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Quantum effects on the polarization of light propagating in a Kerr medium

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Quantum theory of propagation of elliptically polarized light in a nonlinear Kerr medium with dissipation is applied to describe changes in the polarization state of the field. Exact analytical formulas describing the degree of polarization and the parameters of the polarization ellipse during the evolution are derived. A number of purely quantum effects that arise during the propagation are discussed. It is shown that the Stokes parameters of light propagating in a Kerr medium can be considered as a good measure of the mean value of the phase-difference cosine or sine, and this relation may be treated as an operational way of measuring such phase quantities.

1. Introduction

When strong light of elliptical polarization propagates through a nonlinear isotropic medium its polarization ellipse performs an intensity-dependent rotation. This effect was first observed by Maker et al. [1] and its classical explanation can be found in textbooks on nonlinear optics [2,3]. To explain the self-induced ellipse rotation there is no need for field quantization. However, if the field propagating in a nonlinear Kerr medium is treated as a quantum field, some new phenomena appear. For instance, the field propagating in such a medium can squeeze its own quantum fluctuations [4]. Quantum properties of the field can also manifest themselves in the polarization of light [5–7] by appearance of the unpolarized component of the field. They can also affect that part of the field which remains polarized.

The polarization state of light propagating in a nonlinear medium can be effectively described in terms of the Stokes parameters which are real numbers in the classical description of the field and become hermitian operators in the quantum description. Quantum fluctuations in the Stokes parameters of strong light propagating in an isotropic nonlinear medium without losses have recently been discussed by Tanaś and Kielich [7]. The Stokes parameters are related to the phase difference of the two orthogonal polarization components forming the elliptical polarization of light, and since they are directly measurable quantities, they can be used to define an operational way of measuring the phase difference. The results obtained on this way can be compared with the results obtained within the Susskind-Glogower [8,9] and the Pegg-Barnett [10–12] phase formalisms.

In this paper we derive exact analytical formulas for the Stokes parameters, which are the expectation values of the Stokes operators, of elliptically polarized light propagating in a Kerr medium with dissipation. To do this we adopt the exact solution of the master equation for two coupled nonlinear oscillators obtained recently by Chaturvedi and Srinivasan [13] to the case of light propagation in a nonlinear medium with linear losses. The Stokes parameters of the outgoing light define its polarization, and the exact quantum formulas for the Stokes parameters reveal a number of new features that are entirely due to the quantum properties of the field:

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the rotation angle of the polarization ellipse is modified, there is a change in the ellipticity of the polarization ellipse (in contrast to the classical field), and there is a change in the degree of polarization. Since the azimuth of the polarization ellipse is directly related to the phase difference between the two circular components of the elliptically polarized field, the measuring of the azimuth (through the measurements of the Stokes parameters) can be considered as a way of measuring the phase difference. At this point, however, we meet the problem of introducing properly defined quantum phase variables [8–12]. Here we shortly discuss some of the differences associated with the understanding of the phase-difference dependent quantities.

2. The master equation and its solution

Quantum properties of elliptically polarized light propagating in an isotropic nonlinear Kerr medium can be described by the following effective interaction hamiltonian [4,5,7]

$$H_1 = \tfrac{1}{2}\hbar\kappa\,(a_+^{\dagger 2}a_+^2 + a_-^{\dagger 2}a_-^2 + 4d a_+^\dagger a_-^\dagger a_- a_+)\,, \tag{1}$$

where $a_\pm$ are the annihilation operators of the circularly right (+) and left (−) polarized modes both of frequency ω, the nonlinear coupling constant κ is real and is given by [4,7]

$$\kappa = \frac{V}{\hbar}\left(\frac{2\pi\hbar\omega}{n^2(\omega)V}\right)^2 2\chi_{xyxy}(\omega)\,, \tag{2}$$

with V denoting the quantization volume, $n(\omega)$ the linear refractive index of the medium, and $\chi_{xyxy}(\omega)$ the third-order nonlinear susceptibility tensor of the medium. The parameter d in eq. (1) is defined by [4]

$$2d = 1 + \chi_{xxyy}(\omega)/\chi_{xyxy}(\omega)\,, \tag{3}$$

and describes the asymmetry of the nonlinear properties of the medium. Ritze [14] has calculated this assymetry parameter for atoms with a degenerate one-photon transition obtaining the results

$$\begin{aligned} d &= (2J-1)(2J+3)/[2(2J^2+2J+1)] && \text{for } J\leftrightarrow J \text{ transitions}\,,\\ &= (2J^2+3)/[2(6J^2-1)] && \text{for } J\leftrightarrow J-1 \text{ transitions}\,. \end{aligned} \tag{4}$$

The coupling between the two modes depends crucially on this asymmetry parameter.

