellipse of the field $-\theta$ is the azimuth of the ellipse denoting the angle between the major axis of the ellipse and the x-axis measured positive from the $+x$-axis towards the $+y$-axis, and η is the ellipticity parameter, $-\pi/4 \le \eta \le \pi/4$, where tan η describes the ratio of the minor and major axes of the ellipse with the sign defining its handedness (plus means right-handed polarization in the helicity convention).

According to Equation (7) the annihilation operator of the elliptically polarized field can be written as

$$a = e_x^* a_x + e_y^* a_y = e_+^* a_+ + e_-^* a_-, \tag{11}$$

where e_x, e_y and $e_\pm$ are given by Equations (9) and (10), and the operators $\alpha_\pm$ are

$$a_\pm = \frac{1}{\sqrt{2}}\left(a_x \mp ia_y\right). \tag{12}$$

Hence, the annihilation operator a of the elliptically polarized light is a superposition of two orthogonal modes in either a Cartesian or a circular basis.

On defining a coherent state of the field with respect to the operator a by the relation

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{13}$$

we have simultaneously

$$|\alpha\rangle = |\alpha_x\rangle_,|\alpha_y\rangle = |\alpha_+\rangle|\alpha_-\rangle, \tag{14}$$

where $|\alpha_x\rangle_,|\alpha_y\rangle =$ and $|\alpha_+\rangle_,|\alpha_-\rangle$ are the coherent states defined with respect to the annihilation operators a_x, a_y and a_+, a_-, respectively. According to (11), (13) and (14) the following relations hold

$$\alpha = e_x^* \alpha_x + e_y^* \alpha_y = e_+^* \alpha_+ + e_-^* \alpha_-, \tag{15}$$

and, due to the normalizations

$$e_x^* e_x + e_y^* e_y = e_+^* e_+ + e_-^* e_- = 1,$$

one obtains

$$\left.\begin{aligned} \alpha_x &= e_x \alpha \\ \alpha_y &= e_y \alpha \end{aligned}\right\} \tag{16}$$

$$\alpha_\pm = e_x \alpha, \tag{17}$$

where e_x, e_y and $e_\pm$ are given by Equations (9) and (10), and

$$|\alpha_x|^2 + |\alpha_y|^2 = |\alpha_+|^2 + |\alpha_-|^2 = |\alpha|^2.$$

So the Cartesian or circular bases can be used alternatively to describe the propagation of elliptically polarized light in a nonlinear Kerr medium. In isotropic media, however, the circular basis is much more advantageous over the Cartesian one.

Relations (15)–(17) together with (8)–(10) allow for the decomposition of a coherent state of elliptically polarized light, with the polarization ellipse described by the azimuth θ and the ellipticity η, into two orthogonal modes being also in a coherent state, and vice versa. However, if the nonlinear interaction between the field and the medium takes place, the resulting state may no longer be a coherent state, even if it was initially. In this case relations (13)–(17) are valid only for the initial coherent states. Quantum evolution of the field propagating through a nonlinear Kerr medium will change these initial states, and the equations of motion will be the subject of the next section.

3. QUANTUM EVOLUTION OF ELLIPTICALLY POLARIZED LIGHT PROPAGATING IN A KERR MEDIUM

Before writing down quantum equations of motion, we remind the main points of the classical description of light propagating through a nonlinear Kerr medium. The classical approach involves the third-order nonlinear polarization of the medium and can be sketched as follows. A monochromatic light field of frequency ω propagating in the medium induces the third-order polarization of this medium at frequency ω which can be written as [2,3]

$$P_i^{(+)}(\omega)=\sum_{jkl} X_{ijkl}(-\omega,-\omega,\omega,\omega)\times E_j^{(-)}(\omega)E_k^{(+)}(\omega)E_l^{(+)}(\omega), \tag{18}$$

where $X_{ijkl}(-\omega,-\omega,\omega,\omega)$ is the third-order nonlinear susceptibility tensor of the medium, and the decomposition of the field into the positive and negative-frequency parts as in Equation (1) has been used; albeit, in the classical description, the field amplitudes $E_i^{(\pm)}(\omega)$ are classical quantities.

For an isotropic medium with a center of inversion, the nonlinear susceptibility tensor $X_{ijkl}(\omega)=X_{ijkl}(-\omega,-\omega,\omega,\omega)$ can be written as follows [2,3]

$$X_{ijkl}(\omega)=X_{xxyy}(\omega)\delta_{ij}\delta_{kl}+X_{xyxy}(\omega)\delta_{ik}\delta_{jl}+X_{xyyx}(\omega)\delta_{il}\delta_{jk} \tag{19}$$

with the additional relation

$$X_{xxxx}(\omega)=X_{yyyy}(\omega)=X_{xxyy}(\omega)+X_{xyxy}(\omega)+X_{xyyx}(\omega). \tag{20}$$

Taking into account the permutation symmetry of the tensor $X_{ijkl}(\omega)$ with respect to its first and second pairs of indices, we have, moreover, $X_{xyxy}(\omega)=X_{xyyx}(\omega)$. The light beam is assumed to propagate along the z axis of the laboratory reference frame.

