Phase properties of fractional coherent states

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The phase properties of the fractional coherent states are discussed from the point of view of the Pegg–Barnett Hermitian phase formalism. Exact analytical formulas for the phase variance are obtained and illustrated graphically. The results can serve as a test for the range of validity of the scaling law for the phase variance.

1. Introduction

Recently, D'Ariano [1] has discussed the possibility of amplitude squeezing through the "photon fractioning" procedure. The idea is to reduce photon number fluctuations at the expense of increased phase fluctuations, and this aim is achieved by introducing multiphoton phase and number operators, which corresponds to the nonunitary scaling of the original phase and number operators. If the scaling transformation is applied to the field states, instead of operators, it leads to the "statistical fractional photon" states [2,3], and, in particular, fractional coherent states. The most interesting are the $1/r$ coherent states for which the photon number variance scales as r^{-2} and the phase variance as r^2, so the uncertainty product remains unchanged. However, these r-dependences of the number and phase variances are true for highly exited states only. Moreover, to describe the phase properties of the field D'Ariano [1] uses the Susskind–Glogower [4,5] nonunitary shift operators $\hat{E}_{\pm}$ introducing the non-Hermitian phase operator $\hat{\phi}$, which for highly excited states and small phase uncertainty can be considered as approximately Hermitian, and it is used to describe phase properties of the $1/r$ coherent states.

At present there is an alternative way to describe the phase properties of such states using the Hermitian phase formalism introduced by Pegg and Barnett [6–8]. In this paper we are going to reexamine the phase properties of the $1/r$ coherent states from the point of view of the Pegg–Barnett phase formalism.

2. Phase properties of 1/r coherent states

To describe the phase properties of the $1/r$ coherent states we use the new Hermitian phase formalism introduced by Pegg and Barnett [6–8], which is based on introducing a finite $(s+1)$-dimensional space Ψ spanned by the number states $|0\rangle$, $|1\rangle$, ..., $|s\rangle$. The Hermitian phase operator operates on this finite space, and after all necessary expectation values have been calculated in Ψ, the value of s is allowed to tend to infinity. A complete orthonormal basis of $s+1$ states is defined on Ψ as

$$|\theta_m\rangle \equiv \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle \, , \qquad (1)$$

where

$$\theta_m \equiv \theta_0 + \frac{2\pi m}{s+1} \quad (m=0, 1, ..., s) \, . \qquad (2)$$

The value of θ_0 is arbitrary and defines a particular basis set of $s+1$ mutually orthogonal phase states.

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The Hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m| , \qquad (3)$$

where the subscript θ indicates the dependence on the choice of θ_0. The phase states (1) are eigenstates of the phase operator (3) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and $\theta_0+2\pi$. The unitary phase operator $\exp(i\hat{\phi}_\theta)$ is defined as the exponential function of the Hermitian operator $\hat{\phi}_\theta$. This operator acting on the eigenstate $|\theta_m\rangle$ gives the eigenvalue $\exp(i\theta_m)$, and it can be written as [6–8]

$$\exp(i\hat{\phi}_\theta) \equiv \sum_{n=0}^{s-1} |n\rangle\langle n+1| + \exp[i(s+1)\theta_0]\,|s\rangle\langle 0| . \qquad (4)$$

It is the last term in (4) that ensures the unitarity of this operator. The first sum reproduces the Susskind–Glogower [4,5] phase operator in the limit $s\to\infty$.

If the field is described by the density operator ρ, the expectation value of the phase operator (3) is given by

$$\langle\hat{\phi}_\theta\rangle = \mathrm{Tr}\{\rho\hat{\phi}_\theta\} = \sum_{m=0}^{s} \theta_m \langle\theta_m|\rho|\theta_m\rangle , \qquad (5)$$

where $\langle\theta_m|\hat{\rho}|\theta_m\rangle$ gives the probability of being found in the phase state $|\theta_m\rangle$. The density of phase states is $(s+1)/2\pi$, so in the continuum limit as s tends to infinity, we can introduce the phase distribution function

$$P(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} \langle\theta_m|\rho|\theta_m\rangle , \qquad (6)$$

where θ_m has been replaced by the continuous phase variable θ. As the phase distribution function $P(\theta)$ is known, all the quantum mechanical phase expectation values can be calculated with this function in a classical-like manner by simply performing integrations over θ. We have, for example,

$$\langle\hat{\phi}_\theta\rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta P(\theta)\,d\theta . \qquad (7)$$

