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Quantum fluctuations in the Stokes parameters of light propagating in a Kerr medium with dissipation

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Abstract. Exact analytical expressions describing the evolution of the expectation values and the variances of the Stokes operators are derived for the elliptically polarized light propagating in a Kerr medium with dissipation. It is shown that quantum fluctuations of the field essentially affect the polarization of the field. The explicit quantum formulas describing the azimuth and the ellipticity of the polarization ellipse as well as the degree of polarization are derived and illustrated graphically. Quantum fluctuations in the Stokes parameters are discussed, and the evolution of the signal-to-noise ratio is shown to be reduced by the quantum field fluctuations. Role of the dissipation is shown explicitly in a fully quantitative way from the exact analytical solutions.

1. Introduction

Optically induced birefringence of an isotropic medium subjected to a strong optical field is a well known fact [1, 2]. Such phenomena like the optical Kerr effect and self-induced ellipse rotation can be explained with recourse to field quantization. On the other hand, if a strong optical field propagating through a nonlinear Kerr medium is treated as a quantum field some new phenomena, like photon antibunching [3–5] and squeezing [6], can appear. Since Kerr media are also considered as suitable candidates for performing quantum non-demolition measurements [7, 8], there is growing interest in revealing those aspects of nonlinear propagation that are directly related to the quantum properties of the field.

The polarization state of light propagating through a nonlinear Kerr medium can be effectively described in terms of the Stokes parameters. The Stokes parameters, which are real numbers in the classical description of the field, become Hermitian operators in the quantum description. On having defined the Stokes operators, which are quantum mechanical observables, one is naturally led to address the problem of quantum fluctuations in these quantities as well as quantum field effects on the polarization state of the field propagating in a Kerr medium. Quantum fluctuations in the Stokes parameters of strong light propagating in an isotropic nonlinear medium have recently been discussed by Tanaś and Kielich [9], who treated the medium as ideally transparent, i.e., without losses. Quantum evolution of

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the field propagating in Kerr medium has been also considered by Agarwal and Puri [10]. Quite recently, Chaturvedi and Srinivasan [11] using the thermofield dynamics notation have found the exact solution of the master equation for coupled nonlinear oscillators. This solution can be adopted to describe propagation of elliptically polarized light in a Kerr medium with dissipation. An approximate approach based on the Heisenberg–Langevin equations of motion for the operators of the two coupled nonlinear oscillators was given earlier by Horák and Peřina [12].

In this paper we apply the solution of Chaturvedi and Srinivasan [11] to study quantum fluctuations of the Stokes parameters and the polarization state of strong light propagating in a nonlinear Kerr medium with dissipation. Even including (linear) losses, the exact analytical formulas describing the expectation values and variances of the Stokes operators are derived. The results obtained in this paper are generalizations of earlier results by Tanaś and Kielich [9] into the case of a medium with dissipation.

2. The master equation and its solution

Quantum properties of elliptically polarized light propagating in an isotropic nonlinear Kerr medium can be described by the following effective interaction Hamiltonian [4, 9, 10]

$$H_1=\tfrac{1}{2}\hbar\kappa(a_+^{\dagger 2}a_+^2+a_-^{\dagger 2}a_-^2+4da_+^{\dagger}a_-^{\dagger}a_-a_+), \tag{1}$$

where $a_\pm$ are the annihilation operators of the circularly right (+) and left (−) polarized modes both of frequency ω, the nonlinear coupling constant κ is real and is given by [4,9]

$$\kappa=\frac{V}{\hbar}\left(\frac{2\pi\hbar\omega}{n^2(\omega)V}\right)^2 2\chi_{xyxy}(\omega), \tag{2}$$

with V denoting the quantization volume, $n(\omega)$ the linear refractive index of the medium, $\chi_{xyxy}(\omega)$ the third-order nonlinear susceptibility tensor of the medium. The parameter d in equation (1) is defined by [4]

$$2d=1+\frac{\chi_{xxyy}(\omega)}{\chi_{xyxy}(\omega)}, \tag{3}$$

and describes the asymmetry of the nonlinear properties of the medium. Ritze [5] has calculated this asymmetry parameter for atoms with a degenerate one-photon transition obtaining the results

$$d=\begin{cases}(2J-1)(2J+3)/[2(2J^2+2J+1)], & \text{for } J\leftrightarrow J \text{ transitions},\\ (2J^2+3)/[2(6J^2-1)], & \text{for } J\leftrightarrow J-1 \text{ transitions}.\end{cases} \tag{4}$$

The coupling between the two modes depends crucially on this asymmetry parameter.