One insertion of the polarization (18) into the Maxwell equations and applying the slowly varying amplitude approximation, one obtains the following equation for the amplitudes of the field [3]

$$\frac{dE_i^{(\pm)}(\omega)}{dz}=\frac{i2\pi\omega}{n(\omega)c}P_i^{(+)}(\omega), \tag{21}$$

where the slowly-varying amplitudes $E_i^{(\pm)}(\omega)$ are assumed to be dependent on z. If the circular basis is introduced, which is the natural basis for isotropic media, with the circular components of the field

$$E_{\pm}^{(+)}(\omega)=\frac{1}{\sqrt{2}}\left[E_x^{(+)}(\omega)\mp iE_y^{(+)}(\omega)\right], \tag{22}$$

the nonlinear polarization takes the form

$$P_{\pm}^{(+)}(\omega)=2X_{xyxy}(\omega)\left|E_{\pm}^{(+)}(\omega)\right|^2E_{\pm}^{(+)}(\omega)+2\left[X_{xxyy}(\omega)+X_{xyxy}(\omega)\right]\left|E_{\mp}^{(+)}(\omega)\right|^2E_{\pm}^{(+)}(\omega), \tag{23}$$

which after insertion into (21) gives

$$\frac{dE_{\pm}^{(+)}(\omega)}{dz}=\frac{i4\pi\omega}{n(\omega)c}\left\{X_{xyxy}(\omega)\left|E_{\pm}^{(+)}(\omega)\right|^2+\left[X_{xxyy}(\omega)+X_{xyxy}(\omega)\right]\left|E_{\mp}^{(+)}(\omega)\right|^2E_{\pm}^{(+)}(\omega)\right. \tag{24}$$

One easily checks that $(d/dz)\left|E_{\pm}^{(+)}(\omega)\right|^2=0$, the intensities $\left|E_{\pm}^{(+)}(\omega)\right|^2$ of both circular components are constants of motion. This is a clear advantage of the circular basis over the Cartesian basis, which allows for the following simple exponential solution of equation (24) [37]:

$$E_{\pm}^{(+)}(\omega;z)=\exp[i\Phi_{\pm}(z)]E_{\pm}^{(+)}(\omega;z=0), \tag{25}$$

where

$$\Phi_{\pm}(z)=\frac{4\pi\omega z}{n(\omega)c}\left\{X_{xyxy}(\omega)\left|E_{\pm}^{(+)}\right|^2+\left[X_{xxyy}(\omega)+X_{xyxy}(\omega)\right]\left|E_{\mp}^{(+)}(\omega)^2\right|\right\} \tag{26}$$

determines the light-intensity-dependent phase of the field (self-phase-modulation or intensity-dependent refractive index). These are well known classical nonlinear effects [2,3], that are not the subject of our interest here.

In this paper we are interested in quantum phase properties of the field propagating in a Kerr medium; so we need quantum equations of motions for the field. Equations of that type, the Heisenberg equations of motion for the field operators, can be obtained from the following effective interaction Hamiltonian [7]:

$$H_I = \frac{1}{2}\hbar k\left\{a_+^{\dagger 2}a_+^2 + a_-^{\dagger 2}a_-^2 + 4da_+^\dagger a_-^\dagger a_- a_+\right\}, \tag{27}$$

where the nonlinear coupling constant k is real and is given by

$$k = \frac{V}{\hbar}\left(\frac{2\pi\hbar\omega}{n^2(\omega)V}\right)^2 2X_{xyxy}(\omega) \tag{28}$$

with V being the quantization volume. We have introduced in (27) a nonlinear asymmetry parameter d defined as

$$2d = 1 + \frac{X_{xxyy}(\omega)}{X_{xyxy}(\omega)} \tag{29}$$

If the nonlinear susceptibility tensor X is symmetric with respect to all its indices, the asymmetry parameter d is equal to unity. Otherwise $d \neq 1$ and describes the asymmetry of the nonlinear properties of the medium. When the medium is an assembly of independent, identical molecules the asymmetry parameter d is related to the hyperpolarizability of individual molecules [7]. Ritze [6] has calculated this asymmetry parameter for atoms with a degenerate one-photon transition and obtained the results

$$d = \begin{cases} (2J-1)(2J+3)/\left[2\left(2J^2+2J+1\right)\right] \\ \quad \text{for } J \leftrightarrow J \text{ transitions} \\ \left(2J^2+3\right)/\left[2\left(6J^2-1\right)\right] \\ \quad \text{for } J \leftrightarrow J- \text{ transitions} \end{cases} \tag{30}$$

The operators $a_\pm$ in the Hamiltonian (27) are the annihilation operators for the circularly right and left-polarized modes.

Using the interaction Hamiltonian (27) and the commutation rules (3), one can easily write down the Heisenberg equations of motion describing the time evolution of the field operators. Here, we consider the travelling wave case instead of the field in a cavity; so we replace the time t by $-n(\omega)z/c$, and we obtain the following equation:

$$\frac{da_\pm(z)}{dz} = i\frac{n(\omega)}{c}k\left[a_\pm^\dagger(z)a_\pm(z) + 2da_\mp^\dagger(z)a_\mp(z)\right]a_\pm(z). \tag{31}$$

When the relation, obtained from (5),

$$E_\pm^{(+)}(\omega) = i\left[\frac{2\pi\hbar\omega}{n^2(\omega)V}\right]^{1/2} a_\pm \tag{32}$$

is applied, Equation (31) takes the form (24), that makes the quantum-classical correspondence quite transparent, but now we deal with the quantum field.

Our approach is based on the discrete-mode approach, and the transition from the cavity modes to the travelling waves suffers from the cavity-size dependence of the results. Recently, the quantum theory of optical wave propagation without recourse to cavity quantization has been formulated [38], and the exact solution for quantum self-phase modulation within this new approach has been obtained [39].

