Phase properties of the two-mode squeezed vacuum states

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The phase properties of the two-mode squeezed vacuum states are re-examined from the point of view of the Hermitian phase formalism introduced by Pegg and Barnett. The joint probability distribution for the phases of the two models is obtained, and the phase properties associated with this distribution are discussed thoroughly.

1. Introduction

Squeezed states of light have become a subject of intensive studies in recent years (see, for example, special issues of two optical journals [1] devoted to this subject). A number of successful experiments [2–8] in which squeezed states have been produced still strengthened motivation for such studies. Squeezed states have phase-sensitive noise properties, and it is interesting to study their phase properties. The phase properties of squeezed states have been investigated by Sanders et al. [9], Yao [10] and Fan and Zaidi [11], who used the Susskind and Glogower [12] phase formalism which involves non-unitary phase operators. Recently Pegg and Barnett [13–15] have introduced a new Hermitian phase formalism which successfully overcomes the troubles inherent in the Susskind–Glogower phase formalism and enables one to study finer details of the phase properties of quantum fields. Such quantities as expectation values and variances of Hermitian phase operators or phase distribution functions are now available for investigation. Vaccaro and Pegg [16] have investigated phase properties of the single-mode squeezed states from the point of view of the new Pegg–Barnett phase formalism. Recently Gronbech-Jensen et al. [17] have made comparisons of the phase properties of single-mode squeezed states obtained according to different phase formalisms, including the Pegg–Barnett formalism. However, the single-mode squeezed states differ essentially from the two-mode squeezed states discussed extensively by Caves and Schumaker [18]. The first observation of squeezing [2] was, in fact, observation of the two-mode squeezed vacuum. The phase properties of the two-mode squeezed vacuum have been shortly discussed by Fan and Zaidi [11] within the framework of the Susskind–Glogower phase formalism.

In this paper we re-examine the phase properties of the two-mode squeezed vacuum states applying the Pegg–Barnett Hermitian phase formalism. The joint probability distribution for phases of the two modes is obtained and its properties discussed. It is shown that this joint probability distribution is a function of the sum of the phases only. This implies some specific phase properties of the two-mode squeezed vacuum which are discussed in the paper.

2. Phase properties of the two-mode squeezed vacuum state

The two-mode squeezed vacuum state is defined by applying the two-mode squeeze operator $S(r, \varphi)$ to the two-mode vacuum state [18],

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$$|0,0\rangle_{(r,\varphi)} \equiv S(r,\varphi)|0,0\rangle\,, \tag{1}$$

where the two-mode squeeze operator is defined by [18]

$$S(r,\varphi) \equiv \exp[r(e^{-2i\varphi}a_+a_- - e^{2i\varphi}a_+^\dagger a_-^\dagger)]\,, \tag{2}$$

with

$$0 \leqslant r < \infty\,, \quad -\pi/2 < \varphi \leqslant \pi/2\,, \tag{3}$$

which define the strength and the phase of squeezing. The operators $a_\pm$ and $a_\pm^\dagger$ are the annihilation and creation operators of the two modes of the electromagnetic field. The squeeze operator unitarily transforms $a_\pm$ into a linear combination of $a_\pm$ and $a_\mp^\dagger$ (Bogoliubov transformation):

$$S(r,\varphi)a_\pm S^+(r,\varphi) = a_\pm \cosh r + a_\mp^\dagger e^{2i\varphi}\sinh r\,, \tag{4}$$

which is essential in obtaining squeezing. Since we are interested only in the two-mode squeezed vacuum states, the action of the squeeze operator (2) on the two-mode vacuum is given by [18]

$$\begin{aligned}|0,0\rangle_{(r,\varphi)} &= S(r,\varphi)|0,0\rangle \\ &= (\cosh r)^{-1}\exp(-e^{2i\varphi}\tanh r\, a_+^\dagger a_-^\dagger)|0,0\rangle \\ &= (\cosh r)^{-1}\sum_{n=0}^{\infty}(-e^{2i\varphi}\tanh r)^n|n,n\rangle\,. \end{aligned}\tag{5}$$

The phase φ, which describes the orientation of the quadrature phase uncertainty ellipse, is chosen in (5) just for convenience in discussing the quadrature phase squeezing. This choice, however, is not convenient when phase fluctuations are considered. So, we prefer here a choice of the phase φ that compensates the minus sign. This means a shift by $\pi/2$ of the phase φ. With this new choice of the phase, the two-mode vacuum state can be rewritten as

$$|0,0\rangle_{(r,\varphi)} = (\cosh r)^{-1}\sum_{n=0}^{\infty}(e^{2i\varphi}\tanh r)^n|n,n\rangle\,, \tag{6}$$

where we have retained the notation φ for the new notion of the phase. The reason for such a choice of the phase φ will become clear in the discussion of the phase properties of the state (6).