If there is no damping in the system, the interaction Hamiltonian (1) can be directly applied to derive the Heisenberg equations of motion for the field operators which, after replacing the time t by $-n(\omega)z/c$ to deal with the field propagation instead of the field in a cavity, have the simple exponential solutions of the form [5]

$$a_\pm(\tau)=\exp\{i\tau[a_\pm^{\dagger}(0)a_\pm(0)+2da_\mp^{\dagger}(0)a_\mp(0)]\}a_\pm(0), \tag{5}$$

where

$$\tau = \frac{n(\omega)\kappa z}{c}. \tag{6}$$

To describe the field state evolution the evolution operator $U(\tau)$ can be used, which is given by

$$\begin{aligned} U(\tau) &= \exp\{\mathrm{i}\frac{\tau}{2}[a_+^{\dagger 2}a_+^2 + a_-^{\dagger 2}a_-^2 + 4da_+^\dagger a_-^\dagger a_- a_+]\} \\ &= \exp\{\mathrm{i}\frac{\tau}{2}[\hat{n}_+(\hat{n}_+ - 1) + \hat{n}_-(\hat{n}_- - 1) + 4d\hat{n}_+\hat{n}_-]\}, \end{aligned} \tag{7}$$

where we have introduced the number operators $\hat{n}_\pm = a_\pm^\dagger a_\pm$ for the two circularly polarized modes. If the initial state of the field is a coherent state of elliptically polarized light one obtains [10]

$$\begin{aligned} |\psi(\tau)\rangle = U(\tau)|\alpha_+, \alpha_-\rangle = \sum_{n+,n-} b_{n+}^{(+)} b_{n-}^{(-)} \exp\Big\{ \mathrm{i}(n_+\phi_+ + n_-\phi_- \\ + \mathrm{i}\frac{\tau}{2}[n_+(n_+ - 1) + n_-(n_- - 1) + 4dn_+n_-]\Big\} |n_+, n_-\rangle, \end{aligned} \tag{8}$$

where

$$b_{n\pm}^{(\pm)} = \exp(-|\alpha_\pm|^2/2)\frac{|\alpha_\pm|^{n\pm}}{(n_\pm!)^{1/2}}, \qquad \alpha_\pm = |\alpha_\pm| \exp(\mathrm{i}\phi_\pm), \tag{9}$$

$\alpha_\pm$ are the amplitudes of the initial coherent states of the two modes, $|\alpha_\pm|^2$ are the mean numbers of photons, and $\phi_\pm$ are the phases of $\alpha_\pm$. Properties of such states have been discussed by Agarwal and Puri [10].

When dissipation is introduced into the system, the state of the field can no longer be described by a pure state, like equation (8), and the density operator description is necessary. A standard way of introducing the dissipation into the system is to couple it to a reservoir of oscillators and after well known steps, to write down a master equation describing the evolution of the system with dissipation. For the system of two coupled nonlinear oscillators obtained by the Hamiltonian (1), the corresponding master equation in the interaction picture has the following form [11]

$$\partial\rho\partial t = \frac{1}{\mathrm{i}\hbar}[H_I, \rho] + \sum_{i=+,-}\left\{\frac{\gamma_i}{2}([a_i\rho, a_i^\dagger] + [a_i, \rho a_i^\dagger]) + \gamma_i\bar{n}_i[[a_i, \rho], a_i^\dagger]\right\}, \tag{10}$$

where γ_i are the damping constants and $\bar{n}_i$ are the mean numbers of thermal photons.

The exact solution to the master equation (10) has been found by Chaturvedi and Srinivasan [11]. In this paper we adopt their solution to describe propagation of light in a Kerr medium with dissipation. We assume that the reservoir is at zero temperature ($\bar{n}_i = 0$) and that the field is initially in a coherent state. Moreover, we replace the time evolution of the density matrix by the coordinate dependence (assuming that the field propagates along the z coordinate). This means that the damping constants $\gamma_\pm$ can be interpreted as the absorption coefficients (per unit length) related to the linear absorption of the medium. With these assumptions the solution to the master equation (10), in the number state basis, has the following form

$$\begin{aligned}\rho_{m_+,m_-;n_+,n_-}(\tau) &= \langle m_+, m_-|\rho(\tau)|n_+, n_-\rangle \\ &= b^{(+)}_{m_+} b^{(+)}_{n_+} b^{(-)}_{m_-} b^{(-)}_{n_-} \exp\left[-\frac{\lambda\tau}{2}(\sigma_+ + \sigma_-) + \Gamma_{\delta_+,\delta_-}(\tau)\right] \\ &\quad \times \exp\left\{\mathrm{i}\left[\delta_+\phi_+ + \delta_-\phi_- + \frac{\tau}{2}[\delta_+(\sigma_+ + 2d\sigma_- - 1) \right.\right. \\ &\quad \left.\left. + \delta_-(\sigma_- + 2d\sigma_+ - 1)] + \Lambda_{\delta_+,\delta_-}(\tau)\right]\right\},\end{aligned} \tag{11}$$

where the $b^{(\pm)}_{n_\pm}$ are given by (9), and we have introduced the following notation

$$\left.\begin{aligned}\sigma_\pm &= m_\pm + n_\pm, \\ \delta_\pm &= m_\pm - n_\pm,\end{aligned}\right\} \tag{12}$$

$$\Gamma_{m,n}(\tau) = \lambda[A^{(+)}_{m,n}(\tau) + A^{(-)}_{n,m}(\tau)] + \eta_{m,n}B^{(+)}_{m,n}(\tau) + \eta_{n,m}B^{(-)}_{n,m}(\tau), \tag{13}$$

$$\Lambda_{m,n}(\tau) = \eta_{m,n}A^{(+)}_{m,n}(\tau) + \eta_{n,m}A^{(-)}_{n,m}(\tau) - \lambda[B^{(+)}_{m,n}(\tau) + B^{(-)}_{n,m}(\tau)], \tag{14}$$

$$\eta_{m,n} = m + 2dn, \tag{15}$$

$$A^{(\pm)}_{m,n}(\tau) = \frac{|\alpha_\pm|^2\lambda}{\lambda^2 + \eta^2_{m,n}}[1 - \exp(-\lambda\tau)\cos(\eta_{m,n}\tau)], \tag{16}$$

$$B^{(\pm)}_{m,n}(\tau) = \frac{|\alpha_\pm|^2\lambda}{\lambda^2 + \eta^2_{m,n}}\exp(-\lambda\tau)\sin(\eta_{m,n}\tau). \tag{17}$$

In equations (11–17) τ is given by (6), and

$$\lambda = \gamma_+/\kappa = \gamma_-/\kappa \tag{18}$$

describes the relative (with respect to the nonlinear coupling κ) damping constant assumed to be the same for both modes of the field.