If there is no damping in the system, the interaction hamiltonian (1) can be directly applied to derive the Heisenberg equations of motion for the field operators which, after replacing the time t by $-n(\omega)z/c$ to deal with the field propagation instead of the field in a cavity, take the form

$$\mathrm{d}a_\pm(\tau)/\mathrm{d}\tau = \mathrm{i}[a_\pm^\dagger(\tau)a_\pm(\tau) + 2d a_\mp^\dagger(\tau)a_\mp(\tau)]a_\pm(\tau)\,, \tag{5}$$

where

$$\tau = n(\omega)\kappa z/c\,. \tag{6}$$

Since the numbers of photons $a_\pm^\dagger a_\pm$ are constants of motion for a system without dissipation, eq. (5) has a simple exponential solution that can be directly applied to calculate the Stokes parameters [7].

To deal with the system with dissipation we adopt the master equation solution obtained by Chaturvedi and Srinivasan [13]. We assume the zero-temperature reservoir and the field being initially in a coherent state. Moreover, we replace the time evolution of the density matrix by the coordinate dependence assuming that the field propagates along the z coordinate. Thus the damping constants $\gamma_\pm$ that appear in the master equation are to be interpreted as the absorption coefficients (per unit length) related to the linear absorption of the medium. With these assumptions the Chaturvedi and Srinivasan [13] solution for the density matrix in the number state basis takes the form

$$\rho_{m_+,m_-;n_+,n_-}(\tau)=\langle m_+, m_- |\hat{\rho}(\tau)| n_+, n_- \rangle = b_{m_+}^{(+)} b_{n_+}^{(+)} b_{m_-}^{(-)} b_{n_-}^{(-)} \exp[-\tfrac{1}{2}\lambda\tau(\sigma_+ + \sigma_-) + \Gamma_{\delta_+,\delta_-}(\tau)]$$
$$\times \exp\{\mathrm{i}[\delta_+\varphi_+ + \delta_-\varphi_- + \tfrac{1}{2}\tau[\delta_+(\sigma_+ + 2d\sigma_- - 1) + \delta_-(\sigma_- + 2d\sigma_+ - 1)] + \Lambda_{\delta_+,\delta_-}(\tau)]\}, \tag{7}$$

where the notation used is the following

$$b_{n_\pm}^{(\pm)} = \exp(-|\alpha_\pm|^2/2)\frac{|\alpha_\pm|^{n_\pm/2}}{\sqrt{n_\pm!}}, \quad \alpha_\pm = |\alpha_\pm|\exp(\mathrm{i}\varphi_\pm), \tag{8}$$

$$\sigma_\pm = m_\pm + n_\pm, \quad \delta_\pm = m_\pm - n_\pm, \tag{9}$$

$$\Gamma_{m,n}(\tau) = \lambda[A_{m,n}^{(+)}(\tau) + A_{n,m}^{(-)}(\tau)] + \eta_{m,n}B_{m,n}^{(+)}(\tau) + \eta_{n,m}B_{n,m}^{(-)}(\tau), \tag{10}$$

$$\Lambda_{m,n}(\tau) = \eta_{m,n}A_{m,n}^{(+)}(\tau) + \eta_{n,m}A_{n,m}^{(-)}(\tau) - \lambda[B_{m,n}^{(+)}(\tau) + B_{n,m}^{(-)}(\tau)], \tag{11}$$

$$\eta_{m,n} = m + 2dn, \tag{12}$$

$$A_{m,n}^{(\pm)}(\tau) = \frac{|\alpha_\pm|^2\lambda}{\lambda^2 + \eta_{m,n}^2}[1 - \exp(-\lambda\tau)\cos(\eta_{m,n}\tau)], \tag{13}$$

$$B_{m,n}^{(\pm)}(\tau) = \frac{|\alpha_\pm|^2\lambda}{\lambda^2 + \eta_{m,n}^2}\exp(-\lambda\tau)\sin(\eta_{m,n}\tau)]. \tag{14}$$

$$\lambda = \gamma_+/\kappa = \lambda_-/\kappa, \tag{15}$$

where the indices plus and minus denote the right and left circular components of the initial field which is in the coherent state $|\alpha_+, \alpha_-\rangle$ with the complex amplitudes α_+ and α_-, λ is the relative (with respect to the nonlinearity of the medium) absorption of the medium for both modes (no circular dichroism), and τ is given by eq. (6).