Before starting our discussion of the phase properties of the two-mode squeezed vacuum states, let us recall the main points of the Pegg–Barnett [13–15] phase formalism, which we will use in this paper. Their approach is based on introducing a finite $(s+1)$-dimensional subspace Ψ spanned by the number states $|0\rangle, |1\rangle, ..., |s\rangle$. The Hermitian phase operator operates on this finite subspace, and after all necessary expectation values have been calculated in Ψ, the value of s is allowed to tend to infinity. A complete orthonormal basis of $s+1$ states is defined on Ψ as

$$|\theta_m\rangle \equiv (s+1)^{-1/2}\sum_{n=0}^{s}\exp(in\theta_m)|n\rangle\,, \tag{7}$$

where

$$\theta_m \equiv \theta_0 + 2\pi m/(s+1)\,, \quad m=0,1,...,s\,. \tag{8}$$

The value of θ_0 is arbitrary and defines a particular basis set of $s+1$ mutually orthogonal phase states. The Hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s}\theta_m|\theta_m\rangle\langle\theta_m|\,. \tag{9}$$

The phase states (7) are eigenstates of the phase operator (9) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and θ_0 2π. The unitary phase operator $\exp(i\hat{\theta}_\theta)$ can be defined as the exponential function of the Hermitian operator $\hat{\phi}_\theta$. This operator acting on the eigenstate $|\theta_m\rangle$ gives the eigenvalue $\exp(i\theta_m)$, and can be written as [13–15]

$$\exp(i\hat{\phi}_\theta) \equiv \sum_{n=0}^{s-1}|n\rangle\langle n+1| + \exp[i(s+1)\theta_0]|s\rangle\langle 0|\,. \tag{10}$$

It is the last term in (10) that assures the unitarity of this operator. The first sum reproduces the Susskind–Glogower phase operator in the limit $s\to\infty$.

The expectation value of the phase operator (9) in a state $|\psi\rangle$ is given by

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \sum_{m=0}^{s}\theta_m|\langle\theta_m|\psi\rangle|^2\,, \tag{11}$$

where $|\langle\theta_m|\psi\rangle|^2$ gives the probability of being found in the phase state $|\theta_m\rangle$. The density of phase states is $(s+1)/2\pi$, so in the continuum limit as s tends to infinity, we can write eq. (11) as

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta P(\theta)\,\mathrm{d}\theta\,, \tag{12}$$

where the continuum phase distribution $P(\theta)$ is introduced by

$$P(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} |\langle\theta|\psi\rangle|^2\,, \tag{13}$$

where θ_m has been replaced by the continuous phase variable θ. As the phase distribution function $P(\theta)$ is known, all the quantum mechanical phase expectation values can be calculated with this function in a classical-like manner. The choice of the value of θ_0 defines the 2π range window of the phase values.

Generalization of the phase formalism to the two-mode case is straightforward, and for the two-mode vacuum state (6) we have

$$\begin{aligned}
&\langle\theta_{m+}|\langle\theta_{m-}|0,0\rangle_{(r,\varphi)} \\
&= (s_+ +1)^{-1/2}(s_- +1)^{-1/2}(\cosh r)^{-1} \\
&\times \sum_{n_+,n_-=0}^{s_+,s_-} \exp[-\mathrm{i}(n_+\theta_{m+} + n_-\theta_{m-})] \\
&\times \sum_{n=0}^{\infty} (\mathrm{e}^{2\mathrm{i}\varphi}\tanh r)^n \langle n_+, n_-|n,n\rangle \\
&= (s_+ +1)^{-1/2}(s_- +1)^{-1/2}(\cosh r)^{-1} \\
&\times \sum_{n=0}^{\min(s_+,s_-)} \{\exp[\mathrm{i}(2\varphi - \theta_{m-} - \theta_{m+})]\tanh r\}^n\,.
\end{aligned} \tag{14}$$