The solution (11) is exact and, despite the complexity of $\Gamma_{\delta_+,\delta_-}(\tau)$ and $\Lambda_{\delta_+,\delta_-}(\tau)$, its structure is quite transparent. If there is no absorption in the medium, $\lambda = 0$, both $\Gamma_{\delta_+,\delta_-}(\tau)$ and $\Lambda_{\delta_+,\delta_-}(\tau)$ are zero, and the density matrix factorizes into components of the field state (8). The solution (11) will be used to calculate the expectation values and variances of the Stokes operators.

3. Quantum fluctuations in the Stokes parameters

The changes of the polarization of initially elliptically polarized light when it propagates through a nonlinear Kerr medium can be easily accounted for with the use of the Stokes parameters. In quantum treatment of the two-mode field considered in this paper, the following Hermitian Stokes operators can be defined [13]

$$\left.\begin{aligned}S_0 &= a^\dagger_+ a_+ + a^\dagger_- a_-, \\ S_1 &= a^\dagger_+ a_- + a^\dagger_- a_+, \\ S_2 &= -i(a^\dagger_+ a_- - a^\dagger_- a_+), \\ S_3 &= a^\dagger_+ a_+ - a^\dagger_- a_-,\end{aligned}\right\} \tag{19}$$

where $a_\pm$ ($a^\dagger_\pm$) are the annihilation (creation) operators of the two circularly polarized modes. If the boson commutation relations are applied for $a_\pm$ and $a^\dagger_\pm$, it is easy to check that the Stokes operators themselves satisfy the commutation relations

$$\begin{aligned} &[S_1, S_2]=2\mathrm{i}S_3, \qquad \text{and cyclic interchange of indices,} \\ &[S_i, S_0]=0, \qquad i=1,2,3. \end{aligned} \tag{20}$$

Moreover, we have

$$S_1^2+S_2^2+S_3^2=S_0(S_0+2). \tag{21}$$

The quantum mechanical expectation values of the Stokes operators (19) are the Stokes parameters describing the polarization of the light beam, For elliptically polarized light the parameters of the polarization ellipse are given by [9]

$$\left.\begin{aligned} \tan 2\theta &= \langle S_2\rangle/\langle S_1\rangle, \\ \tan 2\eta &= \langle S_3\rangle/(\langle S_1\rangle^2+\langle S_2\rangle^2)^{1/2}, \end{aligned}\right\} \tag{22}$$

where θ is the azimuth of the polarization ellipse denoting the angle between the major axis of the polarization ellipse and the x axis, and η is the ellipticity parameter, $-\pi/4\leqslant\eta\leqslant\pi/4$; $\tan\eta$ describes the ratio of the semi-minor axis and the semi-major axis of the polarization ellipse and the sign defines its handedness (plus indicates right-handed polarization on the helicity convention).

The degree of polarization of the field can be defined as

$$P=\frac{(\langle S_1^2\rangle+\langle S_2^2\rangle+\langle S_3^2\rangle)^{1/2}}{\langle S_0\rangle}. \tag{23}$$

For the initial coherent state of elliptically polarized light the Stokes parameters have the values

$$\left.\begin{aligned} \langle\alpha_+,\alpha_-|S_0|\alpha_+,\alpha_-\rangle &= |\alpha_+|+|\alpha_-|^2=|\alpha|^2, \\ \langle\alpha_+,\alpha_-|S_1|\alpha_+,\alpha_-\rangle &= 2\mathrm{Re}\,(\alpha^*_+\alpha_-)=|\alpha|^2\cos 2\eta\cos 2\theta, \\ \langle\alpha_+,\alpha_-|S_2|\alpha_+,\alpha_-\rangle &= 2\mathrm{Im}\,(\alpha^*_+\alpha_-)=|\alpha|^2\cos 2\eta\sin 2\theta, \\ \langle\alpha_+,\alpha_-|S_3|\alpha_+,\alpha_-\rangle &= |\alpha_+|^2-|\alpha_-|^2=|\alpha|^2\sin 2\eta, \end{aligned}\right\} \tag{24}$$

where $|\alpha|^2$ is the total mean number of photons in the field while $\theta=(\phi_- - \phi_+)/2$ and η define the polarization ellipse. The degree of polarization P is in this case equal to unity. This means that the coherent state of the field corresponds to a classical, fully polarized field. However, the non-commutability of the Stokes operators puts well known limits on measurements of the physical quantities represented by these operators. For example, according to the commutation relations (20), we have the following Heisenberg uncertainty relation

$$[\langle(\Delta S_1)^2\rangle\langle(\Delta S_2)\rangle]^{1/2}\geqslant|\langle S_3\rangle|. \tag{25}$$

Quantum fluctuations in the Stokes parameters of light propagating in a Kerr medium without losses have been discussed by Tanaś and Kielich [9]. In this paper we wish to generalize those results including dissipation into the system. The exact solution (11) to the master equation enables us to derive exact analytical formulas for the expectation values and variances of the Stokes operators as well as the characteristics of the field polarization for the medium with dissipation.