The solution given by eq. (7) is exact and, despite the complexity of $\Gamma_{\delta_+,\delta_-}(\tau)$ and $\Lambda_{\delta_+,\delta_-}(\tau)$ its structure us quite transparent. If there is no absorption in the medium, $\lambda = 0$, both $\Gamma_{\delta_+,\delta_-}(\tau)$ and $\Lambda_{\delta_+,\delta_-}(\tau)$ are zero, and the solution simplifies considerably. We shall use the complete solution (7) to calculate the expectation values of the Stokes operators and phase variables.

3. Quantum effects in the field polarization

The polarization of initially elliptically polarized light propagating through a nonlinear Kerr medium is changed due to the nonlinear interaction, and the changes can be easily accounted for with the use of the Stokes parameters. In quantum treatment of the two-mode field considered by us, the following hermitian Stokes operators can be defined [15]

$$\hat{S}_0 = a_+^\dagger a_+ + a_-^\dagger a_-, \quad \hat{S}_1 = a_+^\dagger a_- + a_-^\dagger a_+, \quad \hat{S}_2 = -\mathrm{i}(a_+^\dagger a_- - a_-^\dagger a_+), \quad \hat{S}_3 = a_+^\dagger a_+ - a_-^\dagger a_-, \tag{16}$$

where $a_\pm$ ($a_\pm^\dagger$) are the annihilation (creation) operators of the two circularly polarized modes. The quantum mechanical expectation values of the Stokes operators (16) are the Stokes parameters describing the polarization of the light beam. For the initial coherent state $|\alpha_+, \alpha_-\rangle$ of elliptically polarized light the Stokes parameters have the values

$$S_0(0) = \langle \alpha_+, \alpha_- |\hat{S}_0| \alpha_+, \alpha_- \rangle = |\alpha_+|^2 + |\alpha_-|^2 = |\alpha|^2,$$
$$S_1(0) = \langle \alpha_+, \alpha_- |\hat{S}_1| \alpha_+, \alpha_- \rangle = 2\mathrm{Re}(\alpha_+^* \alpha_-) = |\alpha|^2\cos 2\eta \cos 2\theta,$$
$$S_2(0) = \langle \alpha_+, \alpha_- |\hat{S}_2| \alpha_+, \alpha_- \rangle = 2\mathrm{Im}(\alpha_+^* \alpha_-) = |\alpha|^2\cos 2\eta \sin 2\theta,$$
$$S_3(0) = \langle \alpha_+, \alpha_- |\hat{S}_3| \alpha_+, \alpha_- \rangle = |\alpha_+|^2 - |\alpha_-|^2 = |\alpha|^2\sin 2\eta, \tag{17}$$

where $|\alpha|^2$ is the total mean number of photons in the field, $\theta=-(\varphi_+-\varphi_-)/2$ defines the azimuth of the polarization ellipse, i.e., the angle between the major axis of the polarization ellipse and the x axis, and η defines the ellipticity parameter, $-\pi/4\leq\eta\leq\pi/4$; tan η is the ratio of the semi-minor axis and the semi-major axis of the polarization ellipse and the sign defines its handedness (plus means the right-handed polarization in the helicity convention).

During the propagating in a nonlinear medium the polarization of the field can change and the actual parameters defining the polarization ellipse are then given by [7]

$$\tan 2\theta(\tau)=S_2(\tau)/S_1(\tau), \qquad \tan 2\eta(\tau)=S_3(\tau)/[S_1^2(\tau)+S_2^2(\tau)]^{1/2}, \tag{18}$$

where

$$S_i(\tau)=\mathrm{Tr}\{\hat{S}_i\hat{\rho}(\tau)\} \tag{19}$$

are the expectation values of the Stokes operators for the resulting field. Another quantity defining the polarization is the degree of polarization defined as

$$P(\tau)=[S_1^2(\tau)+S_2^2(\tau)+S_3^2(\tau)]^{1/2}/S_0(\tau)\,. \tag{20}$$

For the initial state $|\alpha_+,\alpha_-\rangle$, according to (17), the degree of polarization $P(0)$ is equal to unity, i.e., the initial field is completely polarized.

