

Phase properties of a damped anharmonic oscillator

Ts. Gantsog

Department of Theoretical Physics, Mongolian State University, Ulan Bator 210646, Mongolia

R. Tanaś

Nonlinear Optics Division, Institute of Physics, Adam Mickiewicz University, 60-780 Poznań, Poland

(Received 21 February 1991)

Phase properties of a damped anharmonic oscillator are studied within the Hermitian phase formalism of Pegg and Barnett. Exact analytical formulas for the phase distribution function, the expectation value and variance of the phase operator, and the expectation value and variance of the cosine of the phase operator are obtained assuming the zero-temperature reservoir and a coherent initial state of the system. It is shown that quantum periodicity in the evolution of phase quantities is destroyed by damping. The effect of damping on the formation of discrete superpositions of coherent states is discussed. A comparison is made between different phase approaches.

I. INTRODUCTION

A simple exactly solvable model of an anharmonic oscillator is a very instructive example in which many aspects of the nonlinear dynamics have been studied.^{1–20} Tanaś¹ has shown that a high degree of squeezing can be obtained in the system. Milburn² has discussed the evolution of the quasiprobability distribution function $Q(\alpha, \alpha^*, t)$ for the anharmonic oscillator showing periodic recurrences of its initial form due to quantum dynamics. Milburn and Holmes³ have shown that dissipation in the model rapidly destroys the quantum recurrence effects. Kitagawa and Yamamoto,⁴ who also considered the quasiprobability distribution $Q(\alpha, \alpha^*, t)$, have shown that contours of this distribution have “crescent” shape, and squeezing obtained in the model differs from the ordinary “elliptic” squeezing. Squeezing and its graphical representation in the anharmonic oscillator model have recently been discussed by Tanaś, Miranowicz, and Kielich.²⁰

Yurke and Stoler⁵ have shown that the states produced in the anharmonic-oscillator model become a superposition of a finite number of coherent states under a proper choice of the evolution time. Tombesi and Mecozzi⁶ have obtained the superposition states for the two-mode case and arbitrary initial state of the field. Miranowicz, Tanaś, and Kielich¹⁹ have recently shown that superpositions with not only even but also odd numbers of components can be obtained, and that the maximum number of well-distinguishable states is proportional to the field amplitude.

The anharmonic-oscillator model admits exact analytical solution even in the presence of dissipation that was first shown by Milburn and Holmes³ for the Q function with initially coherent-state distribution. This approach has been generalized by Peřinova and Lukš,^{10,17} Daniel and Milburn,¹³ and Milburn, Mecozzi, and Tombesi.¹⁶

Recently, Gerry,²¹ and Gantsog and Tanaś^{22,23} have discussed phase properties of the states produced in the

anharmonic-oscillator model from the point of view of a new Hermitian phase formalism introduced by Pegg and Barnett.^{24–26} A comparison between the Q function and the phase distribution function $P(\theta)$ in the description of the superpositions of coherent states obtained in the system has been recently given by Tanaś *et al.*²⁷ The phase properties discussed so far were obtained for the model without losses.

In this paper we study phase properties of the anharmonic-oscillator states for the system with dissipation. The Pegg-Barnett^{24–26} phase formalism is employed to calculate the phase probability distribution, the mean value and variance of the phase operator, and the mean values and variances of the sine and cosine functions of the phase operators. The exact analytical formulas describing phase properties of the system with damping are obtained, and the evolution of the phase expectation values for different values of the damping constant is illustrated graphically. It is shown that, like in the Q -function case,^{3,13} the quantum periodicity of the phase quantities is destroyed by the damping. The process of formation of discrete superpositions of coherent states in the system with damping is illustrated by considering the shape of the phase distribution function in polar coordinates. The evolution of the cosine function of the phase operator and its variance is calculated, and a comparison is made between the results obtained within different phase formalisms.

II. QUANTUM DYNAMICS

The system can be described by the Hamiltonian

$$H = H_S + H_I + H_R, \quad (1)$$

where

$$H_S = \hbar\omega a^\dagger a + \hbar\frac{\kappa}{2}a^\dagger{}^2a^2 \quad (2)$$

is the anharmonic-oscillator Hamiltonian,

$$H_I = \sum_i (g_i a b_i^\dagger + g_i^* a^\dagger b_i) \quad (3)$$

represents the coupling to a reservoir of oscillators, and H_R is the free Hamiltonian for the reservoir. In Eqs. (2) and (3) a is the annihilation operator for a cavity mode at frequency ω , κ is the nonlinear coupling (anharmonicity), proportional to the third-order nonlinear susceptibility, if the model is used to describe the interaction of the field with a nonlinear Kerr medium. Damping of the system is modeled by its coupling to a reservoir of oscillators. By use of standard techniques of the quantum theory of damping,²⁸ the following master equation is obtained in the Markov approximation and in the interaction picture

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & -i \frac{\kappa}{2} [a^\dagger a^2, \rho] + \frac{\gamma}{2} ([a \rho, a^\dagger] + [a, \rho a^\dagger]) \\ & + \gamma \bar{n} [a, \rho], a^\dagger, \end{aligned} \quad (4)$$

where γ is the damping constant, \bar{n} is the mean number of thermal photons, $\bar{n} = [\exp(\hbar\omega/kT) - 1]^{-1}$, for the temperature T of the reservoir.