Since the numbers of photons $a_\pm^\dagger a_\pm$ in the two modes are constants of motion, equation (31) has the simple exponential solution [6,7]

$$a_\pm(\tau) = \exp\left\{i\tau\left[a_\pm^\dagger(0)a_\pm(0) + 2da_\pm^\dagger(0)a_\pm(0)\right]\right\}a_\pm(0), \tag{33}$$

where we have introduced the notation

$$\tau = \frac{n(\omega)kz}{c} \tag{34}$$

The solutions (33) are exact operator solutions for the field operators of light propagating through a nonlinear isotropic Kerr medium without dissipation. These equations were used for calculations of such quantum effects as photon antibunching [6] and squeezing [7].

To describe the evolution of the field states we can use the evolution operator $U(\tau)$ which, according to (27) and (34) and after replacement $t = n(\omega)z/c$, has the form

$$U(\tau) = \exp\left\{i\frac{\tau}{2}\left[\hat{n}_+(\hat{n}_+ - 1) + \hat{n}_-(\hat{n}_- - 1) + 4d\hat{n}_+\hat{n}_-\right]\right\}, \tag{35}$$

where we have introduced the number operators $\hat{n}_\pm = a_\pm^\dagger a_\pm$ for the two circularly polarized modes. The resulting state of the field is thus given by

$$|\psi(\tau)\rangle = U(\tau)|\psi(0)\rangle, \tag{36}$$

where $|\psi(\tau)\rangle$ is the initial state of the field. If the initial state of the field is a coherent state of elliptically polarized light, one obtains [28]

$$\begin{aligned} |\psi(\tau)\rangle &= U(\tau)|\alpha_+\alpha_-\rangle \\ &= \sum_{n_+n_-} b_{n_+}^{(+)} b_{n_-}^{(-)} \exp\{i(n_+\varphi_+ + n_-\varphi_-) \\ &\quad + i\frac{\tau}{2}[n_+(n_+ - 1) + n_-(n_- - 1) + 4dn_+n_-]\}|n_+, n_-\rangle, \end{aligned} \tag{37}$$

where

$$b_{n_\pm}^{(\pm)} = \exp\left(-|\alpha_\pm|^2/2\right)\frac{|\alpha_\pm|^{n_\pm}}{\sqrt{n_\pm!}} \tag{38}$$

and the state $|n_+, n_-\rangle = |n_+\rangle|n_-\rangle$ is the Fock state. We have used here $\alpha_\pm = |\alpha_\pm|\exp(i\varphi_\pm)$.

If the dissipation is present in the system the pure state description of the field is no longer valid, the mean numbers of photons $\langle a_\pm^\dagger a_\pm\rangle$ are no longer constants of motion, and the above formulas do not properly describe the field evolution. Nevertheless, even including damping, the master equation for two coupled nonlinear oscillators has the exact solution [32] which, on assumption of zero temperature reservoir and initially coherent state of the field, can be easily adopted to the travelling wave situation with linear losses. In the presence of damping we have, instead of the solution (37) for the field state, the following solution for the matrix elements of the field density operator:

$$\begin{aligned} \rho_{m_+,m_-;n_+,n_-}(\tau) &= \langle m_+, m_-|\rho(\tau)|n_+, n_-\rangle \\ &= b_{m+}^{(+)} b_{n+}^{(+)} b_{m-}^{(-)} b_{n-}^{(-)} \\ &\quad \times \exp\left\{i\left[\left(\varphi_+ - \frac{\tau}{2}\right)(m_+ - n_+) \right.\right. \\ &\quad \left.\left. + \left(\varphi_- - \frac{\tau}{2}\right)(m_- - n_-)\right]\right\} \\ &\quad \times f_{m_+-n_+;m_--n_-}^{(m_++n_+)/2}(\tau) f_{m_--n_-;m_+-n_+}^{(m_-+n_-)/2}(\tau) \\ &\quad \times \exp\left\{N_+\lambda\frac{1 - f_{m_+-n_+;m_--n_-}(\tau)}{\lambda - i\eta_{m_+-n_+,m_--n_-}}\right\} \\ &\quad \times \exp\left\{N_-\lambda\frac{1 - f_{m_--n_-;m_+-n_+}(\tau)}{\lambda - i\eta_{m_--n_-,m_+-n_+}}\right\} \end{aligned} \tag{39}$$

where τ is given by (34), and we have introduced the following notation

$$\lambda = \gamma_+/\kappa = \gamma_-/\kappa, \tag{40}$$

with γ_+ and γ_- being the damping constants for the two modes,

$$\eta_{n,m} = n + 2dm, \tag{41}$$

$$f_{m;n}(\tau) = \exp\left[(-\lambda - i\eta_{m,n})\tau\right], \tag{42}$$

$b_{n_\pm}^{(\pm)}$ are given by (38), and $\varphi_\pm$ are the phases of the initial coherent states Amplitudes $\alpha_\pm$ while $N_\pm = |\alpha_\pm|^2$ are the mean number of photons. The dissipation is assumed to be equal for both modes and its value (relative with respect to the coupling constant κ) is described by λ.

The solution (39) is exact, and it enables calculations of all one-time expectation values of the field operators. In this paper it is used to calculate quantum phase properties of the field propagating in a Kerr medium with dissipation.