The Stokes parameters of the resulting field can be calculated according to (19) with the density matrix given by (7). The results are the following

$$S_0(\tau)=\sum_{n,m}(n+m)\rho_{n,m;\,n,m}(\tau)=(|\alpha_+|^2+|\alpha_-|^2)\exp(-\lambda\tau)=|\alpha|^2\exp(-\lambda\tau)\,, \tag{21}$$

$$\begin{aligned}S_1(\tau)&=2\mathrm{Re}\sum_{n,m}\sqrt{(n+1)(m+1)}\rho_{n,m+1;\,n+1,m}(\tau)\\&=2|\alpha_+||\alpha_-|\exp\{-\lambda\tau-\Gamma_{-1,1}(\tau)+(|\alpha_+|^2+|\alpha_-|^2)[\exp(-\lambda\tau)\cos\tilde{\tau}-1]\}\\&\quad\times\cos\{\varphi_+-\varphi_-+(|\alpha_+|^2-|\alpha_-|^2)\,exp(-\lambda\tau)sin\,\tilde{\tau}-\Lambda_{-1,1}(\tau)\}\\&=|\alpha|^2\cos 2\eta\exp\{-\lambda\tau+\Gamma_{-1,1}(\tau)+|\alpha|^2[\exp(-\lambda\tau)\cos\tilde{\tau}-1]\}\\&\quad\times\cos\{2\theta-|\alpha|^2\sin 2\eta\exp(-\lambda\tau)\sin\tilde{\tau}+\Lambda_{-1,1}(\tau)\}\,,\end{aligned} \tag{22}$$

$$S_2(\tau)=2\mathrm{Im}\sum_{n,m}\sqrt{(n+1)(m+1)}\rho_{n,m+1;\,n+1,m}(\tau) \tag{23}$$

$$\begin{aligned}&=|\alpha|^2\cos 2\eta\exp\{-\lambda\tau+\Gamma_{-1,1}(\tau)+|\alpha|^2[\exp(-\lambda\tau)\cos\tilde{\tau}-1]\}\\&\quad\times\sin\{2\theta-|\alpha|^2\sin 2\eta\exp(-\lambda\tau)\sin\tilde{\tau}+\Lambda_{-1,1}(\tau)\}\,,\end{aligned} \tag{23}$$

$$S_3(\tau)=\sum_{n,m}(n-m)\rho_{n,m;\,n,m}(\tau)=|\alpha|^2\sin 2\eta\exp(-\lambda\tau)\,, \tag{24}$$

where $\tilde{\tau}=(1-2\mathrm{d})\tau$.

Formulas (21)–(24) are exact analytical results describing the Stokes parameters of light propagating in a Kerr medium with dissipation. For $\lambda=0$ they go over into the earlier results of Tanaś and Kielich [7]. On inserting (21)–(24) into (18) and (20) we get

$$\tan 2\theta(\tau)=\tan[2\theta-|\alpha|^2\sin 2\eta\exp(-\lambda\tau)\sin\tilde{\tau}+\Lambda_{-1,1}(\tau)]\,, \tag{25}$$

$$\tan 2\eta(\tau)=\exp\{-|\alpha|^2[\exp(-\lambda\tau)\cos\tilde{\tau}-1]\}\tan 2\eta\,, \tag{26}$$

$$P^2(\tau)=\sin^2 2\eta+\cos^2 2\eta\exp\{2|\alpha|^2[\exp(-\lambda\tau)\cos\tilde{\tau}-1]+2\Gamma_{-1,1}(\tau)\}\,, \tag{27}$$