The choice of the value of θ_0 defines the 2π range window of the phase values. Taking into account the definition (1), we can rewrite the phase distribution (6) as

$$P(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} \langle\theta_m|\rho|\theta_m\rangle = \frac{1}{2\pi} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \exp[-i(n-k)(\theta_0+\theta)]\rho_{nk} , \qquad (8)$$

where $\rho_{nk} = \langle n|\hat{\rho}|k\rangle$ are the matrix elements of the density operator $\hat{\rho}$ in the number state basis.

At this point we are able to study phase properties of $1/r$ coherent states the density operator of which is given by [1]

$$\hat{\rho}_\omega^{(1/r)} = \exp(-|\omega|^2) \times \sum_{\lambda=0}^{r-1}\sum_{l,m=0}^{\infty} |m\rangle \frac{\omega^{mr+\lambda}\,\omega^{*lr+\lambda}}{\sqrt{(mr+\lambda)!(lr+\lambda)!}} \langle l| , \qquad (9)$$

where ω is a complex number. On introducing the notation

$$b_n = \exp(-|\omega|^2/2) \frac{|\omega|^n}{\sqrt{n!}} , \quad \omega = |\omega| \exp(i\varphi) , \qquad (10)$$

the density operator matrix elements can be written as

$$\rho_{ml} = \langle m|\hat{\rho}_\omega^{(1/r)}|l\rangle = \sum_{\lambda=0}^{r-1} b_{mr+\lambda} b_{lr+\lambda} \exp[ir\varphi(m-l)] , \qquad (11)$$

and the phase distribution for such states is, according to (8), given by

$$P(\theta) = \frac{1}{2\pi} \sum_{\lambda=0}^{r-1}\sum_{m,l=0}^{\infty} \exp[-i(m-l)(\theta+\theta_0-r\varphi)] \times b_{mr+\lambda} b_{lr+\lambda} . \qquad (12)$$

Assuming $\theta_0 = r\varphi - \pi$, we symmetrize the phase distribution with respect to the phase $r\varphi$ and define the θ values window from $-\pi$ to $+\pi$, so the phase distribution takes the simpler form

$$P(\theta) = \frac{1}{2\pi} \sum_{\lambda=0}^{r-1} \left| \sum_{m=0}^{\infty} \exp(-im\theta) b_{mr+\lambda} \right|^2 = \frac{1}{2\pi} \left(1 + 2 \sum_{\lambda=0}^{r-1} \sum_{m>l} b_{mr+\lambda} b_{lr+\lambda} \cos[(m-l)\theta] \right) , \qquad (13)$$

with the normalization

$$\int_{-\pi}^{\pi} P(\theta)\,\mathrm{d}\theta=1\ . \qquad (14)$$

Knowing the phase distribution function (13) we are able to derive exact analytical formulas for the expectation value and the variance of the Hermitian phase operator in the $1/r$ coherent state. The results are

$$\langle\hat{\phi}_\theta\rangle=\mathrm{Tr}\{\hat{\rho}_\omega^{(1/r)}\hat{\phi}_\theta\}=r\varphi+\int_{-\pi}^{\pi}\theta P(\theta)\,\mathrm{d}\theta=r\varphi\,, \qquad (15)$$

$$\langle(\Delta\hat{\phi}_\theta)^2\rangle=\mathrm{Tr}\{\hat{\rho}_\omega^{(1/r)}(\Delta\hat{\phi}_\theta)^2\}=\int_{-\pi}^{\pi}\theta^2P(\theta)\,\mathrm{d}\theta$$

$$=\tfrac{1}{3}\pi^2+4\sum_{\lambda=0}^{r-1}\sum_{m>l}\frac{(-1)^{m-l}}{(m-l)^2}b_{mr+\lambda}b_{lr+\lambda}\,. \qquad (16)$$