The expectation values of the Stokes operators are given by

$$\begin{aligned}\langle S_0\rangle &= \sum_{n,m=0}^{\infty}(n+m)\rho_{n,m;n,m}(\tau)\\ &= \sum_{n,m=0}^{\infty}(n+m)b_n^{(+)2}b_m^{(-)2}\exp[-\lambda\tau(n+m)]\\ &\quad\times\exp\{(|\alpha_+|^2+|\alpha_-|^2)[1-\exp(\lambda\tau)]\}\\ &= |\alpha|^2\exp(-\lambda\tau),\end{aligned} \tag{26}$$

$$\begin{aligned}\langle S_1\rangle &= 2\mathrm{Re}\sum_{n,m=0}^{\infty}[(n+1)(m+1)]^{1/2}\rho_{n,m+1>n+1,m}(\tau)\\ &= 2|\alpha_+||\alpha_-|\exp\{-\lambda\tau+\Gamma_{-1,1}(\tau)\\ &\quad+(|\alpha_+|^2+|\alpha_-|^2)\{\exp(-\lambda\tau)\cos[(1-2d)\tau]-1\}\}\\ &\quad\times\cos\{\phi_+-\phi_-+(|\alpha_+|^2-|\alpha_-|^2)\exp(-\lambda\tau)\sin[(1-2d)\tau]-\Lambda_{-1,1}(\tau)\}\\ &= |\alpha|^2\cos 2\eta\exp\{-\lambda\tau+\Gamma_{-1,1}(\tau)+|\alpha|^2\{\exp(-\lambda\tau)\cos[(1-2d)\tau]-1\}\}\\ &\quad\times\cos\{2\theta-|\alpha|^2\sin 2\eta\exp(-\lambda\tau)\sin[(1-2d)\tau]+\Lambda_{-1,1}(\tau)\},\end{aligned} \tag{27}$$

$$\begin{aligned}\langle S_2\rangle &= 2\mathrm{Im}\sum_{n,m=0}^{\infty}[(n+1)(m+1)]^{1/2}\rho_{n,m+1;n+1,m}(\tau)\\ &= |\alpha|^2\cos 2\eta\exp\{-\lambda\tau+\Gamma_{-1,1}(\tau)+|\alpha|^2\{\exp(-\lambda\tau)\cos[(1-2d)\tau]-1\}\}\\ &\quad\times\sin\{2\theta-|\alpha|^2\sin 2\eta\exp(-\lambda\tau)\sin[(1-2d)\tau]+\Lambda_{-1,1}(\tau)\},\end{aligned} \tag{28}$$

$$\langle S_3\rangle = \sum_{n,m=0}^{\infty}(n-m)\rho_{n,m;n,m}(\tau)=|\alpha|^2\sin 2\eta\exp(-\lambda\tau), \tag{29}$$

where $|\alpha|^2$, $2\theta=\phi_--\phi_+$, and η are the parameters defining initial state of the field; λ describes the dissipation of the medium, and $\Gamma_{-1,1}(\tau)$ and $\Lambda_{-1,1}(\tau)$ are given by (13) and (14) or (A 3) and (A 4). It is to be kept in mind that the latter two quantities depend on τ, and that they vanish for $\lambda=0$. For $\lambda=0$ the results (26–29) go over into the earlier results of Tanaś and Kielich [9].

According to (22) and (23), from (26–29) we obtain

$$\tan 2\theta(\tau)=\tan\{2\theta-|\alpha|^2\sin 2\eta\exp(-\lambda\tau)\sin[(1-2d)\tau]+\Lambda_{-1,1}(\tau)\}, \tag{30}$$

$$\tan 2\eta(\tau)=\exp\{-|\alpha|^2\exp(-\lambda\tau)\cos[(1-2d)\tau]-1\}\tan 2\eta, \tag{31}$$

$$P^2(\tau)=\sin^2 2\eta+\cos^2 2\eta\exp\{2|\alpha|^2\{\exp(-\lambda\tau)\cos[(1-2d)\tau]-1\}+2\Gamma_{-1,1}(\tau)\}. \tag{32}$$

Formulas (30–32) are exact quantum expressions describing the evolution of the polarization state of elliptically polarized field propagating in a Kerr medium with dissipation. It is evident from (30–32) (see also the Appendix) that all polarization parameters depend on $1-2d$, that is, they crucially depend on the asymmetry parameter d of the nonlinear medium. For $d=1/2$, i.e. for $1/2\leftrightarrow 1/2$ transitions, the polarization state of the field does not change. For the fully symmetric susceptibility tensor $\chi(\omega)$, $d=1$, and there are changes in the polarization of the field. From (30) and (A 2) we obtain the following expression describing the rotation of the polarization ellipse

$$\begin{aligned}\theta(\tau)&=\theta-\tfrac{1}{2}\sin 2\eta\,\frac{1-2d}{\lambda^2+(1-2d)^2}\,\{(1-2d)|\alpha|^2\exp(-\lambda\tau)\sin[(1-2d)\tau]\\&\quad+\lambda[|\alpha|^2-|\alpha|^2\exp(-\lambda\tau)\cos[(1-2d)\tau]]\}\\&=\theta-\tfrac{1}{2}\sin 2\eta\,\frac{1-2d}{\lambda^2+(1-2d)^2}\,\{(1-2d)S_0(\tau)\sin[(1-2d)\tau]\\&\quad+\lambda\{S_0(0)-S_0(\tau)\cos[(1-2d)\tau]\}\},\end{aligned}\tag{33}$$

where

$$S_0(\tau)=\langle S_0\rangle=|\alpha|^2\exp(-\lambda\tau),\tag{34}$$

according to (26).