4. QUANTUM PHASE FLUCTUATIONS AND CORRELATIONS

To study phase properties of elliptically polarized light propagating in a Kerr medium we use the new Hermitian phase formalism introduced by Pegg and Barnett [33]–[35]. Their idea is used on introducing, for one mode of the field, a finite (s + 1)-dimensional space Ψ spanned by the number states $|0\rangle, |1\rangle, \ldots, |s\rangle$. The Hermitian phase operator operates on this finite space, and after all necessary expectation values have been calculated in Ψ, the value of s is allowed to tend to infinity. A complete orthonormal basis (s + 1) states is defined on Ψ as

$$|\theta_m\rangle \equiv \frac{1}{\sqrt{s+1}}\sum_{n=0}^{s} \exp(in\theta_m)|n\rangle, \tag{43}$$

where

$$\theta_m \equiv \theta_0 + \frac{2\pi m}{s+1}, (m = 0, 1, \ldots s). \tag{44}$$

The value of θ_0 is arbitrary and defines a particular basis set of $(s + 1)$ mutually orthogonal phase states. The Hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m|, \tag{45}$$

where the subscript θ indicates the dependence on the choice of θ_0. The phase states (43) are eigenstates of the phase operator (45) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and $\theta_0 + 2\pi$. The unitary phase operator $\exp(i\hat{\phi}_\theta)$ is defined as the exponential function of the Hermitian operator $\hat{\phi}_\theta$. This operator acting on the eigenstate $|\theta_m\rangle$ gives the eigenvalue $(i\theta_m)$, and it can be written as [33]–[35]

$$\exp(i\hat{\phi}_\theta) \equiv \sum_{n=0}^{s-1} |n\rangle\langle n+1| + \exp[i(s+1)\theta_0]|s\rangle\langle 0|. \tag{46}$$

This is the last term in (46) that assures the unitarity of this operator. The first sum reproduces the Susskind-Glogower [40,41] phase operator in the limit $s \to \infty$.

If the field is described by the density operator ρ, the expectation value of the phase operator (45) is given by

$$\langle\hat{\phi}_\theta\rangle = Tr\{\rho\hat{\phi}_0\} = \sum_{m=0}^{s} \theta_m \langle\theta_m|\rho|\theta_m\rangle, \tag{47}$$

where $\langle\theta_m|\rho|\theta_m\rangle$ gives a probability of being found in the phase state $|\theta_m\rangle$. The density of phase states is $(S + 1)/2\pi$; so in the continuum limit as s tends to infinity we can write (47) as

$$\langle\hat{\phi}_\theta\rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta P(\theta) d\theta, \tag{48}$$

where the continuum phase distribution $P(\theta)$ is introduced by

$$P(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} \langle\theta_m|\rho|\theta_m\rangle. \tag{49}$$

where θ_m has been replaced by the continuous phase variable θ. As the phase distribution function $P(\theta)$ is known, all the quantum mechanical phase expectation values can be calculated with this function in a classical-like manner by performing integrations over θ.

Taking into account the definition (43), we have

$$P(\theta) = \lim_{s\to\infty} \frac{1}{2\pi} \sum_{n=0}^{s} \sum_{k=0}^{s} \exp[-i(n-k)\theta_m]\rho_{nk}. \tag{50}$$

If we symmetrize the phase distribution with respect to a phase φ by taking

$$\theta_0 = \varphi - \frac{\pi s}{s+1} \tag{51}$$

and introducing a new phase label $\mu = m - s/2$, which goes in integer steps from $-s/2$ to $s/2$, the phase distribution becomes symmetric in μ, and we get

$$P(\theta) = \frac{1}{2\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \exp[-i(m-n)(\varphi+0)\theta_m]\rho_{mn}. \tag{52}$$

Now, all integrals over θ are taken in the symmetric range between $-\pi$ and π, and the phase distribution $P(\theta)$ is normalized so that

$$\int_{-\pi}^{\pi} P(\theta) d\theta = 1. \tag{53}$$

All the above formulas defining phase properties of the field can be easily extended into the two-mode case we are interested in. Proceeding along the same lines, we arrive at the following formula for the joint phase probability distribution $P(\theta_+,\theta_-)$ which is symmetrized with respect to the phases φ_+ and φ_-

$$P(\theta_+,\theta_-) = \frac{1}{(2\pi)}2 \sum_{m_\pm,n_\pm}^{\infty} \exp\{-i[(m_+ - n_+)(\varphi_+ + \theta_+) + (m_- - n_-)(\varphi_- + \theta_-)]\}\rho_{m_+,m_-;n+,n_-}(\tau). \tag{54}$$

On inserting into (54) the solution (39) for the density matrix, we finally obtain the joint phase probability distribution for the continuous phase variables θ_+ and θ_- describing phases of the two modes. This gives us the following formula

$$P(\theta_+,\theta_-)=\frac{1}{(2\pi)^2}\sum_{m_\pm,n_\pm}^{\infty} b_{m_+}^{(+)}b_{n_+}^{(+)}b_{m_-}^{(-)}b_{n_-}^{(-)} \times\exp\left\{-\frac{\lambda\tau}{2}(\sigma_+ +\sigma_-)+\Gamma(\delta_+,\delta_-)\right\} \times\cos\{\delta_+\theta_+ +\delta_-\theta_- -\frac{\tau}{2}[\delta_+(\sigma_+ +2d\sigma_- -1) +\delta_-(\sigma_- +2d\sigma_+ -1)]-\Lambda(\delta_+,\delta_-)\}, \tag{55}$$