The striking feature of the quantum solutions (25)–(27) is that for $\lambda=0$ they are periodic in $\tilde{\tau}=(1-2d)\tau$. This periodic behaviour is destroyed by the dissipation. The essential consequence of the quantum treatment of the field are changes in the degree of polarization and the ellipticity of the beam which are purely quantum effects. There are no such changes for classical fields. The quantum field fluctuations are responsible for the lowering of the degree of polarization [5–7] and making that part of the field which remains polarized to be closer to the circular polarization. The evolution of the degree of polarization is shown in fig. 1 for different values of the mean number of photons $|\alpha|^2$, $\eta=0$, $\lambda=0$(a), and $\lambda=0.1$ (b). As it is seen, the degree of polarization rapidly falls down to zero for $|\alpha|^2 \gg 1$ when the initial polarization is linear ($\eta=0$). For elliptical polarization, however, there is a lower bound for the degree of polarization equal to $|\sin 2\eta|$, as seen from eq. (27), which means that there is no change at all for the circular polarization. If there is no dissipation (a) the degree of polarization shows quantum periodic recurrences. The dissipation destroys the periodicity of the evolution, which is clearly seen from the picture (b). The ellipticity $\eta(\tau)$ calculated according to (26) is plotted in fig. 2 for various field intensities, $\eta=\pi/8$, $\lambda=0$ (a), and $\lambda=0.1$ (b). The value of $\eta(\tau)$ rapidly approaches $\pi/4$ if $|\alpha|^2 \gg 1$, i.e. the polarized part of the field becomes circularly polarized. Again the dissipation removes quantum periodicity of the evolution. Both effects illustrated in figs. 1 and 2 are purely quantum; were the field classical, there would be no changes at all. In fig. 3 the quantum evolution of the $\theta(\tau)$ is plotted. This quantity describes the self-induced ellipse rotation, which is linear in τ for classical fields and the medium without dissipation [2,3]. The quantum result obtained from (25) is periodic for $\lambda=0$. The classical result is obtained from (25) for $\tau \ll 1$ after replacing $\sin\tilde{\tau}$ with $\tilde{\tau}$. Thus all the quantities defining the polarization of the

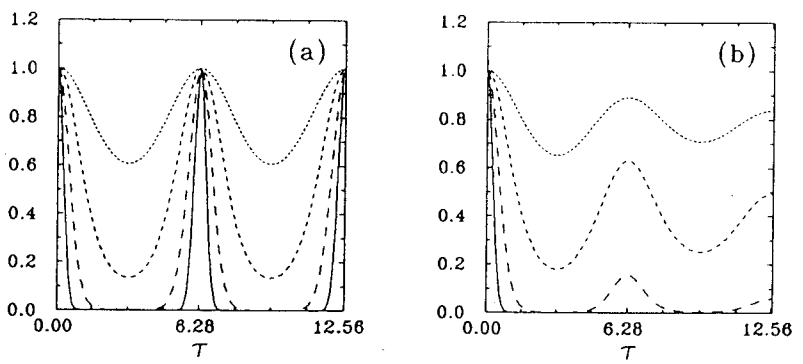

Fig. 1. Evolution of the degree of polarization $P(\tau)$, for $d=1$, $\eta=0$, different $|\alpha|^2$ (0.25 dotted line, 1 short dashes, 4 long dashes, 16 solid); (a) $\lambda=0$, and (b) $\lambda=0.1$.

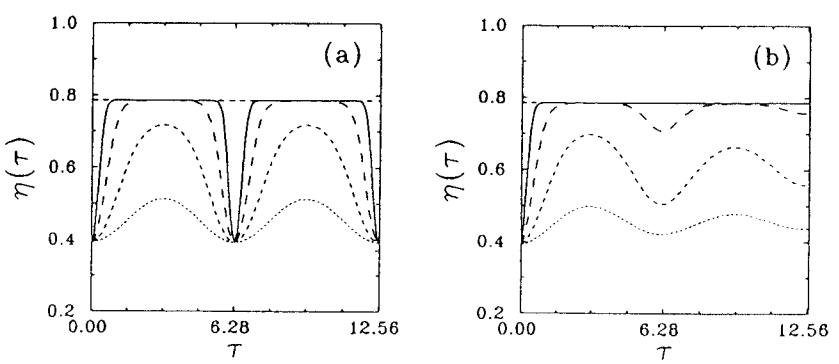

Fig. 2. Evolution of the ellipticity $\eta(\tau)$, for $d=1$, $\eta=\pi/8$, and the other parameters are the same as in fig. 1.