Because of the presence of the phase φ the two-mode squeezed vacuum can be treated as a partial phase state, and it is convenient to choose the phase value windows symmetrized with respect to the phase φ. This means choosing $\theta_0^\pm$ as

$$\theta_0^\pm = \varphi - \frac{\pi s_\pm}{s_\pm + 1}\,, \tag{15}$$

and introduction of the new phase values

$$\theta_{\mu\pm} = \theta_{m\pm} - \varphi\,, \tag{16}$$

where the new phase labels $\mu_\pm$ run in unit steps between the values $-s_\pm/2$ and $s_\pm/2$. Taking into account (15) and (16), after taking the modulus square of (14) and performing the continuum limit transition by making the replacements

$$\sum_{\mu_\pm=-s_\pm/2}^{s_\pm/2} \frac{2\pi}{s_\pm+1} \to \int_{-\pi}^{\pi} \mathrm{d}\theta_\pm\,, \tag{17}$$

we arrive at the continuous joint probability distribution for the continuous variables θ_+ and θ_-, which has the form

$$\begin{aligned}
P(\theta_+,\theta_-) = \frac{1}{(2\pi)^2}\bigg(1 + 2(\cosh r)^{-2} \sum_{n>k} (\tanh r)^{n+k} \\
\times \cos[(n-k)(\theta_+ + \theta_-)]\bigg)\,.
\end{aligned} \tag{18}$$

The distribution (18) is normalized such that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P(\theta_+,\theta_-)\,\mathrm{d}\theta_+\,\mathrm{d}\theta_- = 1\,. \tag{19}$$

From the form (18) of the joint probability distribution of the phases θ_+ and θ_- it is obvious that this distribution depends on the sum of the two phases only. This is an important phase property of the two-mode squeezed vacuum, which reflects the fact of strong correlations between the two modes.

Integrating $P(\theta_+,\theta_-)$ over one of the phases gives the marginal phase distribution $P(\theta_+)$ or $P(\theta_-)$ for the phase θ_+ or θ_-,

$$P(\theta_+) = \int_{-\pi}^{\pi} P(\theta_+,\theta_-)\,\mathrm{d}\theta_- = \frac{1}{2\pi}\,, \tag{20}$$

$$P(\theta_-) = P(\theta_+) = \frac{1}{2\pi}\,. \tag{21}$$

Thus the phases θ_+ and θ_- of the individual modes are uniformly distributed. The expectation values of the phase operators, defined by (9), can be calculated according to (12) with the phase distributions given by (21). We have

$$\langle\hat{\phi}_{\theta+}\rangle = \varphi + \int_{-\pi}^{\pi} \theta_+ P(\theta_+)\,\mathrm{d}\theta_+ = \varphi\,, \tag{22}$$

and similarly

$$\langle\hat{\phi}_{\theta-}\rangle = \langle\hat{\phi}_{\theta+}\rangle = \varphi\,. \tag{23}$$

As a consequence, we have

$$\langle\hat{\phi}_{\theta+} + \hat{\phi}_{\theta-}\rangle = 2\varphi, \quad \langle\hat{\phi}_{\theta+} - \hat{\phi}_{\theta-}\rangle = 0\,. \tag{24}$$

This means that the expectation value of the phase-sum-operator is related to the phase 2φ defining the two-mode squeezed vacuum state (6). Our choice of the phase φ is now more transparent. Of course, for the variances of the individual phases we obtain

$$\langle(\Delta\hat{\phi}_{\theta+})^2\rangle = \langle(\Delta\hat{\phi}_{\theta-})^2\rangle = \pi^2/3\,. \tag{25}$$

Thus, the two-mode squeezed vacuum has very specific phase properties: individual phases as well as the phase difference are random and the only non-random phase is the phase sum.