The exact solution to the master equation (4) is possible for both the “quiet”^{3,10} ($\bar{n}=0$) and “noisy”^{13,17} ($\bar{n}\neq 0$) reservoirs, and was applied for studying the evolution of the Q function and squeezing. We shall apply it to study phase properties of the field. In the general case the solution to the master equation (4) can be written in terms of the matrix elements of the density operator ρ as^{13,17}

$$\begin{aligned} \rho_{nm}(t) = & \exp \left[\left(\frac{\gamma}{2} + i\kappa(n-m) \right) t \right] E_{n-m}^{n+m+1}(t) \sum_{l=0}^{\infty} F \left[-n, -m, l+1; \frac{4\bar{n}(\bar{n}+1)}{\Delta^2} \sinh^2 \left(\frac{\gamma \Delta t}{2} \right) \right] \\ & \times \frac{1}{l!} \left[\frac{(\bar{n}+1)}{\bar{n}} g_{n-m}(t) \right]^l \left[\frac{(n+l)!(m+l)!}{n!m!} \right]^{1/2} \rho_{n+l, m+l}(0), \end{aligned} \quad (5)$$

where $F(-n, -m, l+1; x)$ is the hypergeometric function, and

$$g_{n-m}(t) = \frac{2\bar{n}}{\Omega + \Delta \coth(\gamma \Delta t / 2)}; \quad (6)$$

$$E_{n-m}(t) = \frac{\Delta}{\Omega \sinh(\gamma \Delta t / 2) + \Delta \cosh(\gamma \Delta t / 2)}; \quad (7)$$

$$\Omega \equiv \Omega_{n-m} = 1 + 2\bar{n} + i \frac{\kappa}{\gamma} (n-m), \quad (8)$$

$$\Delta \equiv \Delta_{n-m} = [\Omega^2 - 4\bar{n}(\bar{n}+1)]^{1/2}.$$

For the reservoir at zero temperature ($T=0$), we have $\bar{n}=0$, and the solution (5) simplifies considerably. In this case one obtains¹⁰

$$\begin{aligned} \rho_{nm}(\tau) = & \exp \left[i \frac{\tau}{2} (n-m) \right] f_{n-m}^{(n+m)/2}(\tau) \\ & \times \sum_{l=0}^{\infty} \frac{1}{l!} \left[\frac{\lambda [1 - f_{n-m}(\tau)]}{\lambda + i(n-m)} \right]^l \\ & \times \left[\frac{(n+l)!(m+l)!}{n!m!} \right]^{1/2} \rho_{n+l, m+l}(0), \end{aligned} \quad (9)$$

where we have introduced the notation

$$\tau = \kappa t, \quad \lambda = \gamma / \kappa, \quad (10)$$

$$f_{n-m}(\tau) = \exp \{ -[\lambda + i(n-m)]\tau \}.$$

For $\lambda=0$ the solution (9) becomes

$$\rho_{nm}(\tau) = \exp \left[-i \frac{\tau}{2} [n(n-1) - m(m-1)] \right] \rho_{nm}(0), \quad (11)$$

and describes the dynamics of the lossless anharmonic oscillator. From Eq. (11) it is clear that the diagonal matrix elements of the field density matrix do not change if there is no damping. This means that also the photon statistics remains the same as it was at the beginning.¹ The nondiagonal elements are related to the nonlinear change of the field phase, and this change is responsible for squeezing in the system.¹⁻⁴

If the initial state of the field is a coherent state $|\alpha_0\rangle$ then we have

$$\rho_{n+l, m+l}(0) = b_{n+l} b_{m+l} \exp[i(n-m)\varphi_0], \quad (12)$$

where

$$b_n = \exp(-N/2) \frac{N^{n/2}}{\sqrt{n!}}, \quad (13)$$

and we have assumed

$$\alpha_0 = |\alpha_0| \exp(i\varphi_0) = N^{1/2} \exp(i\varphi_0). \quad (14)$$

With such assumptions formula (9) can be further simplified, and we get

$$\begin{aligned} \rho_{nm}(\tau) = & b_n b_m \exp \left[i(n-m) \left(\varphi_0 + \frac{\tau}{2} \right) \right] f_{n-m}^{(n+m)/2}(\tau) \\ & \times \exp \left[N \lambda \frac{1 - f_{n-m}(\tau)}{\lambda + i(n-m)} \right], \end{aligned} \quad (15)$$

where $f_{n-m}(\tau)$ is given by Eq. (10). Having the solutions for $\rho_{nm}(\tau)$ in hand, we are able to calculate any field characteristics at time $\tau = \kappa t$. In this paper we are interested in quantum phase properties of the field.

III. PHASE PROPERTIES

To study phase properties of the damped anharmonic oscillator we use the new Hermitian phase formalism introduced by Pegg and Barnett.²⁴⁻²⁶ Their formalism is based on introducing a finite $(s+1)$ -dimensional space Ψ spanned by the number states $|0\rangle, |1\rangle, \dots, |s\rangle$. The Hermitian phase operator operates on this finite space, and after all necessary expectation values have been calculated in Ψ , the value of s is allowed to tend to infinity. A complete orthonormal basis of $s+1$ states is defined on Ψ as

$$|\theta_m\rangle \equiv \frac{1}{\sqrt{s+1}} \sum_{n=0}^s \exp(in\theta_m) |n\rangle, \quad (16)$$

where

$$\theta_m \equiv \theta_0 + \frac{2\pi m}{s+1}, \quad m=0, 1, \dots, s. \quad (17)$$