where, for brevity, we have comprised the summaton indices into the following combinations:

$$\left.\begin{array}{l}\sigma_+ = m_+ + n_\pm\\ \delta_\pm = m_+ - n_\pm\end{array}\right\}, \tag{56}$$

and we defined the quantities

$$\Gamma(m,n)=\lambda\left[A_{m,n}^{(+)}(\tau)+A_{m,n}^{(-)}(\tau)\right] +\eta_{m,n}B_{m,n}^{(+)}(\tau)+\eta_{n,m}B_{n,m}^{(-)}(\tau), \tag{57}$$

$$\Lambda(m,n)=\eta_{m,n}A_{m,n}^{(+)}(\tau)+\eta_{m,n}A_{m,n}^{(-)}(\tau) -\lambda\left[B_{m,n}^{(+)}(\tau)+B_{n,m}^{(-)}(\tau)\right], \tag{58}$$

where $\eta_{m,n}$ is given by (41), and

$$A_{m,n}^{(\pm)}(\tau)=\frac{N_\pm\lambda}{\lambda^2+\eta_{m,n}^2}\left[1-\exp(-\lambda\tau)\cos(\eta_{m,n}\tau)\right], \tag{59}$$

$$B_{m,n}^{(\pm)}(\tau)=\frac{N_\pm\lambda}{\lambda^2+\eta_{m,n}^2}\exp(-\lambda\tau)\sin(\eta_{m,n}\tau)]. \tag{60}$$

Formula (55) is the exact analytical expression describing the joint probability distribution $P(\theta_+,\theta_-)$, and it allows calculations of all phase expectation values by simple integrations over θ_+ and θ_- in the symmetrical range between $-\pi$ and π.

Despite the complexity of $\Gamma(\delta_+,\delta_-)$ and $\Lambda(\delta_+,\delta_-)$, the structure of formula (55) is quite transparent. If there is no dissipation in the system, $\lambda = 0$ and both these quantities are also equal to zero. In this case, formula (55) goes over into our earlier result [42]. Another limit is the case of no coupling between the two modes, i.e., the case $d = 0$, when the expressions for $\Gamma(\delta_+,\delta_-)$ and $\Lambda(\delta_+,\delta_-)$ split into sums of separate terms for the "plus" and "minus" modes and the phase distribution $P(\theta_+,\theta_-)$ can be factorized into the individual mode distributions. However, either of them still includes the dissipation. The one-mode case with dissipation has been studied by us elsewhere [43].

On intergrating the distribution function $P(\theta_+,\theta_-)$ over one of the phases $P(\theta_-)$ or $P(\theta_+)$ one obtains the marginal distributions for the individual phases. The result is

$$P(\theta_+)=\frac{1}{2\pi}\{1+2\sum_{n>m} b_n^{(+)}b_m^{(+)} \times\exp\left\{-N_-\left[1-\mathrm{Re}\left[f_{0;n-m}(\tau)\right]\right] -\frac{\lambda\tau}{2}(n+m)+\Gamma_{n-m}^{(+)}(\tau)\right\} \times\cos\{(n-m)\theta_+ -\frac{\tau}{2}[n(n-1)-m(m-1)] +N_-\,\mathrm{Im}\left[f_{0;n-m}(\tau)\right]-\Lambda_{n-m}^{(+)}(\tau)\}\}, \tag{61}$$

where

$$\left.\begin{array}{l}\Gamma_{n-m}^{(+)}(\tau)=\Gamma(n-m,0),\\ \Lambda_{n-m}^{(+)}(\tau)=\Lambda(n-m,0).\end{array}\right\} \tag{62}$$

The distribution $P(\theta_-)$ can be obtained from (61) by interchanging the indices plus and minus and taking into account that

$$\left.\begin{array}{l}\Gamma_{n-m}^{(-)}(\tau)=\Gamma(0,n-m),\\ \Lambda_{n-m}^{(-)}(\tau)=\Lambda(0,n-m).\end{array}\right\} \tag{63}$$

Knowing the phase distribution (61) allows calculations of the expectation values and variances of the Hermitian phase operators by performing appropriate integrations. We have, for example,

$$\langle\hat{\phi}\rangle = Tr\{\rho\hat{\phi}_+\} = \varphi_+ + \int_{-\pi}^{\pi}\theta_+P(\theta_+)d\theta_+ = \varphi_+ + \langle\theta_+\rangle, \tag{64}$$

where

$$\begin{aligned}\langle\theta_+\rangle &= \int_{-\pi}^{\pi} \theta_+ P(\theta_+)d\theta_+ \\ &= 2\sum_{n>m} b_n^{(+)} b_m^{(+)} \frac{(-)^{n-m}}{n-m} \\ &\quad \times \exp\Big\{-N_-\big[1-\mathrm{Re}\big[f_{0;n-m}(\tau)\big]\big] \\ &\quad -\frac{\lambda\tau}{2}(n+m)+\Gamma^{(+)}_{n-m}(\tau)\Big\} \\ &\quad \times \sin\Big\{-\frac{\tau}{2}\big[n(n-1)-m(m-1)\big] \\ &\quad + N_-\,\mathrm{Im}\big[f_{0;n-m}(\tau)\big]-\Lambda^{(+)}_{n-m}(\tau)\Big\},\end{aligned} \tag{65}$$

and the variance is given by

$$\langle(\Delta\hat{\phi}_+)^2\rangle = \langle\theta_+^2\rangle - \langle\theta_+\rangle^2 \tag{66}$$