Formula (33) is the exact quantum formula, which evolves in two different 'time-scales' (in fact length-scales): one associated with the nonlinearity of the medium (τ), and another one related to the dissipation ($\lambda\tau$). For $\lambda=0$, formula (33) reproduces the earlier obtained result [9], which is still the quantum result and is periodic in $(1-2d)\tau$. Only for $\tau\ll 1$, when $\sin[(1-2d)\tau]\approx(1-2d)\tau$, the classical result is obtained. Another limit is $\lambda\gg 1$, in which only the term proportional to λ in the large braces contributes if the terms of order λ^{-1} are retained. Again, the transition to the classical field is obtained by taking $\tau\ll 1$, i.e., by replacing $\cos[(1-2d)\tau]$ with unity. This gives us the quantity $[S_0(0)-S_0(\tau)]/\lambda$ that appears in the solution of the problem for the classical field and the medium with dissipation. It is interesting to note that, despite the fact that this classical result has been obtained from the quantum formula under assumption $\lambda\gg 1$, putting $\lambda=0$ in the resulting formula leads to the same classical result which is obtained from the first term in the large braces of equation (33) for $\lambda=0$ and $\tau\ll 1$. Thus, the classical result without dissipation can be obtained from equation (33) in two different ways. For $\lambda\approx 1$, the effects of nonlinearity and dissipation are equally important, and in this case formula (33) must be used in its full form. In figure 1 the evolution of $\theta(\tau)$ is shown for different values of $\lambda, \eta=\pi/8, d=1$, and $|\alpha|^2=0{\cdot}25, 1, 4, 16$.

The ellipticity $\eta(\tau)$ given by equation (31) exhibits the τ dependence that is a purely quantum effect (we have assumed the same absorption for both circular

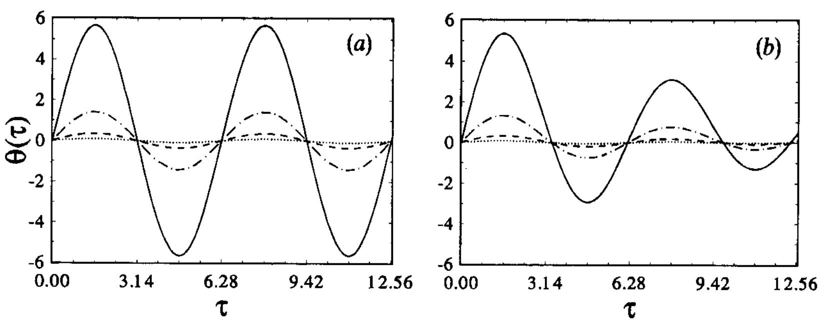

Figure 1. Evolution of the azimuth $\theta(\tau)$ for $\eta=\pi/8$. The other parameters are taken the same for all the figures, and they are: $d=1$, $|\alpha|^2=0{\cdot}25$ (dotted line), 1 (dashed line), 4 (dashed-dotted line), 16 (solid line). Figure (a) is for $\lambda=0$, and (b) for $\lambda=0{\cdot}1$.

components of the field). The exponential does not appear for classical fields. For $\lambda=0$, equation (31) goes over into the earlier result [9]. The evolution of $\eta(\tau)$ is drawn in figure 2 for different values of $\lambda, \eta=\pi/8, d=1$, and $|\alpha|^2=0{\cdot}25, 1, 4, 16$. As is seen from figure 2, the ellipticity $\eta(\tau)$ approaches $\pi/4$ when $|\alpha|^2 \gg 1$. This means the circular polarization of the field. Thus the quantum fluctuations of the field cause that the field which remains polarized during the propagation can only be circularly polarized. For $\lambda=0$ there is a quantum periodicity of the evolution which is removed by the dissipation. For $\eta=0$, according to (31), there is no change in the ellipticity of the field which suggests that initially linear polarization remains linear during the evolution. This is true when speaking about the polarized part of the field. However, one can easily check using equation (32), that in this case the degree of polarization rapidly drops to zero, and there is practically no polarized part of the field. The degree of polarization $P(\tau)$ is plotted in figure 3 for $\eta=0, d=1$, and different values of $|\alpha|^2$ and λ. The reduction of the degree of polarization is quite evident, and the dissipation prevents the recurrence of the initial degree of polarization. From equation (32) it is seen that there is a lower bound for the degree of polarization equal to $|\sin 2\eta|$. This means, for example, that for $\eta=\pi/8$ taken in figures 1 and 2 the degree of polarization $P(\tau)$ cannot fall below the value $1/\sqrt{2}$, i.e., the field retains quite a bit of its polarization. For the circular polarization the degree of polarization does not change at all. So, the effect of quantum field fluctuations is most dramatic when linearly polarized light propagates through an isotropic nonlinear medium. The changes in the degree of polarization for media without dissipation were discussed earlier [9, 10, 14]. We have also discussed the relation of the Stokes parameters with the phase properties of the field [15].