The value of θ_0 is arbitrary and defines a particular basis set of $s+1$ mutually orthogonal phase states. The Hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^s \theta_m |\theta_m\rangle \langle \theta_m|, \quad (18)$$

where the subscript θ indicates the dependence on the choice of θ_0 . The phase states (16) are eigenstates of the phase operator (18) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and $\theta_0 + 2\pi$. The unitary phase operator $\exp(i\hat{\phi}_\theta)$ is defined as the exponential function of the Hermitian operator $\hat{\phi}_\theta$. This operator acting on the eigenstate $|\theta_m\rangle$ gives the eigenvalue $\exp(i\theta_m)$, and can be written as²⁴⁻²⁶

$$\exp(i\hat{\phi}_\theta) \equiv \sum_{n=0}^{s-1} |n\rangle \langle n+1| + \exp[i(s+1)\theta_0] |s\rangle \langle 0|. \quad (19)$$

This is the last term in (19) that ensures the unitarity of this operator. The first sum reproduces the Susskind-Glogower^{29,30} phase operator in the limit $s \rightarrow \infty$.

The expectation value of the phase operator (18) in a pure state $|\psi\rangle$ is given by

$$\langle \psi | \hat{\phi}_\theta | \psi \rangle = \sum_{m=0}^s \theta_m |\langle \theta_m | \psi \rangle|^2, \quad (20)$$

where $|\langle \theta_m | \psi \rangle|^2$ gives a probability of being found in the phase state $|\theta_m\rangle$. The density of phase states is $(s+1)/2\pi$, so in the continuum limit as s tends to infinity, we can write Eq. (20) as

$$\langle \psi | \hat{\phi}_\theta | \psi \rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta P(\theta) d\theta, \quad (21)$$

where the continuum phase distribution $P(\theta)$ is introduced by

$$P(\theta) = \lim_{s \rightarrow \infty} \frac{s+1}{2\pi} |\langle \theta_m | \psi \rangle|^2, \quad (22)$$

where θ_m has been replaced by the continuous phase variable θ . As the phase distribution function $P(\theta)$ is known, all the quantum-mechanical phase expectation

values can be calculated with this function in a classical-like manner. The choice of the value of θ_0 defines the 2π range window of the phase values.

If the field is described by the density operator ρ , Eq. (20) reads

$$\langle \hat{\phi}_\theta \rangle = \text{Tr}(\rho \hat{\phi}_\theta) = \sum_{m=0}^s \theta_m \langle \theta_m | \rho | \theta_m \rangle, \quad (23)$$

and the phase distribution function $P(\theta)$ is given, instead of Eq. (22), by

$$P(\theta) = \lim_{s \rightarrow \infty} \frac{s+1}{2\pi} \langle \theta_m | \rho | \theta_m \rangle. \quad (24)$$

After taking into account Eq. (16), we can write Eq. (24) as

$$\begin{aligned} P(\theta) &= \lim_{s \rightarrow \infty} \frac{s+1}{2\pi} \langle \theta_m | \rho | \theta_m \rangle \\ &= \lim_{s \rightarrow \infty} \frac{1}{2\pi} \sum_{n=0}^s \sum_{k=0}^s e^{-i(n-k)\theta_m} \rho_{nk}(t). \end{aligned} \quad (25)$$

If we symmetrize the phase distribution with respect to the phase φ_0 by taking

$$\theta_0 = \varphi_0 - \frac{\pi s}{s+1} \quad (26)$$

and introducing a new phase label

$$\mu = m - \frac{s}{2}, \quad (27)$$

which runs in integer steps from $-s/2$ to $s/2$, the phase distribution becomes symmetric in μ , and we get

$$\begin{aligned} P(\theta) &= \lim_{s \rightarrow \infty} \frac{1}{2\pi} \sum_{n=0}^s \sum_{m=0}^s e^{-i(n-m)(\varphi_0 + \theta_\mu)} \rho_{nm}(t) \\ &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-i(n-m)(\varphi_0 + \theta)} \rho_{nm}(t). \end{aligned} \quad (28)$$

Now, all integrals over θ are taken in the symmetric range between $-\pi$ and π . The phase distribution $P(\theta)$ is normalized such that

$$\int_{-\pi}^{\pi} P(\theta) d\theta = 1. \quad (29)$$

To find the phase distribution for the damped anharmonic oscillator considered in this paper it suffices to insert into Eq. (28) the corresponding solution for $\rho_{nm}(t)$, which is given by Eq. (5) in the most general case of noisy reservoir, by Eq. (9) in the case of quiet reservoir, and by Eq. (15) in the case of quiet reservoir and initially coherent state of the field. Here, we shall discuss the simplest and probably most important case of zero temperature reservoir and coherent state of the initial field. On inserting Eq. (15) into Eq. (28), we obtain for the phase distribution $P(\theta)$ the following formula:

$$P(\theta) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_n b_m \exp \left[-i(n-m) \left(\theta - \frac{\tau}{2} \right) \right] \\ \times f_{n-m}^{(n+m)/2}(\tau) \\ \times \exp \left[N\lambda \frac{1-f_{n-m}(\tau)}{\lambda + i(n-m)} \right], \quad (30)$$