with

$$\begin{aligned}\langle\theta_+^2\rangle &= \int_{-\pi}^{\pi} \theta_+^2 P(\theta_+)d\theta_+ \\ &= \frac{\pi^2}{3} + 4\sum_{n>m} b_n^{(+)} b_m^{(+)} \frac{(-)^{n-m}}{(n-m)^2} \\ &\quad \times \exp\Big\{-N_-\big[1-\mathrm{Re}\big[f_{0;n-m}(\tau)\big]\big] \\ &\quad -\frac{\lambda\tau}{2}(n+m)+\Gamma^{(+)}_{n-m}(\tau)\Big\} \\ &\quad \times \cos\Big\{-\frac{\tau}{2}\big[n(n-1)-m(m-1)\big] \\ &\quad + N_-\,\mathrm{Im}\big[f_{0;n-m}(\tau)\big]-\Lambda^{(+)}_{n-m}(\tau)\Big\}\end{aligned} \tag{67}$$

Formulas (65) and (67) are generalizations of our earlier results [42].

Equation (65) is the quantum formula describing the intensity-dependent phase shift, and for the medium without losses it can be compared with the classical expression (26). This shift depends, in general, on the intensities of both modes and on the asymmetry parameter d. For $d = 0$ there is no coupling between the two modes and then the phase shift for the "plus" mode does not depend on the other mode intensity (N_-). Classically, as it is evident from (26), the intensity-dependent phase shift is linear in z (or in τ). Quantum mechanical formula (65), even for $\lambda = 0$, involves nonlinear τ dependence. In fact, for 2d being an integer the mean phase is periodic in τ in case $\lambda = 0$. However, for $\tau << 1$ the phase shift is practically linear in τ. The range of small τ values is most easily accessible from the experimental point of view. In Figure 1 we illustrate the evolution of the mean phase for different values of the mean number of photons (intensity) and for $\lambda = 0$ (a), $\lambda = 10$ (b). For $\tau << 1$ the linear dependence on τ is clearly visible, and the rate of increase is proportional to the intensity. For larger τ some oscillations appear in the mean phase evolution which are washed out by damping.

Evolution of the phase variance given by (67) is plotted in Figure 2. For $\tau << 1$ without damping the variance is growing as τ^2, and the higher is the mean number of photons the faster is the growth. This means that for strong fields the phase is rapidly randomized, i.e. the phase variance approaches the value $\pi^2/3$, which is the value for uniformly distributed phase. The presence of dissipation in the system causes the randomization to proceed smoothly.

When the two modes are coupled ($d \neq 1$), some degree of correlation between them can arise during the evolution. The phase correlations can, for example, manifest themselves in the variance of the phase-difference (or phase-sum) operator of the two modes. In the Pegg-Barnett formalism the phase-difference (phase-sum) operator is simply the difference (sum) of the phase operators for the two modes. Thus, to calculate the variance of the phase-difference (phase sum) operator we can use the following relation

$$\begin{aligned}\langle[\Delta(\hat{\phi}_+ \pm \hat{\phi}_-)]^2\rangle &= \langle(\Delta\hat{\phi}_+)^2\rangle + \langle(\Delta\hat{\phi}_-)^2\rangle \\ &\quad \pm 2\{\langle\hat{\phi}_+\hat{\phi}_-\rangle - \langle\hat{\phi}_+\hat{\phi}_-\rangle\}\end{aligned} \tag{68}$$

The variances $\left\langle(\Delta\hat{\phi}_+)^2\right\rangle$ and $\left\langle(\Delta\hat{\phi}_-)^2\right\rangle$ can be calculated according to (66) and (67) and their counterparts for the "minus" mode obtained by interchanging "+" and "–". The last term describing the correlation between the phases of the two modes can be written as

$$\begin{aligned}C+-(\tau) &= \langle\hat{\phi}_+\hat{\phi}_-\rangle - \langle\hat{\phi}_+\rangle\langle\hat{\phi}_-\rangle \\ &= \langle\theta_+\theta_-\rangle - \langle\theta_+\rangle\langle\theta_-\rangle,\end{aligned} \tag{69}$$