To study the quantum fluctuations in the Stokes parameters, we calculate the variances of the Stokes operators and look at their behaviour during the evolution. For the expectation values of the squares of the Stokes operator we get the relations

$$
\begin{aligned}
\langle S_{1,2}^2\rangle = {} & \pm 2\,\mathrm{Re}\sum_{n,m}[(n+1)(n+2)(m+1)(m+2)]^{1/2}\rho_{n,m+2;n+2,m}(\tau) \\
& + \sum_{n,m}(n+m+2nm)\rho_{n,m;n,m} \\
= {} & \pm\tfrac{1}{2}|\alpha|^4\cos^2 2\eta\exp\{|\alpha|^2[\exp(-\lambda\tau)\cos[2(1-2d)\tau]-1] \\
& -2\lambda\tau+\Gamma_{-2,2}(\tau)\}\cos\{4\theta-|\alpha|^2\sin 2\eta\exp(-\lambda\tau) \\
& \times\sin[2(1-2d)\tau]+\Lambda_{-2,2}(\tau)\} \\
& +\tfrac{1}{2}|\alpha|^4\cos^2 2\eta\exp(-2\lambda\tau)+|\alpha|^2\exp(-\lambda\tau),
\end{aligned}
\tag{35}
$$

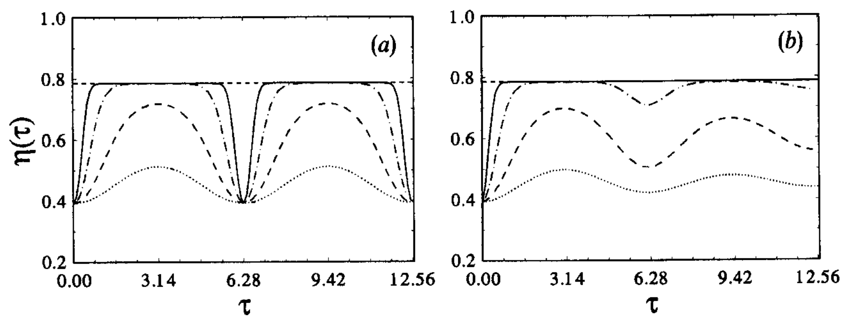

Figure 2. Evolution of the ellipticity $\eta(\tau)$, for $\eta=\pi/8$.

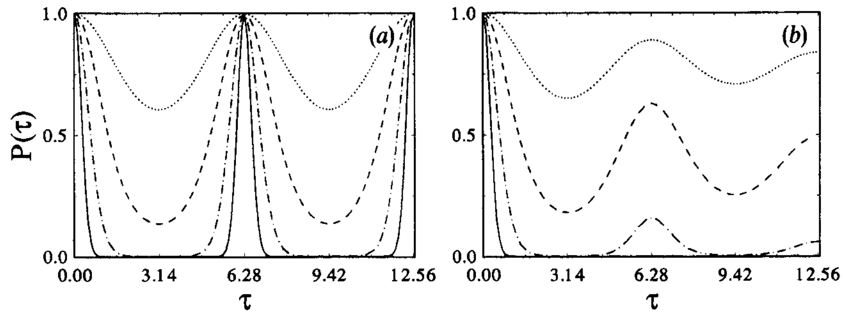

Figure 3. Evolution of the degree of polarization $P(\tau)$, for $\eta=0$.

$$\langle S_3^2\rangle=\sum_{n,m}(n+m)^2\rho_{n,m;n,m}(\tau)$$
$$=|\alpha|^4\sin^2 2\eta\exp(-2\lambda\tau)+|\alpha|^2\exp(-\lambda\tau). \quad (36)$$

Formulas (35) and (36) together with (26–29) and (A 1–A 6) allow calculations of the variances of the Stokes parameters. The variances are intensity dependent, i.e., the quantum noise related with the measurement of the Stokes parameters is also intensity dependent. However, it is rather the relative noise, or the signal-to-noise ratio, that are interesting from the experimental point of view. One can ask the question: How will the quantum fluctuations of the field affect the signal-to-noise ratio? Our exact analytical formulas will immediately give the answer to this question. Let us define the signal-to-noise ratio for the measurements of the Stokes parameters as

$$R_i(\tau)=\frac{|\langle S_i\rangle|}{(\langle S_i^2\rangle-\langle S_i\rangle^2)^{1/2}}, \qquad (i=1,2,3). \quad (37)$$

The evolution of the signal-to-noise ratio (37) is plotted in figures 4–7 for various sets of the parameters. In figure 4 there are plots of R_1 (τ) for $\eta=0, d=1, |\alpha|^2=0{\cdot}25, 1, 4, 16$, and (*a*) $\lambda=0$, (*b*) $\lambda=0{\cdot}1$. For $\lambda=0$, i.e. the absence of damping in the system, the evolution of the signal-to-noise ratio is periodic. The initial value of the ratio is $|\alpha|$, as it should be for the coherent state $|\alpha\rangle$. However, owing to the quantum fluctuations of the field the ratio rapidly falls down if the field is strong. That is, even without damping in the system, the signal-to-noise ratio deteriorates drastically making the measurement of $\langle S_1\rangle$ less certain. The presence of damping deteriorates the signal to-noise ratio still further, as is seen from figure 4 (*b*), and it removes the periodicity of the evolution. For the linear polarization of the initial field, $\eta=0$ and $\sin\eta=0$, and both $\langle S_2\rangle$ and $\langle S_3\rangle$ are zero all the time. That is, the only signal we can measure in the case of linear polarization is $\langle S_1\rangle$.