$$P(\theta) = \frac{1}{2\pi} \left[1 + 2 \sum_{\substack{n,m \\ n>m}} b_n b_m \exp \left[-\frac{\lambda\tau}{2}(n+m) + \lambda A_{n-m}(\tau) + (n-m)B_{n-m}(\tau) \right] \right] \\ \times \cos \left[(n-m)\theta + \frac{\tau}{2}[n(n-1) - m(m-1)] + (n-m)A_{n-m}(\tau) - \lambda B_{n-m}(\tau) \right], \quad (31)$$

where we have introduced the notation

$$A_{n-m}(\tau) = \frac{N\lambda}{\lambda^2 + (n-m)^2} \{ 1 - e^{-\lambda\tau} \cos[(n-m)\tau] \}, \quad (32)$$

$$B_{n-m}(\tau) = \frac{N\lambda}{\lambda^2 + (n-m)^2} e^{-\lambda\tau} \sin[(n-m)\tau]. \quad (33)$$

If there is no damping in the system ($\lambda=0$), $A_{n-m}(\tau)$ and $B_{n-m}(\tau)$ are zero, and formula (31) goes over into our earlier result.²³ It is clear that damping will cause the decay of the modulation terms and will lead to the uniform phase distribution. The quantum periodicity present in the lossless system will be, like in the Q -function case,^{3,13} destroyed. To illustrate the effect of damping on the phase distribution, we have plotted in Fig. 1 the distribution $P(\theta)$ for various values of τ and λ , and $N=4$. It is seen that as the evolution proceeds, i.e., τ increases, the peak of the phase distribution is shifted, and even for $\lambda=0$ it is broadened. So, even without damping, the nonlinear evolution of the system leads at its initial stage to the randomization of the phase. However, if there is no damping, the evolution is periodic, and after $\tau=2\pi$ the phase distribution acquires its initial form. The effect of damping is to speed up the randomization of the phase, i.e., to broaden the phase distribution, and in effect to destroy the quantum periodicity of the evolution. It is clear from formula (31) that $P(\theta)=1/(2\pi)$ when all the interference terms have been gradually eliminated.

It is known^{5,6} that under appropriate choice of τ the anharmonic-oscillator states evolve into a superposition of coherent states. Miranowicz, Tanaš, and Kielich¹⁹ have shown that superpositions with k components appear when $\tau=2\pi/k$ ($k=2,3,\dots$). They have also shown that the number of well distinguishable coherent states in the superposition is proportional to $|\alpha_0|$. Recently, Gantsog and Tanaš^{22,23} have shown that such superpositions are clearly indicated by the phase distribution $P(\theta)$, which for the superposition of k states has the k -fold rotational symmetry if plotted in the polar coordi-

where $f_{n-m}(\tau)$ is given by Eq. (10) and b_n by Eq. (13). The phase φ_0 disappeared from Eq. (30) owing to our choice of the phase window.

After a minor rearrangement formula (30) for the phase distribution can be written as

ates. The splitting of $P(\theta)$ is already seen in Fig. 1(d), where for small λ a number of small peaks appears. The effect of damping on the formation of the superpositions of coherent states is shown in Fig. 2. The k -fold rotational symmetry of the phase distribution $P(\theta)$ is well resolved for $\lambda=0$ and is gradually destroyed as λ increases. According to estimates made by Miranowicz, Tanaš, and Kielich,¹⁹ and Tanaš *et al.*²⁷ the maximum number of well-resolved states in the superposition is equal to $k_{\max} \approx 2|\alpha_0| = 2\sqrt{N}$, which for $N=4$ taken in the figure gives $k_{\max}=4$, and this is the number of peaks in Fig. 2(d). For the superpositions with a higher number of components the k -fold symmetry of the P function (as well as the Q function) is broken and eventually destroyed. This has been shown convincingly by Tanaš *et al.*²⁷ for the system without damping. Since the superpositions of coherent states with k components appear for $\tau=2\pi/k$, the superpositions with a large number of components appear for shorter times τ , and they are therefore less affected by damping. This is clearly seen from Fig. 2. The effect is most dramatic for $\tau=2\pi$ [Fig. 2(a)], where for $\lambda=0$ there is a periodic recurrence of the initial phase distribution, while for $\lambda \neq 0$ the distribution is rapidly symmetrized. To make this statement a bit more quantitative, we introduce a parameter ("visibility of fringes")

$$v = \frac{P_{\max}(\theta) - P_{\min}(\theta)}{P_{\max}(\theta) + P_{\min}(\theta)}, \quad (34)$$

which allows for quantitative assessment of the resolution of the superposition components: unity means perfect resolution, and zero, no resolution at all. For all the superpositions shown in Fig. 2, this parameter is very close to unity when $\lambda=0$. For $\lambda=0.2$, the parameter v takes the values (a) $v=0.15$, (b) $v=0.25$, (c) $v=0.32$, and (d) $v=0.41$. So, the resolution is best for the four-peak structure. This suggests that, in the presence of damping, there are better chances to detect the superpositions with larger numbers of components. The best situation is, probably, for the maximum number of well-distinguishable states.