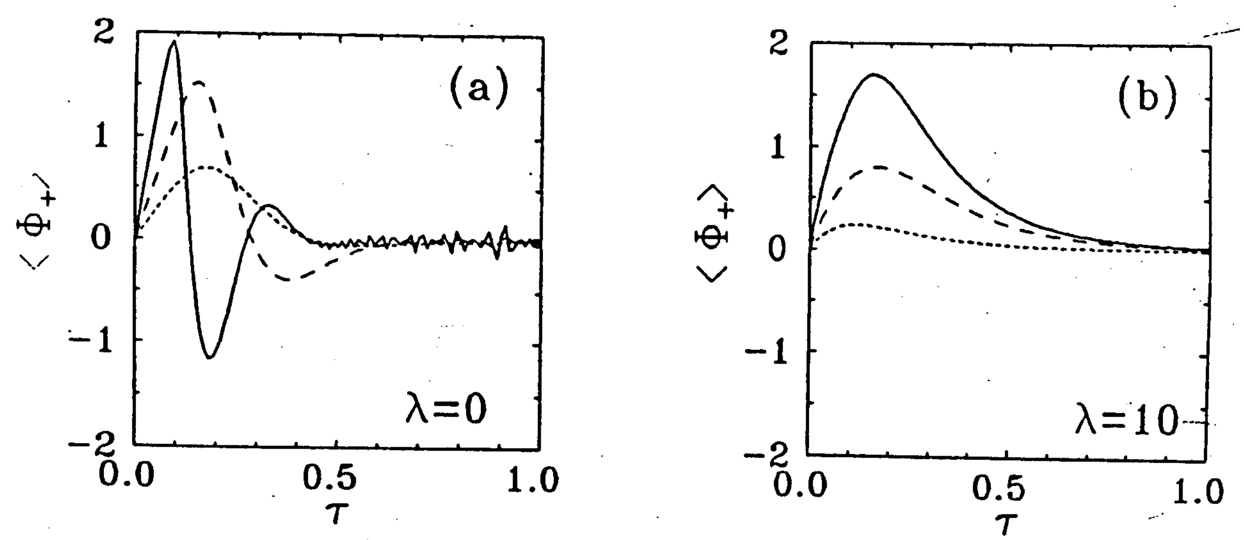

Figure 1. Evolution of the mean phase $\langle\hat{\phi}_+\rangle$, for (a) $\lambda = 0$, and (b) $\lambda = 10$. Other parameters are $N_- = 4$, $d = 1$, and the curves are plotted for $N_+ = 0.25$ (short-dash), $N_+ = 4$ (long-dash), and $N_+ = 16$ (solid). These values and curve descriptions are used in all the figures.

where

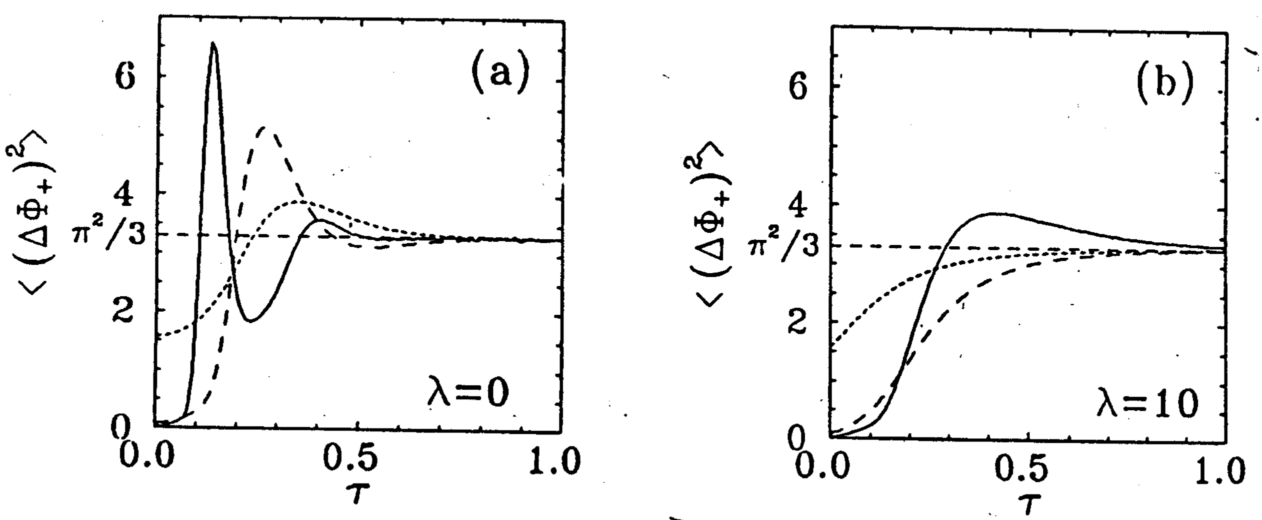

Figure 2. Evolution of the phase variance $\left\langle\left(\Delta\hat{\phi}_+\right)^2\right\rangle$, for (a) $\lambda = 0$, and (b) $\lambda = 10$.

where $\langle\theta_+\rangle$ and $\langle\theta_-\rangle$ are given by (65), and

$$\begin{aligned}\langle\theta_-\rangle &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\theta_+\theta_- P(\theta_+,\theta_-)\,d\theta_+ d\theta_- \\ &= -\sum_{m_\pm,n_\pm}{}' b_{m_+}^{(+)} b_{n_+}^{(+)} b_{m_-}^{(-)} b_{n_-}^{(-)} \frac{(-1)^{\delta_+}}{\delta_+}\frac{(-1)^{\delta_-}}{\delta_-} \\ &\times \exp\left\{-\frac{\lambda\tau}{2}(\phi_+ + \phi_-) + \Gamma(\delta_+,\delta_-)\right\} \\ &\times \cos\left\{-\frac{\tau}{2}\left[\delta_+(\sigma_+ + 2d\sigma_- - 1)\right]\right. \\ &\left. +\delta(\sigma_- + 2d\sigma_+ - 1) + \Lambda(\delta_+,\delta_-)\right\},\end{aligned} \tag{70}$$

where the notation is the same as in formula (55). The prime over the summation symbol means that the terms with $\delta_+ = 0$ and $\delta_- = 0$ do not enter into the sum.