For elliptical polarization with $\eta=\pi/8$, we have the situation illustrated in figures 5–7, where $R_1(\tau)$, $R_2(\tau)$, and $R_3(\tau)$ are plotted for $\eta=\pi/8$, all other parameters being the same as in figure 4. The signal-to-noise ratio $R_1(\tau)$ is lower initially than for linear polarization, it is now $|\alpha||\cos 2\eta|$ instead of $|\alpha|$, there are some oscillations at the initial stages of the evolution if the field is strong, but deterioration of the ratio owing to quantum fluctuations and dissipation is evident. The ratio $R_2(\tau)$, shown in figure 6, grows up initially indicating the conjugate character of S_2 with respect to S_1, as one could expect from the commutation relation (20) and the uncertainty relation (25). However, at later times quantum fluctuations of the field deteriorate this ratio as well. Of course, the dissipation still worsens the situation.

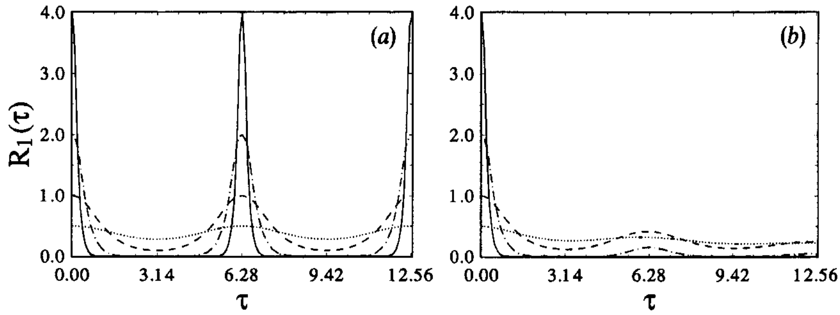

Figure 4. Evolution of the signal-to-noise ratio $R_1(\tau)$, for $\eta=0$.

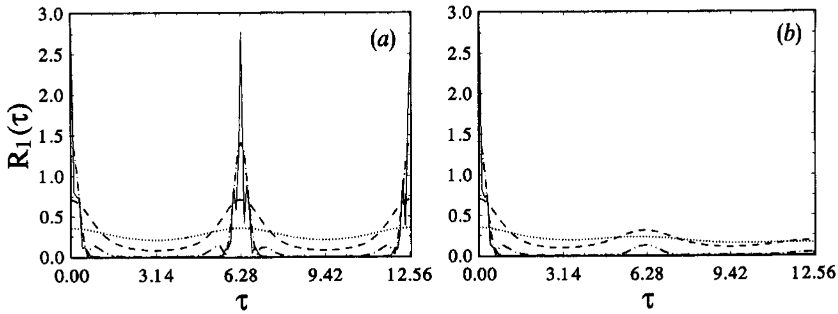

Figure 5. Same as figure 4, but for $\eta=\pi/8$.

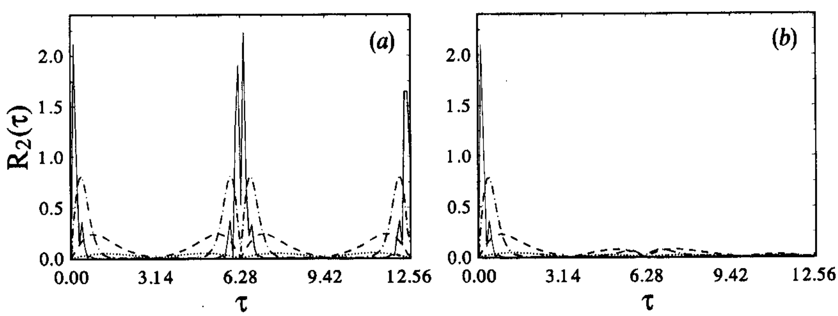

Figure 6. Evolution of $R_2(\tau)$, for $\eta=\pi/8$.

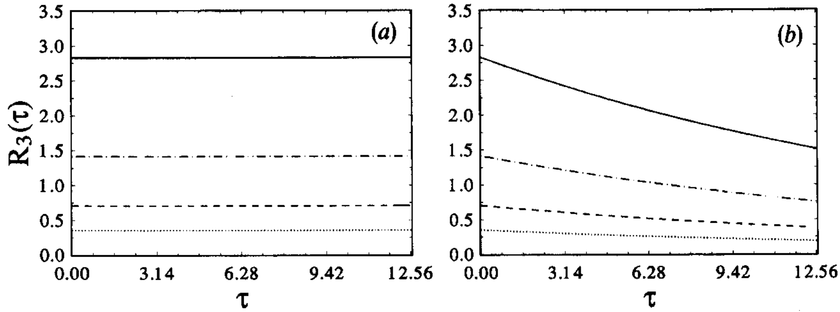

Figure 7. Evolution of $R_3(\tau)$, for $\eta=\pi/8$.