The mean phase given by (15) is r times the phase of the ordinary coherent state, so the scaling law considered by D'Ariano [1] works exactly in this case. As regards the phase variance given by (16), it is clear that the scaling law does not work exactly, and can only be met for highly excited states. However, our formula (16) obtained within the Pegg–Barnett phase formalism is exact and valid for any value of $|\omega|$. The value $\frac{1}{3}\pi^2$ is the phase variance for a state with uniformly distributed phase, e.g. the vacuum state. This is the situation when $|\omega|=0$, or for given $|\omega|$ $r\to\infty$. To illustrate the phase properties of the $1/r$ coherent states we plot in fig. 1 the phase distribution function $P(\theta)$ for $|\omega|=2$ and various values of r. It is clear that as r increases the phase distribution becomes more and more uniform. In fig. 2 the phase variance is plotted against r for different values of $|\omega|$. As r increases the phase variance asymptotically approaches the value $\frac{1}{3}\pi^2$ characteristic for uniformly distributed phase. The scaling law $\langle(\Delta\hat{\phi}_\theta)^2\rangle\approx r^2$ means the parabolic shape of the curves, which is really seen in the figure for $|\omega|\gg1$ and not too large r, so that $|\omega|/r>1$.

In the bright limit $|\omega|\gg1$, the phase probability distribution (13) can be approximated by the Gaussian distribution. This can be done replacing the Poissonian weighting factor by the Gaussian distribution [7]

$$\exp(-|\omega|^2)\frac{|\omega|^{2n}}{n!}$$

$$\approx\frac{1}{\sqrt{2\pi|\omega|^2}}\exp\left(-\frac{(|\omega|^2-n)^2}{2|\omega|^2}\right) \qquad (17)$$

and performing integrations instead of summations. This gives us

$$D=\sum_{m=0}^{\infty}\exp(-im\theta)b_{mr+\lambda}$$

$$=\sum_{m=0}^{\infty}\exp(-im\theta)\exp(-|\omega|^2)\frac{|\omega|^{mr+\lambda}}{\sqrt{(mr+\lambda)!}}$$

$$\approx\frac{1}{(2\pi|\omega|^2)^{1/4}}$$

$$\times\int\exp\left(-im\theta-\frac{(|\omega|^2-mr-\lambda)^2}{4|\omega|^2}\right)\mathrm{d}m \qquad (18)$$

and in effect we have

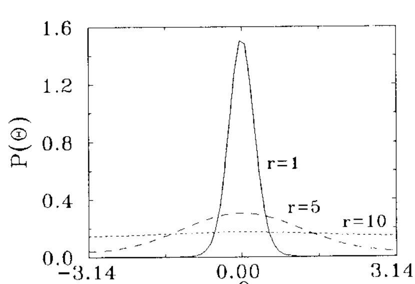

Fig. 1. The phase distribution $P(\theta)$ for $|\omega|=2$ and various r.

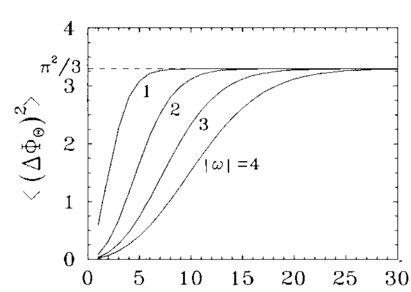

Fig. 2. Plots of the phase variance for various $|\omega|$.

$$P(\theta)=\frac{1}{2\pi}\sum_{\lambda=0}^{r-1}|D|^2$$

$$=\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\theta^2/2\sigma^2)\,, \qquad (19)$$

with

$$\sigma^2=\frac{r^2}{4|\omega|^2}\,. \qquad (20)$$

Thus, in the bright limit the phase variance which is equal to σ^2 scales as $\sim r^2$ in agreement with the D'Ariano results [1].

The photon-number variance for the $1/r$ coherent states can be calculated according to the formula

$$\langle(\Delta\hat{n})^2\rangle=\langle\hat{n}^2\rangle-\langle\hat{n}\rangle^2$$

$$=\sum_{n=0}^{\infty}n^2\rho_{nn}-\left(\sum_{n=0}^{\infty}n\rho_{nn}\right)^2$$

$$=\sum_{n=0}^{\infty}\sum_{\lambda=0}^{r-1}n^2b_{nr+\lambda}^2-\left(\sum_{n=0}^{\infty}\sum_{\lambda=0}^{r-1}nb_{nr+\lambda}^2\right)^2. \qquad (21)$$

In fig. 3 we have plotted the photon-number variance, evaluated according to (21), against r for different values of $|\omega|$. In fig. 4 the number–phase uncertainty product $\langle(\Delta\hat{n})^2\rangle\langle(\Delta\hat{\phi}_\theta)^2\rangle$ is plotted against r for various $|\omega|$. Of course, in the bright limit the Gaussian approximation of the Poissonian weighting factors leads to the D'Ariano scaling results $\langle\hat{n}\rangle\approx r^{-1}$, and $\langle(\Delta\hat{n})^2\rangle\approx r^{-2}$, which retain the number–phase uncertainty product unchanged.