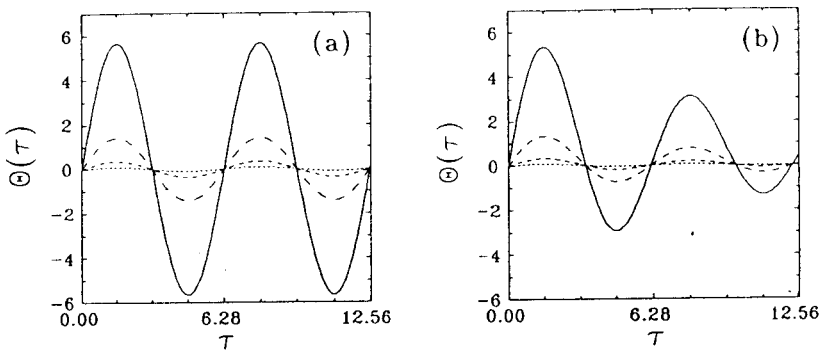

Fig. 3. Evolution of the azimuth $\theta(\tau)$, for $d=1$, $\eta=\pi/8$, and the other parameters are the same as in fig. 1.

outgoing beam exhibit quantum features in the long τ limit, despite the fact that they are defined by the Stokes parameters which are the first order (in intensity) field correlation functions. In real physical situation we have rather $\tau \ll 1$, but even in this case the quantum effects can be observed when $|\alpha|^2 \gg 1$.

4. Relation to the phase properties of the field

Classically the azimuth of the polarization ellipse is directly related to the phase difference between the two circular components of the field, i.e., $\theta = -(\varphi_+ - \varphi_-)/2$, which is also true for the initial coherent state of the field. Thus the rotation of the polarization ellipse can be interpreted as a change of this phase difference, and $\theta(\tau)$ could be considered as a measure of the phase difference for the resulting field. However, in quantum description of the field the questions arise: "How to define the quantum phase variables?", and "What can we actually measure?". The Stokes parameters we discuss in this paper can be measured by simply measuring the intensity of light that passed through a combination of the optical elements such as quarter-wavelength plates and/or polarizers. Knowing the Stokes parameters we can calculate $\theta(\tau)$ according to (18) as

$$\theta(\tau) = \tfrac{1}{2}\tan^{-1}[S_2(\tau)/S_1(\tau)] \,. \tag{28}$$

However, this quantity is not the quantum mechanical expectation value of the phase-difference operator. The hermitian phase formalism of Pegg and Barnett [10–12] allows for direct calculation of the phase-difference operator expectation value. Phase properties of elliptically polarized light propagating in a Kerr medium have been studied by us elsewhere [16,17], and here we only recall the results we need. In the Pegg-Barnett [10–12] approach the phase-difference operator is simply the difference of the phase operators for the two components of the field, and for the medium with dissipation the expectation value of the phase difference is given by [17]

$$\begin{aligned}
-\tfrac{1}{2}(\langle\hat{\phi}_+\rangle - \langle\hat{\phi}_-\rangle) &= -\tfrac{1}{2}(\varphi_+ - \varphi_-) \\
&+ \sum_{n>m} \frac{(-1)^{n-m}}{n-m} \{ b_n^{(+)} b_m^{(+)} \exp[-\tfrac{1}{2}\lambda\tau(n+m) - |\alpha_-|^2[1-\exp(-\lambda\tau)\cos(2d\tau)] + \Gamma_{n-m,0}(\tau)] \\
&\times \sin[\tfrac{1}{2}\tau[n(n-1)-m(m-1)] - |\alpha_-|^2 \exp(-\lambda\tau)\sin(2d\tau) + \Lambda_{n-m,0}(\tau)] \\
&- b_n^{(-)} b_m^{(-)} \exp[-\tfrac{1}{2}\lambda\tau(n+m) - |\alpha_+|^2[1-\exp(-\lambda\tau)\cos(2d\tau)] + \Gamma_{0,n-m}(\tau)] \\
&\times \sin[\tfrac{1}{2}\tau[n(n-1)-m(m-1)] - |\alpha_+|^2 \exp(-\lambda\tau)\sin(2d\tau) + \Lambda_{0,n-m}(\tau)]\} \,.
\end{aligned} \tag{29}$$