The behaviour of $R_3(\tau)$ is different: if there is no damping $R_3(\tau)$ is constant and equal to $|\alpha|\,|\sin 2\eta|$, and it is damped when $\lambda \neq 0$. This is shown in figure 7. So, the quantum fluctuations of the field do not deteriorate the precision of measurement of the Stokes parameter $\langle S_3 \rangle$. This can be explained by the fact that this Stokes parameter describes the circularly polarized part of the field which is not affected during the propagation in the isotropic nonlinear medium. The only evolution is the result of the linear damping of the field. The quantum character of the field propagating in an isotropic Kerr medium manifests itself most strongly when the field is linearly polarized.

4. Conclusions

We have considered the quantum field of elliptical polarization propagating in an isotropic, nonlinear Kerr medium with dissipation finding the exact analytical formulas describing the evolution of the Stokes parameters, which define the polarization of the field, and the evolution of their variances, which describe quantum fluctuations in the Stokes parameters. We applied the exact solution of the master equation obtained recently by Chaturvedi and Srinivasan [11] to find the evolution of the Stokes parameters and their variances. The quantum character of the field essentially affects the polarization of light propagating in the medium. In the case of linear polarization there is a rapid decrease of the degree of polarization. For the elliptical polarization there is a lower bound for the degree of polarization equal to $|\sin 2\eta|$, which means that the circular polarization remains unchanged during the evolution. The results for the azimuth of the polarization ellipse, its ellipticity, and the degree of polarization are illustrated graphically for various sets of parameters. There are quantum effects that can be found in the evolution of these quantitites despite the fact that they are defined by the expectation values of the Stokes operators that are linear in intensity.

The quantum noise of the Stokes parameters of light propagating in a Kerr medium is discussed in detail. The evolution of the signal-to-noise ratio for the measurements of the Stokes parameters is studied. It is shown that quantum fluctuations of the strong field drastically deteriorate the signal-to-noise ratio for the measurements of the Stokes parameters $\langle S_1 \rangle$ and $\langle S_2 \rangle$ related with the linear polarization of the field. The parameter $\langle S_3 \rangle$, related to the circular polarization of the field, is not affected by the quantum fluctuations. As one could expect, the dissipation in the system lowers the signal-to-noise ratio in any case. Our exact formulas that include linear damping allow for the quantitative assessment of its destructive role in the quantum evolution of the field.

Although the damping is linear, the final results are intensity dependent, and the influence of dissipation on pure quantum effects is more evident when the coherent excitation is large. This is a result of mixing of two physical mechanisms that lead to the final results: (i) the nonlinear interaction leading (through κ) to the intensity dependent pure quantum effects and (ii) the linear damping associated with λ that damps these quantum effects. If the coherent excitation is small, the quantum effects are small, and there is 'nothing' to be damped. In this case the effect of damping is less evident. Thus, our results give an interesting example how the nonlinear and linear processes can be mixed into a nonlinear final result.

We should also emphasize that the results obtained in this paper are based on the assumption that the measured quantities are the expectation values of the Stokes operators (which are Hermitian operators, i.e. observables) and their variances. The expectation values of the Stokes operators, i.e. the Stokes parameters can be actually measured by measuring the intensity of light that passed through a combination of the optical elements such as polarizers and quarter-wavelength plates. In this context, we have to keep in mind that our results for the azimuth $\theta(\tau)$ and the ellipticity $\eta(\tau)$ are in fact calculated as the inverse trigonometric functions of the expectation values of the Stokes operators, and not as the expectation values of the Hermitian operators representing these observables. Since, generally, the expectation value of a function of operators is different from the function of the expectation values of the operators, the results obtained from the measurements of the Stokes parameters can differ from the results obtained, for example, from the measurements of the expectation values of the Hermitian phase operators, as we have shown elsewhere [15]. So, the results of this paper must be associated with the measurements of the Stokes parameters which, in fact, can be measured in practice.

Appendix

For convenience, in this Appendix we write down the explicit expressions for $\Gamma_{m,n}(\tau)$ and $\Lambda_{m,n}(\tau)$ needed for our calculations in this paper. Let us define the quantities

$$A_k(\tau)=\lambda\frac{|\alpha|^2\lambda}{\lambda^2+[k(1-2d)]^2}\{1-\exp(-\lambda\tau)\cos[k(1-2d)]\}, \tag{A 1}$$

$$B_k(\tau)=\frac{|\alpha|^2\lambda}{\lambda^2+[k(1-2d)]^2}\exp(1-\lambda\tau)\sin[k(1-2d)]. \tag{A 2}$$

Using (A1) and (A2), from the general expressions (13–17), we can derive the following relations:

$$\Gamma_{-1,1}(\tau)=\lambda A_1(\tau)+(1-2d)B_1(\tau), \tag{A3}$$

$$\Lambda_{-1,1}(\tau)=\sin 2\eta\,[-(1-2d)A_1(\tau)+\lambda B_1(\tau)], \tag{A4}$$

$$\Gamma_{-2,2}(\tau)=\lambda A_2(\tau)+2(1-2d)B_2(\tau), \tag{A5}$$

$$\Lambda_{-2,2}(\tau)=\sin 2\eta\,[-2(1-2d)A_2(\tau)+\lambda B_2(\tau)]. \tag{A6}$$

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