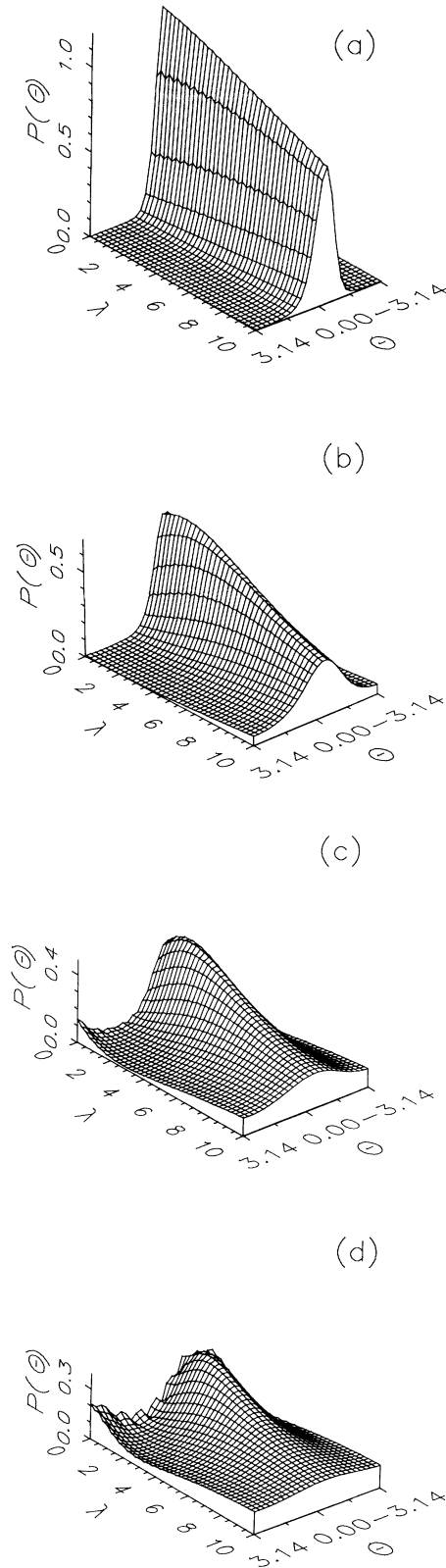


FIG. 1. Plots of the phase distribution function $P(\theta; \lambda)$ for different values of the evolution time τ : (a) $\tau=0.1$, (b) $\tau=0.3$, (c) $\tau=0.5$, (d) $\tau=0.7$. The initial mean number of photons $N=4$.

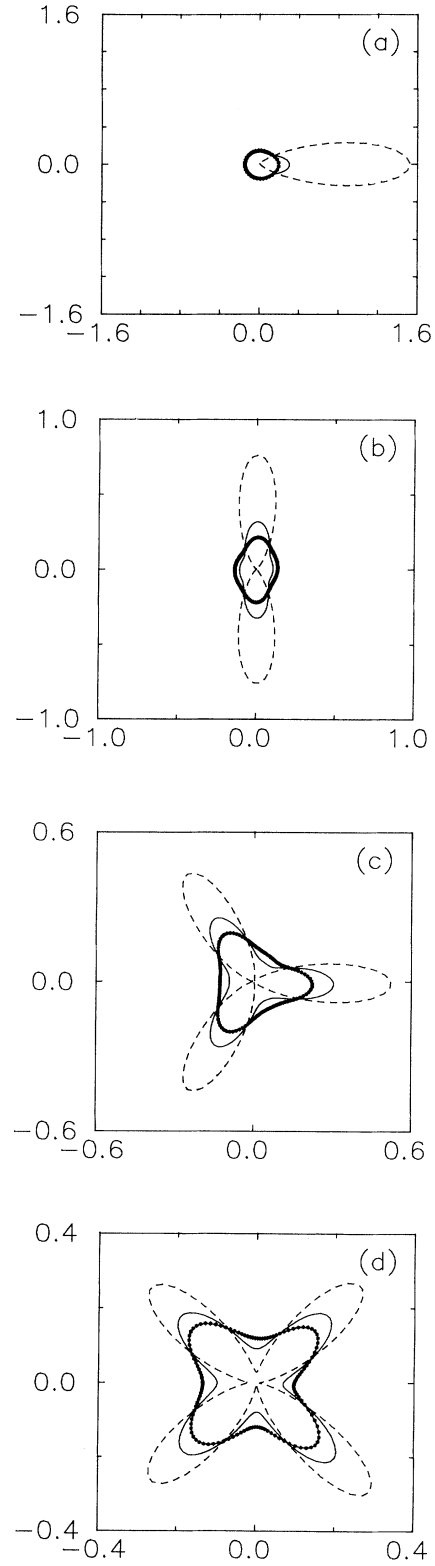


FIG. 2. Plots of $P(\theta)$ in polar coordinates for $\tau=2\pi/k$ and the damping parameter: $\lambda=0$ (dashed line), $\lambda=0.1$ (solid line), and $\lambda=0.2$ (bold line). The times are (a) $\tau=2\pi$, (b) $\tau=2\pi/2$, (c) $\tau=2\pi/3$, (d) $\tau=2\pi/4$; $N=4$.