This expression is quite different from $\theta(\tau)$ obtained from (25). For comparison we plot the evolution of both

quantities in fig. 4, for $|\alpha|^2=16$, $\eta=\pi/8$, $d=1$, and $\lambda=0$. The two quantities behave quite differently for long τ. However, for $|\alpha|^2 \gg 1$ and $\tau \ll 1$, i.e. in the classical limit, they are indistinguishable as is already seen from fig. 4, where $|\alpha|^2=16$. Of course, the difference between $\theta(\tau)$ and the corresponding phase-difference expectation value is not unexpected because $\theta(\tau)$ is calculated as the $\tan^{-1}$ function of the measured Stokes parameters and, generally, for any operator quantity a function of its mean value is different from the mean value of its function. Thus the measurement of the Stokes parameter, say $S_1(\tau)$, gives us the expectation value of a function of both the amplitude and phase variables. Here, the measured phase concept [18] which relates the measurement of the appropriately normalized field quadrature to the mean value of the phase cosine (or sine) can be invoked. From the form of expression (22) and (23) one can expect that $S_1(\tau)$ is related to the mean value of the cosine (and $S_2(\tau)$ to the sine) of the phase difference. To check this hypothesis we recall the expression for the expectation value of the phase-difference cosine obtained within the Pegg-Barnett [10–12] formalism (which for physical states is the same as in the Susskind-Glogower [8,9] approach). The formula obtained by us earlier [17] is as follows

$$\langle \cos(\hat{\phi}_+ - \hat{\phi}_-) \rangle = \mathrm{Re} \sum_{n,m} \rho_{n,m+1;\, n+1,m}(\tau) = \exp[-\lambda\tau + \Gamma_{-1,1}(\tau)]$$
$$\times \sum_{n,m} b_n^{(+)} b_{n+1}^{(+)} b_m^{(-)} b_{m+1}^{(-)} \exp[-\lambda\tau(n+m)] \cos[\varphi_+ - \varphi_- + (n-m)\tilde{\tau} - \Lambda_{-1,1}(\tau)] \,. \tag{30}$$

A comparison of (30) and (22) immediately shows that both expressions are defined by the same matrix elements of the density matrix, and this is the presence of $[(n+1)(m+1)]^{1/2}$ in (22) that allows to sum up the series in (22). Its absence in (30) prevents the summation to be performed there. Replacing $b_{n+1}^{(\pm)}$ with $b_n^{(\pm)}$ in (30) makes it possible to sum up the series giving the same expression as (22), apart from the factor $2|\alpha_+||\alpha_-|$. Thus it is tempting to consider the properly normalized $S_1(\tau)$ as a measure of the phase-difference cosine. What is the proper normalization in this case? From the form of eqs. (22) and (23) one would think of the normalization by $[S_1^2(\tau)+S_2^2(\tau)]^{1/2}$ leading to $\cos 2\theta(\tau)$. However, the comparison of (22) and (30) prompts us that the proper normalization is rather by $[S_1^2(0)+S_2^2(0)]^{1/2}$ which leads to the formula that should be very close to the expectation value of the phase-difference cosine. In fact, for $|\alpha|^2 \gg 1$ both formulas give indistinguishable results. In fig. 5 we have shown the evolution of three different quantities: $\langle \cos(\hat{\phi}_+ - \hat{\phi}_-) \rangle$ calculated according to (30), the measured phase-difference cosine given by $S_1(\tau)/[S_1^2(0)+S_2^2(0)]^{1/2}$, and $\cos(\langle \hat{\phi}_+ \rangle - \langle \hat{\phi}_- \rangle)$. It is seen that even for $|\alpha|^2=4$ taken in the figure the agreement between $\langle \cos(\hat{\phi}_+ - \hat{\phi}_-) \rangle$ and $S_1(\tau)/[S_1^2(0)+S_2^2(0)]^{1/2}$ is quite good, and it becomes much better when $|\alpha|^2$ increases. The third quantity, $\cos(\langle \hat{\phi}_+ \rangle - \langle \hat{\phi}_- \rangle)$, behaves quite differently for long τ. However, for the most important from the experimental point of view case, i.e. $\tau \ll 1$ and $|\alpha|^2 \gg 1$, all three quantities evolve in a similar way, as seen from fig. 6, where the initial stage of the evolution is shown for $|\alpha|^2=16$, $\eta=\pi/8$, $d=1$, and $\lambda=0$.