An example of the joint phase probability distribution $P(\theta_+, \theta_-)$ is shown in fig. 1. The ridge which is parallel to the diagonal of the phase window square reflects the dependence of $P(\theta_+, \theta_-)$ on $\theta=\theta_+ + \theta_-$ only. There is another ridge that is split into two pieces which appear in the corners of the phase window square. If the values of $\theta_0^{\pm}$, given by (15), were chosen differently, the second ridge would appear in the distribution in its full form. Because of the symmetry of the distribution $P(\theta_+, \theta_-)$, we can plot a function $P(\theta=\theta_+ + \theta_-)$ in the one-dimensional format. This means looking at the section of the distribution $P(\theta_+, \theta_-)$ in a plane perpendicular to the symmetry plane of this distribution. In fig. 2 plots of the function $P(\theta)$ are shown for various values of the squeeze parameter r. It is clearly seen that the distribution becomes narrower as r increases. This means that the sum of the phases becomes less uncertain.

It should be kept in mind, however, that the function $P(\theta)$ itself is not a probability distribution, it is only a convenient, one-dimensional picture of the joint probability distribution $P(\theta_+, \theta_-)$. So, all averages are obtained by performing integrations over

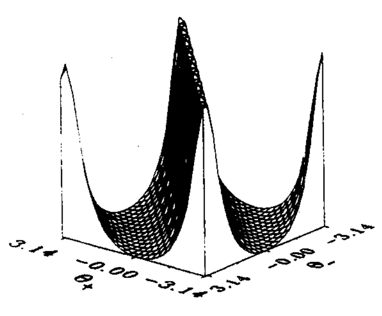

Fig. 1. Plot of the joint probability distribution $P(\theta_+, \theta_-)$ for $r=0.5$.

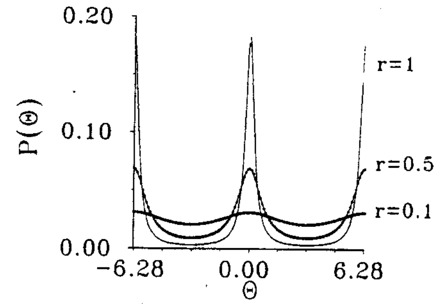

Fig. 2. Plot of the section $P(\theta=\theta_+ + \theta_-)$ against θ for various values of the squeeze parameter r.

θ_+ and θ_- with the distribution $P(\theta_+, \theta_-)$. This means that the side-peaks seen in fig. 2, which are located in the corners of the integration area, do not contribute to the averages in the limit $r\to\infty$, because they are pushed out of the integration area by the limiting process.

The Pegg–Barnett formalism allows us to calculate the variance of the phase-sum-operator, which can be done according to the formula

$$\langle[\Delta(\hat{\phi}_{\theta+} + \hat{\phi}_{\theta-})]^2\rangle = \langle(\Delta\hat{\phi}_{\theta+})^2\rangle + \langle(\Delta\hat{\phi}_{\theta-})^2\rangle + 2(\langle\hat{\phi}_{\theta+}\hat{\phi}_{\theta-}\rangle - \langle\hat{\phi}_{\theta+}\rangle\langle\hat{\phi}_{\theta-}\rangle)\,, \tag{26}$$

where the individual phase variances are given by (25), and the correlation term has appeared in (26). The phase correlation function can be calculated according to

$$\begin{aligned} C_{+-} &= \langle\hat{\phi}_{\theta+}\hat{\phi}_{\theta-}\rangle - \langle\hat{\phi}_{\theta+}\rangle\langle\hat{\phi}_{\theta-}\rangle \\ &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \theta_+\theta_- P(\theta_+, \theta_-)\, d\theta_+\, d\theta_- \\ &= -2(\cosh r)^{-2} \sum_{n>k} \frac{(\tanh r)^{n+k}}{(n-k)^2}\,. \end{aligned} \tag{27}$$

This correlation coefficient is plotted against the squeeze parameter r in fig. 3. It is negative and asymptotically approaches $-\pi^2/3$. The strong negative correlation between the phases of the two modes lowers the variance (26) of the phase-sum-operator; asymptotically, for $r\to\infty$, this variance tends to zero, which means that for very large squeezing the sum of the two phases becomes well defined. This would correspond to the classical behaviour of the fields with well defined phase-sum, of course, insofar as one

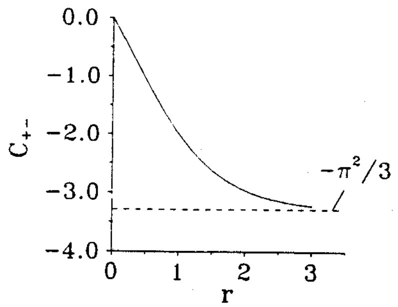

Fig. 3. Plot of the correlation coefficient C_{+-} against the squeeze parameter r.

can speak about classical behaviour of highly squeezed vacuum.