Knowing the phase distribution (31) allows us to calculate the expectation value and the variance of the phase operator (18) according to the formula

$$\langle \hat{\phi}_\theta \rangle = \varphi_0 + \int_{-\pi}^{\pi} \theta P(\theta) d\theta = \varphi_0 + \langle \theta \rangle, \quad (35)$$

where

$$\begin{aligned} \langle \theta \rangle &= \int_{-\pi}^{\pi} \theta P(\theta) d\theta \\ &= \sum_{\substack{n,m \\ n>m}} b_n b_m \frac{(-1)^{n-m}}{n-m} \exp[\Gamma_{nm}(\tau)] \sin[\Lambda_{nm}(\tau)] \end{aligned} \quad (36)$$

with

$$\Gamma_{nm}(\tau) = -\frac{\lambda\tau}{2}(n+m) + \lambda A_{n-m}(\tau) + (n-m)B_{n-m}(\tau), \quad (37)$$

$$\begin{aligned} \Lambda_{nm}(\tau) &= \frac{\tau}{2}[n(n-1) - m(m-1)] \\ &\quad + (n-m)A_{n-m}(\tau) - \lambda B_{n-m}(\tau), \end{aligned} \quad (38)$$

$$\begin{aligned} \langle (\Delta \hat{\phi}_\theta)^2 \rangle &= \langle \theta^2 \rangle - \langle \theta \rangle^2 \\ &= \frac{\pi^2}{3} + 4 \sum_{\substack{n,m \\ n>m}} b_n b_m \frac{(-1)^{n-m}}{(n-m)^2} \exp[\Gamma_{nm}(\tau)] \\ &\quad \times \cos[\Lambda_{nm}(\tau)] - \langle \theta \rangle^2, \end{aligned} \quad (39)$$

where $\langle \theta \rangle$ is given by Eq. (36). For $\lambda=0$, Eqs. (36) and (39) go over into our earlier results,²³ and for $\tau=0$ the results for a coherent state given by Pegg and Barnett²⁶ are recovered. The evolution of the mean phase and its variance is shown in Fig. 3. The nonlinear shift in phase is clearly seen for short times. In this interval of time there is also rapid increase of the phase fluctuation, which is caused by nonlinear but unitary evolution and is observed even for $\lambda=0$. Of course, damping accelerates this process. In the long-time limit the initial values are periodically restored if the evolution is unitary. Damping destroys this periodicity, as it is seen in the figure.

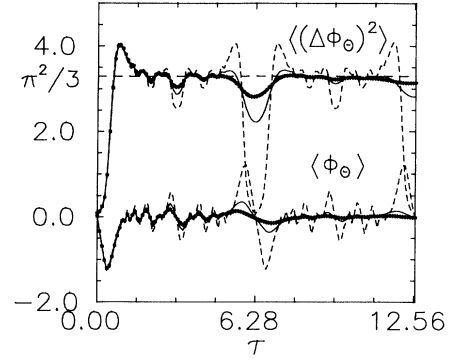


FIG. 3. Evolution of the mean phase and the phase variance, for $N=4$, and $\lambda=0$ (dashed line), $\lambda=0.05$ (solid line), and $\lambda=0.1$ (bold line).

ally restored if the evolution is unitary. Damping destroys this periodicity, as it is seen in the figure.

The phase characteristics of the field that can be compared to the results obtained on the grounds of the Susskind-Glogower²⁹ phase formalism are the sine and cosine functions of the phase operator. To calculate the expectation values and variances of such operators it is convenient to use the following relation, which is true for the “physical states”:^{26,31}

$$\langle \exp(im \hat{\phi}_\theta) \rangle_p = \langle \exp(im \phi_{SG}) \rangle_p, \quad (40)$$

where

$$\exp(im \phi_{SG}) = \sum_{n=0}^{\infty} |n\rangle \langle n+m| \quad (41)$$

is the Susskind-Glogower^{29,30} phase operator. Since the states of the anharmonic oscillator are physical states, we have

$$\begin{aligned} \langle \exp(im \phi_{SG}) \rangle &= \text{Tr}[\rho \exp(im \phi_{SG})] \\ &= \sum_{n=0}^{\infty} \rho_{n+m,n}(\tau) = e^{im(\varphi_0 + \tau/2)} f_m^n(\tau) \\ &\quad \times \exp\{\lambda A_m(\tau) + m B_m(\tau) - i[m A_m(\tau) - \lambda B_m(\tau)]\} \sum_{n=0}^{\infty} b_n b_{n+m} f_m^n(\tau), \end{aligned} \quad (42)$$

where $f_m(\tau)$ is defined by Eq. (10), $A_m(\tau)$ and $B_m(\tau)$ are given by Eqs. (32) and (33). Knowing the relation (42) allows calculations of the cosine and sine functions of the phase operator $\hat{\phi}_\theta$, according to the rules³¹

$$\begin{aligned} \langle \cos \hat{\phi}_\theta \rangle &= \frac{1}{2} \langle \exp(i \hat{\phi}_\theta) + \exp(-i \hat{\phi}_\theta) \rangle \\ &= e^{-\lambda\tau/2} \text{Re}(e^{i\varphi_0} \exp\{\lambda A_1(\tau) + B_1(\tau) - i[A_1(\tau) - \lambda B_1(\tau)]\}) \sum_{n=0}^{\infty} b_n b_{n+1} f_1^n(\tau) \\ &= \exp\left[-\frac{\lambda\tau}{2} + \lambda A_1(\tau) + B_1(\tau)\right] \sum_{n=0}^{\infty} b_n b_{n+1} e^{-n\lambda\tau} \cos[\varphi_0 - n\tau - A_1(\tau) + \lambda B_1(\tau)], \end{aligned} \quad (43)$$

$$\langle \sin \hat{\phi}_\theta \rangle = \exp\left[-\frac{\lambda\tau}{2} + \lambda A_1(\tau) + B_1(\tau)\right] \sum_{n=0}^{\infty} b_n b_{n+1} e^{-n\lambda\tau} \sin[\varphi_0 - n\tau - A_1(\tau) + \lambda B_1(\tau)], \quad (44)$$