To complete the phase properties of the $1/r$ coherent states, we adduce here the results for the cosine and sine functions of the Hermitian phase operator $\hat{\phi}_\theta$. These results can be compared to their counterparts in the Susskind–Glogower approach. Taking advantage of the fact that $1/r$ coherent states are "physical states" [7–9], we can easily calculate the exponential of the phase operator,

$$\langle\exp(im\hat{\phi}_\theta)\rangle$$

$$=\exp(imr\varphi)\sum_{n=0}^{\infty}\sum_{\lambda=0}^{r-1}b_{(n+m)r+\lambda}b_{nr+\lambda}\,. \qquad (22)$$

From (22) one obtains

$$\langle\cos\hat{\phi}_\theta\rangle=\cos(r\varphi)\sum_{n=0}^{\infty}\sum_{\lambda=0}^{r-1}b_{(n+1)r+\lambda}b_{nr+\lambda} \qquad (23)$$

and

$$\langle\cos^2\hat{\phi}_\theta\rangle$$

$$=\tfrac{1}{2}+\tfrac{1}{2}\cos(2r\varphi)\sum_{n=0}^{\infty}\sum_{\lambda=0}^{r-1}b_{(n+2)r+\lambda}b_{nr+\lambda}\,. \qquad (24)$$

Corresponding formulas for the sine function are obtained by replacing the cosine with the sine in (23) and changing the sign of the second term in (24). Of course, we have $\langle\cos^2\hat{\phi}_\theta\rangle+\langle\sin^2\hat{\phi}_\theta\rangle=1$ in the Pegg–Barnett approach.

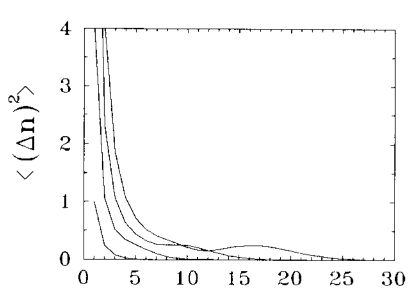

Fig. 3. Plots of the photon-number variance for various $|\omega|$. The values of $|\omega|$ for the subsequent curves counting from left are: 1, 2, 3, 4.

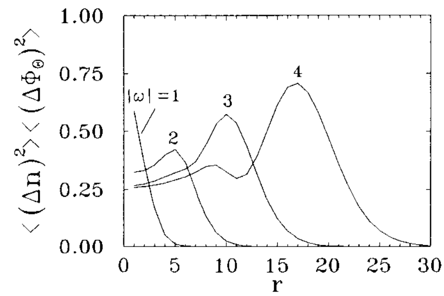

Fig. 4. Plots of the number–phase uncertainty product for various $|\omega|$.

3. Conclusion

In this paper we have discussed phase properties of the $1/r$ coherent states from the point of view of the Hermitian phase formalism of Pegg and Barnett [6–8]. This formalism allows one to get exact analytical formulas describing the variance of the Hermitian phase operator for any value of the state amplitude $|\omega|$. In the bright limit ($|\omega| \gg 1$) the exact formulas obtained within the Pegg–Barnett formalism reproduce the approximate results obtained by D'Ariano [1] who started from the Susskind–Glogower phase formalism. The clear advantage of the Pegg–Barnett approach is the possibility to obtain the exact analytical formula for the variance of the Hermitian phase operator, which next can be approximated for some special limiting cases. We have applied the exact formulas to illustrate some of the phase characteristics of the $1/r$ coherent states. Our results may be of special value for such values of $|\omega|$ and r for which the approximate formulas are not applicable, or they can serve as a test of validity for the approximate results.

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