Generalizing of formula (10) and taking into account the fact that the two-mode squeezed vacuum is a "physical state" [14,15] enables us to calculate the expectation values of the phase exponential operators in the following way,

$$
\begin{aligned}
&{}_{(r,\varphi)}\langle 0,0|\exp(im_+\hat{\phi}_{\theta+})\exp(im_-\hat{\phi}_{\theta-})|0,0\rangle_{(r,\varphi)}\\
&={}_{(r,\varphi)}\langle 0,0|\hat{\exp}(im_+\phi_{+\mathrm{SG}})\\
&\times\hat{\exp}(im_-\phi_{-\mathrm{SG}})|0,0\rangle_{(r,\varphi)}\\
&=(\cosh r)^{-2}\sum_{n,k=0}^{\infty}\sum_{l,m=0}^{\infty}(\mathrm{e}^{2i\varphi}\tanh r)^{l+m}\\
&\times\langle l,l|n,k\rangle\langle n+m_+,k+m_-|m,m\rangle\\
&=(\mathrm{e}^{2i\varphi}\tanh r)^{m_+}\delta_{m_+,m_-}\,,
\end{aligned}
\tag{28}
$$

where the operators

$$
\hat{\exp}(im_\pm\phi_{\pm\mathrm{SG}})=\sum_{n=0}^{\infty}|n\rangle\langle n+m_\pm| \tag{29}
$$

are the Susskind–Glogower phase operators for the two modes. Formula (28) is strikingly simple, and it shows that only exponentials of the phase-sum have expectation values different from zero. Our formula (28) should be compared to the corresponding formula of Fan and Zaidi [11] (notice an error in their formula).

Using our formula (28) we have obtained the following results for the cosine and sine of the phase-sum operator,

$$
\begin{aligned}
&{}_{(r,\varphi)}\langle 0,0|\cos(\hat{\phi}_{\theta+}+\hat{\phi}_{\theta-})|0,0\rangle_{(r,\varphi)}\\
&=\tanh r\cos 2\varphi\,,\\
&{}_{(r,\varphi)}\langle 0,0|\sin(\hat{\phi}_{\theta+}+\hat{\phi}_{\theta-})|0,0\rangle_{(r,\varphi)}\\
&=\tanh r\sin 2\varphi\,,
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&{}_{(r,\varphi)}\langle 0,0|\cos^2(\hat{\phi}_{\theta+}+\hat{\phi}_{\theta-})|0,0\rangle_{(r,\varphi)}\\
&=\tfrac{1}{2}+\tfrac{1}{2}\cos 4\varphi\,(\tanh r)^2\,,\\
&{}_{(r,\varphi)}\langle 0,0|\sin^2(\hat{\phi}_{\theta+}+\hat{\phi}_{\theta-})|0,0\rangle_{(r,\varphi)}\\
&=\tfrac{1}{2}-\tfrac{1}{2}\cos 4\varphi\,(\tanh r)^2\,,
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&{}_{(r,\varphi)}\langle 0,0|\,[\Delta\cos(\hat{\phi}_{\theta+}+\hat{\phi}_{\theta-})]^2|0,0\rangle_{(r,\varphi)}\\
&=2(\cosh r)^{-2}\,,\\
&{}_{(r,\varphi)}\langle 0,0|\,[\Delta\sin(\hat{\phi}_{\theta+}+\hat{\phi}_{\theta-})]^2|0,0\rangle_{(r,\varphi)}\\
&=2(\cosh r)^{-2}\,.
\end{aligned}
\tag{32}
$$

Asymptotically, for very large squeezing ($r\to\infty$, $\tanh r\to 1$, $\cosh r\to\infty$) the expectation values (30) and (31) of the phase-sum-operator become the corresponding functions of the phase 2φ, confirming the relation between the phase-sum and 2φ that is already seen from (24). It is interesting to note that the expectation value of the phase-sum-operator is equal to 2φ irrespective of the value of r, whereas for the sine and cosine functions the correspondence is obtained only asymptotically. The variances (32) then become zero and the sine and cosine of the phase-sum are well defined.