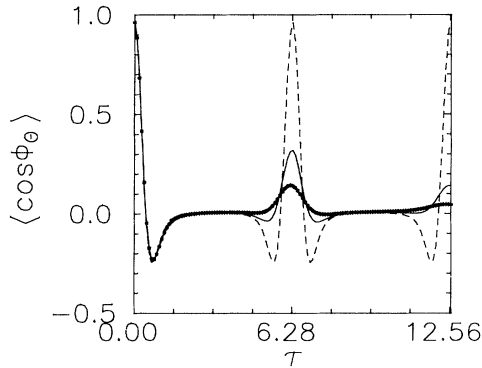


FIG. 4. Plot of the $\langle \cos \hat{\phi}_\theta \rangle$ against τ . The parameters are the same as in Fig. 3.

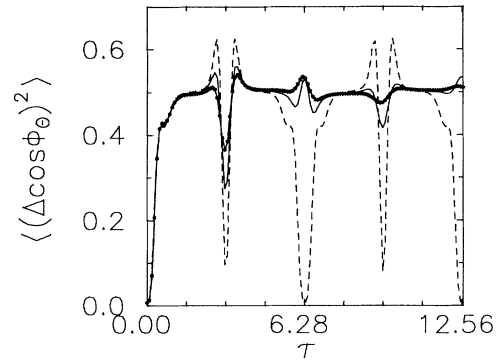


FIG. 5. Plot of the $\langle (\Delta \cos \hat{\phi}_\theta)^2 \rangle$ against τ . The parameters are the same as in Fig. 3.

and

$$\begin{aligned} \left. \begin{aligned} \langle \cos^2 \hat{\phi}_\theta \rangle \\ \langle \sin^2 \hat{\phi}_\theta \rangle \end{aligned} \right\} &= \pm \frac{1}{4} \langle \exp(i2\hat{\phi}_\theta) + \exp(-i2\hat{\phi}_\theta) \pm 2 \rangle \\ &= \left[\begin{aligned} \langle \cos^2 \phi_{SG} \rangle \\ \langle \sin^2 \phi_{SG} \rangle \end{aligned} \right] + \frac{1}{4} \langle (|0\rangle\langle 0|) \rangle \\ &= \frac{1}{2} \pm \frac{1}{2} \exp[-\lambda\tau + \lambda A_2(\tau) + 2B_2(\tau)] \sum_{n=0}^{\infty} b_n b_{n+2} e^{-n\lambda\tau} \cos[2\varphi_0 - (2n+1)\tau - 2A_2(\tau) + \lambda B_2(\tau)]. \end{aligned} \quad (45)$$

To get the Susskind-Glogower result from (45), one has to subtract

$$\frac{1}{4} \langle (|0\rangle\langle 0|) \rangle = \frac{1}{4} \rho_{00}(\tau) = \frac{1}{4} \exp(-N e^{-\lambda\tau}), \quad (46)$$

which for $\lambda\tau \ll 1$ has essentially different from zero values only for $N \ll 1$, and for $\lambda\tau \rightarrow \infty$ tends to $\frac{1}{4}$.

To illustrate the effect of damping on the evolution of the cosine function of the phase operator $\hat{\phi}_\theta$ and its variance, we have plotted in Fig. 4 the evolution of the expect-

tation value of the phase cosine, and in Fig. 5, the evolution of its variance. Again, as in the case of the phase itself (Fig. 3), we see that the quantum periodicity is rapidly removed even for very small λ . Eventually, the mean value approaches zero, and the variance approaches one-half, as it should be for the vacuum state to which the system evolves.

A comparison of the results obtained on the grounds of the Pegg-Barnett,²⁴⁻²⁶ Susskind-Glogower,^{29,30} and so-called measured phase³²⁻³³ formalisms is made in Figs. 6 and 7. In Fig. 6 the cosine variance is plotted for the three cases, and for $N=4$, $\lambda=0.1$. There is no big

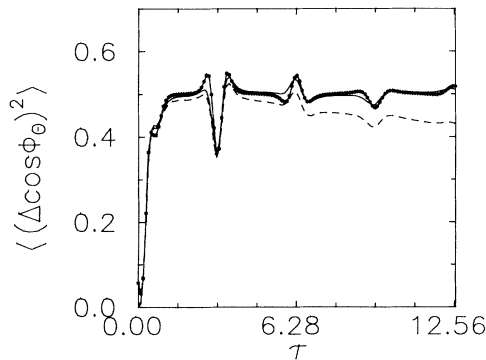


FIG. 6. Plot of the $\langle (\Delta \cos \hat{\phi}_\theta)^2 \rangle$ against τ , for $N=4$, $\lambda=0.1$, obtained within the Pegg-Barnett (solid line), the measured phase (bold line), and the Susskind-Glogower (dashed line) approaches.