The strength of the correlation depends crucially on the value of the asymmetry parameter d. If $d = 0$, the phase distribution $P(\theta_+,\theta_-)$ factorizes and $C_{+-}(\tau) = 0$. The highest values of the correlation are obtained for $d = 1/2$, which means that the minimum of the phase-difference variance, in view of Equation (68), is obtained for $d = 1/2$. The evolution of the correlation coefficient $C_{+-}(\tau)$ is shown in Figure 3, and the corresponding curves for the phase-difference variance are

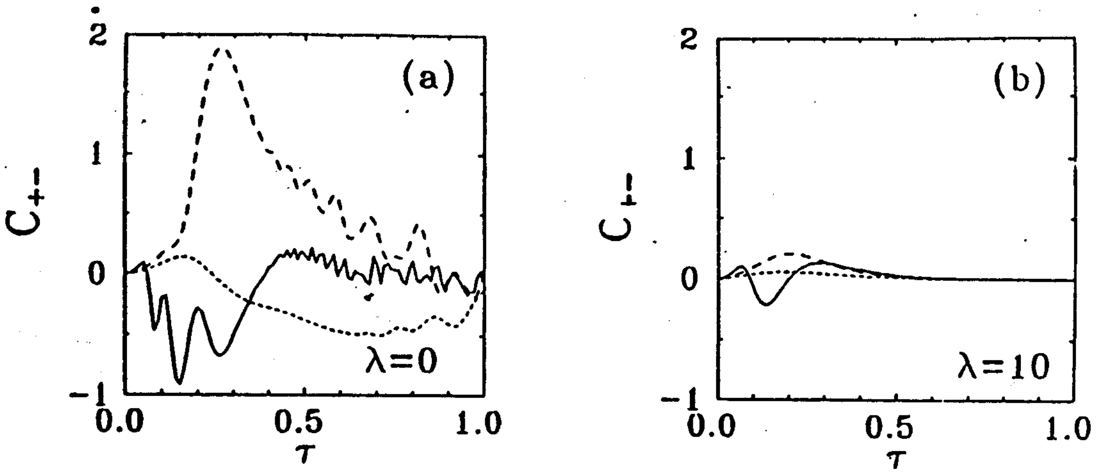

Figure 3. Evolution of the correlation function $C_{+-}(\tau)$, for (a) $\lambda = 0$, and (b) $\lambda = 10$.

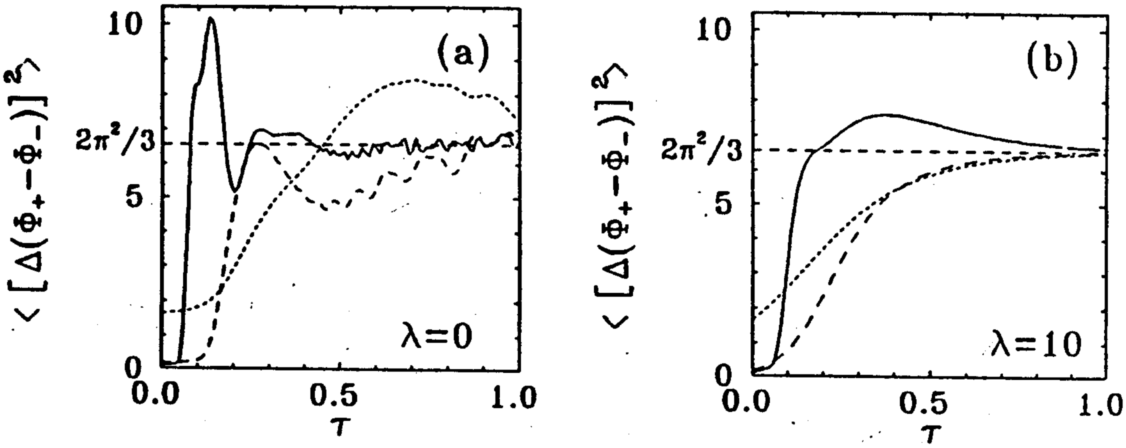

Figure 4. Evolution of the phase-difference variance $\left\langle\left[\Delta\left(\hat{\phi}_+-\hat{\phi}_-\right)\right]^2\right\rangle$, for (a) $\lambda = 0$, and (b) $\lambda = 10$.

plotted in Figure 4. It is seen that the phase correlation can take both positive and negative values depending on the intensity of the field. For high intensities the evolution has oscillatory character. Damping, as expected, makes the evolution smoother. From Figure 4 it is evident that also the phase difference is rapidly randomized when the field intensity is high. The higher the intensity, the faster is the randomization process despite the fact that the correlation can have opposite signs. Generally, a competition between the purely quantum effect of phase randomization and the linear losses of the medium is observed during the propagation of strong light through the medium.

5. CONCLUSION

In this paper we have studied the quantum phase fluctuations and correlations of the elliptically polarized light propagating in a nonlinear Kerr medium with dissipation. The new Hermitian phase formalism of Pegg and Barnett [33]–[35] has been used to describe the phase properties of the field. The exact solution of the master equation for two coupled nonlinear oscillators obtained recently by Chaturvedi and Srinivasan [31,32] has been adopted to describe the propagation of light in a Kerr medium with dissipation. The exact analytical formulas describing the quantum phase fluctuations and correlations of the propagating field have been obtained. The evolution of the mean phase, the phase variance, the inter-mode phase correlation, and the phase-difference variance has been illustrated graphically for various intensities of the initial field, and for the medium without and with losses, The purely quantum effect of the phase randomization is shown to appear owing to the nonlinear coupling. This process becomes very fast for high intensities of light. The dissipation is shown to slow down this process and make the evolution smoother. There is a sort of competition between the quantum effects due to the nonlinear coupling of the field in the medium and the linear losses of the medium. Our exact solution to the problem allows us to find the precise answer regarding the role of dissipation in masking the quantum effects.

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