It should, however, be emphasized that the expectation values calculated according to the Pegg–Barnett formalism depend on the choice of the particular window of the phase eigenvalues. If a different choice from that made in this paper were made, the clear picture of phase properties of the two-mode squeezed vacuum would be disturbed. For example, the value of the correlation coefficient (27) would be different, and the phase-sum variance (26) would not tend to zero asymptotically. However, formulas (28)–(32) because of the way they have been calculated do not, in fact, depend on the choice of the phase window. This gives us an opportunity of choice that introduces consistency in the behaviour of the phase itself and its sine and cosine functions. Another way of making the choice is to minimize the variance (26) of the phase-sum-operator.

3. Conclusions

We have discussed the phase properties of the two-mode squeezed vacuum state from the point of view of the Pegg–Barnett Hermitian phase formalism. The joint probability distribution for the phases of the two modes has been obtained and has been shown to depend only on the sum of the two phases. This implies some specific phase properties of the two-mode squeezed vacuum. The only non-uniformly distributed phase quantity is the phase-sum. Individual phases as well as the phase-difference are uniformly distributed. Strong negative correlation exists between the phases of two modes, which forces the phase-sum-operator variance to tend asymptotically to zero as the squeezing parameter r tends to infinity. This means, asymptotically, the state with the well defined sum of the phases. The sine and cosine functions of the phase-sum-operator have also been obtained and compared to earlier results of Fan and Zaidi [11] obtained within the Susskind–Glogower phase formalism.

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References

[1] J. Mod. Opt., Vol. 34, No. 6/7 (1987); J. Opt. Soc. Am., Vol. B 4, No. 10 (1987).
[2] R.E. Slusher, L.W. Hollberg, B. Yurke, J.C. Mertz and J.F. Valley, Phys. Rev. Lett. 55 (1985) 2409.
[3] R.M. Shelby, M.D. Levenson, S.H. Perlmutter, R.G. Devoe and D.F. Walls, Phys. Rev. Lett. 57 (1986) 691.
[4] L.A. Wu, H.J. Kimble, J.L. Hall and H. Wu, Phys. Rev. Lett. 57 (1986) 2520.
[5] M.G. Raizen, L.A. Orozco, M. Xiao, T.L. Boyd and H.J. Kimble, Phys. Rev. Lett. 59 (1987) 198.
[6] M.W. Maeda, P. Kumar and J. H. Shapiro, Opt. Lett. 12 (1987) 161.
[7] S. Machida, Y. Yamamoto and Y. Jtaya, Phys. Rev. Lett. 58 (1987) 1000.
[8] A. Heidmann, R.J. Horowicz, S. Reynand, E. Giacobino and C. Fabre, Phys. Rev. Lett. 59 (1987) 2555.
[9] B.C. Sanders, S.M. Barnett and P.L. Knight, Opt. Commun. 58 (1986) 290.
[10] D. Yao, Phys. Lett. A 122 (1987) 77.
[11] Fan Hong-Yi and H.R. Zaidi, Opt. Commun. 68 (1988) 143.
[12] L. Susskind and J. Glogower, Physics 1 (1964) 49.
[13] D.T. Pegg and S.M. Barnett, Europhys. Lett. 6 (1988) 483.
[14] D.T. Pegg and S.M. Barnett, Phys. Rev. A 39 (1989) 1665.
[15] S.M. Barnett and D.T. Pegg, J. Mod. Opt. 36 (1989) 7.
[16] J.A. Vaccaro and D.T. Pegg, Opt. Commun. 70 (1989) 529.
[17] N. Gronbech-Jensen, P.L. Christiansen and P.S. Ramanujam, J. Opt. Soc. Am. B 6 (1989) 2423.
[18] C.M. Caves and B.L. Schumaker, Phys. Rev. A 31 (1985) 3068, 3093.