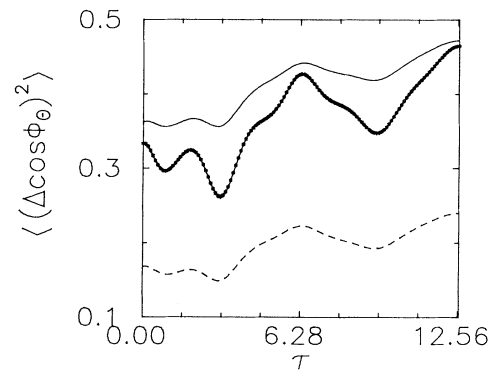


FIG. 7. Same as in Fig. 6, but for $N=0.25$.

difference between them at early stages of the evolution, but the Susskind-Glogower result declines down to reach asymptotically the value $\frac{1}{4}$, instead of the value $\frac{1}{2}$ for the Pegg-Barnett and measured phase results. The difference between the three approaches is more pronounced for $N=0.25$, which is shown in Fig. 7, for $\lambda=0.1$. In this case, the measured phase result is located between the Pegg-Barnett (upper-bound) and the Susskind-Glogower (lower-bound) result. Without damping the results are, of course, periodic and corresponding figures can be found in our earlier paper.²³ In a sense, damping is helpful in distinguishing between the Pegg-Barnett and the Susskind-Glogower results in the long-time limit. This is so, because damping suppresses the mean number of photons in the mode, and the field state becomes close to the vacuum, for which the difference between the two is the largest one.

IV. CONCLUSION

In this paper we have studied the phase properties of the damped anharmonic oscillator using the new Hermitian phase formalism of Pegg and Barnett. The exact analytical formula for the phase probability distribution of the system with damping has been obtained and illus-

trated graphically. It has been shown that quantum interference effects are destroyed rapidly by damping, which is in agreement with the result obtained for the Q function.^{3,13}

The effect of damping on the formation of discrete superpositions of coherent states in the system is also illustrated by plotting the phase distribution function for such superpositions with well-separated components, for different values of the damping parameter. It has been shown that the resolution of the individual states of the superposition in the presence of damping is easier for the superpositions with a larger number of components.

The evolution of the expectation value and variance of the Hermitian phase operator is calculated and illustrated graphically for different values of the damping parameter. It has been shown that damping accelerates the randomization of phase at early stages of evolution and removes quantum periodicity of the evolution. Similar conclusions can be drawn from the evolution of the fluctuations in the cosine function of the phase operator.

Finally, we have compared results for the cosine variance obtained from different phase formalisms in the presence of damping. The long-time divergence between the Pegg-Barnett and the Susskind-Glogower results is shown explicitly.

¹R. Tanaś, in *Coherence and Quantum Optics V*, edited by L. Mandel and E. Wolf (Plenum, New York, 1984), p. 645.

²G. J. Milburn, Phys. Rev. A **33**, 674 (1986).

³G. J. Milburn and C. A. Holmes, Phys. Rev. Lett. **56**, 2237 (1986).

⁴M. Kitagawa and Y. Yamamoto, Phys. Rev. A **34**, 3974 (1986).

⁵B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986).

⁶P. Tombesi and A. Mecozzi, J. Opt. Soc. Am. B **4**, 1700 (1987).

⁷C. C. Gerry, Phys. Rev. A **35**, 2146 (1987).

⁸C. C. Gerry and S. Rodrigues, Phys. Rev. A **36**, 5444 (1987).

⁹G. S. Agarwal, Opt. Commun. **62**, 190 (1987).

¹⁰V. Peřinova and A. Lukš, J. Mod. Opt. **35**, 1513 (1988).

¹¹C. C. Gerry and E. R. Vrscaj, Phys. Rev. A **37**, 4265 (1988).

¹²R. Tanaś, Phys. Rev. A **38**, 1091 (1988).

¹³D. J. Daniel and G. J. Milburn, Phys. Rev. A **39**, 4628 (1989).

¹⁴G. S. Agarwal, Opt. Commun. **72**, 253 (1989).

¹⁵R. Tanaś, Phys. Lett. A **141**, 217 (1989).

¹⁶G. J. Milburn, A. Mecozzi, and P. Tombesi, J. Mod. Opt. **36**, 1607 (1989).

¹⁷V. Peřinova and A. Lukš, Phys. Rev. A **41**, 414 (1990).

¹⁸V. Bužek, J. Mod. Opt. **37**, 303 (1990).

¹⁹A. Miranowicz, R. Tanaś, and S. Kielich, Quantum Opt. **2**, 253 (1990).

²⁰R. Tanaś, A. Miranowicz, and S. Kielich, Phys. Rev. A **43**, 4014 (1991).

²¹C. C. Gerry, Opt. Commun. **75**, 168 (1990).

²²Ts. Gantsog and R. Tanaś, Quantum Opt. (to be published).

²³Ts. Gantsog and R. Tanaś, J. Mod. Opt. (to be published).

²⁴D. T. Pegg and S. M. Barnett, Europhys. Lett. **6**, 483 (1988).

²⁵S. M. Barnett and D. T. Pegg, J. Mod. Opt. **36**, 7 (1989).

²⁶D. T. Pegg and S. M. Barnett, Phys. Rev. A **39**, 1665 (1989).

²⁷R. Tanaś, Ts. Gantsog, A. Miranowicz, and S. Kielich, J. Opt. Soc. Am. B (to be published).

²⁸W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).

²⁹L. Susskind and J. Glogower, Physics **1**, 49 (1964).

³⁰P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).

³¹J. A. Vaccaro and D. T. Pegg, Opt. Commun. **70**, 529 (1989).

³²S. M. Barnett and D. T. Pegg, J. Phys. A **19**, 3849 (1986).

³³R. Lynch, Opt. Commun. **67**, 67 (1988).