In conclusion we can say that the properly normalized Stokes parameters are a good measure of the expec-

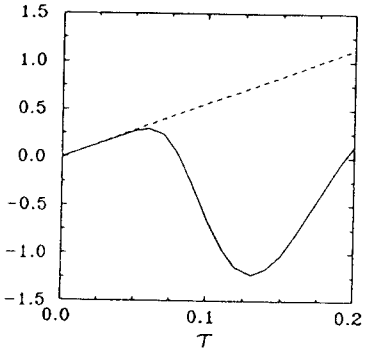

Fig. 4. Comparison of the evolution of $\theta(\tau)$ (dashed line) and $(\langle \hat{\phi}_{k-} \rangle - \langle \hat{\phi}_+ \rangle)/2$ (solid line), for $d=1$, $\eta=\pi/8$, $\lambda=0$, and $|\alpha|^2=16$.

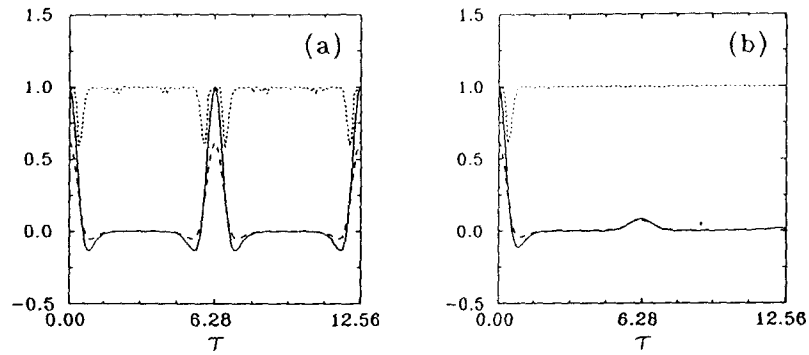

Fig. 5. Comparison of the evolution of $S_1(\tau)/[S_1^2(0)+S_2^2(0)]^{1/2}$ (solid line, $\langle\cos(\hat{\phi}_+-\hat{\phi}_-)\rangle$ (long dashes), and $\cos(\langle\hat{\phi}_+-\hat{\phi}_-\rangle)$ (short dashes), for $d=1$, $\eta=\pi/8$, $|\alpha|^2=4$, and (a) $\lambda=0$, (b) $\lambda=0.1$.

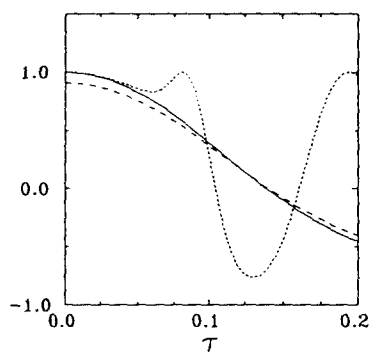

Fig. 6. The same as in fig. 5 but for $d=1$, $\eta=\pi/8$, $|\alpha|^2=16$, and $\lambda=0$.

tation value of the phase-difference cosine (or sine), if $|\alpha|^2 \gg 1$, for any values of τ. For $\tau \ll 1$ they can also be considered as a good approximation for $\cos(\langle\hat{\phi}_+\rangle-\langle\hat{\phi}_-\rangle)$ (or $\sin(\langle\hat{\phi}_+\rangle-\langle\hat{\phi}_-\rangle)$).

5. Conclusion

In this paper we have studied the quantum effects in the polarization of elliptically polarized light propagating in a Kerr medium with dissipation. Exact analytical formulas describing the degree of polarization and the parameters of the polarization ellipse have been obtained for the medium with dissipation and initially elliptical polarization of the field. It has been shown that owing to quantum fluctuations of the field, the initially polarized light becomes partially polarized. For elliptically polarized initial light there is, however, a lower bound for the degree of polarization P which is equal to $|\sin 2\eta|$. Another interesting feature of quantum evolution is the fact that the ellipticity $\eta(\tau)$ approaches $\pm\pi/4$, i.e., that part of the field which remains polarized approaches a circular polarization. Presence of dissipation in the medium removes the quantum periodicity of the evolution. Our exact analytical formulas allow for giving precise answers regarding the role of dissipation in the quantum evolution.

We have made a comparison between the Stokes parameters which are directly measurable quantities and the phase properties of the field. We have shown that the appropriately normalized Stokes parameters can be considered as a measure of the expectation values of the phase-difference cosine (or sine).

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