Quantum phase fluctuations in the second-harmonic generation

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Quantum phase fluctuations in the second-harmonic generation are examined from the point of view of the Hermitian phase formalism introduced by Pegg and Barnett. The joint probability distribution as well as the variances for the phases of the fundamental and second-harmonic modes are calculated numerically. Their evolution is illustrated graphically, and the phase properties of the generated light are discussed.

1. Introduction

The process of converting laser light of frequency ω into its second harmonic of frequency 2ω – the second-harmonic generation – is the first optical nonlinear process observed in a laboratory with the use of lasers [1]. In the quantum picture of the process we deal here with a nonlinear process in which two photons are annihilated and one photon with doubled frequency is created. The quantum states of the field generated in the process exhibit a number of unique quantum features such as photon antibunching [2] and squeezing [3,4] for both fundamental and second-harmonic modes (for a review and references see ref. [5]). Recently, Nikitin and Masalov [6] have discussed the properties of the quantum state of the fundamental mode calculating numerically the quasiprobability distribution function $Q(\alpha, \alpha^*)$ for this mode. They have suggested that the quantum state of the fundamental mode evolves, in the course of the second-harmonic generation, into a superposition of two macroscopically distinguishable states. Such superpositions of well separated coherent states that appear in the evolution of the anharmonic oscillator [7] are clearly indicated by the splitting of the function Q into separate peaks [8]. Recently, Gantsog and Tanaś [9,10] have shown, using the new Pegg–Barnett [11–13] Hermitian phase formalism, that such superpositions are clearly visible from the phase distribution functions. The new phase formalism of Pegg and Barnett enables one also to pose the question of quantum phase fluctuations in various nonlinear optical processes [9,10,14,15].

In this paper, we consider the problem of quantum phase fluctuations of the field generated in the process of second-harmonic generation. The joint phase probability distribution function for the two modes is calculated and its evolution illustrated graphically. It is shown that at the initial stage of the evolution the second-harmonic mode acquires a relatively well defined phase, which is randomized at later stages of the evolution. The phase variances for the second-harmonic as well as the fundamental modes are also calculated showing clearly the appearance of the minimum in the variance for the second-harmonic mode. The splitting of the phase distribution function into separate peaks, suggesting the appearance of the superposition of two or more states, is also demonstrated. In fact, the sequence of more and more peaks that appear in the phase distribution for later times leads to the randomization of the phase, i.e., to making the phase distribution more and more uniform.

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2. Quantum evolution of the field state

To describe the second-harmonic generation process we use the following model Hamiltonian,

$$H=H_0+H_I$$
$$=\hbar\omega a^+a+2\hbar\omega b^+b+\hbar g(b^+a^2+ba^{+2}) , \qquad (1)$$

where a (a^+) and b (b^+) are the annihilation (creation) operators of the fundamental mode of frequency ω and the second-harmonic mode at frequency 2ω, respectively. The coupling constant g, which is real, describes the coupling between the two modes. Since H_0 and H_I commute, there are two constants of motion: H_0 and H_I. H_0 determines the total energy stored in both modes, which is conserved by the interaction H_I. This allows us to factor out $\exp(-iH_0t/\hbar)$ from the evolution operator, in fact to drop it altogether, and to write the resulting state of the field as

$$|\psi(t)\rangle=\exp(-iH_It/\hbar)|\psi(0)\rangle , \qquad (2)$$

where $|\psi(0)\rangle$ is the initial state of the field. If the Fock states are used as basis states, the interaction Hamiltonian H_I is not diagonal in such a basis. To find the state evolution, we apply the numerical method of diagonalization of H_I. Such a method has been used several times in the context of second-harmonic generation [6,16–18].

The typical initial conditions for the second-harmonic generation are: a coherent state of the fundamental mode and the vacuum of the second-harmonic mode. The initial state of the field can thus be written as

$$|\psi(0)\rangle=\sum_{n=0}^{\infty} b_n|n, 0\rangle , \qquad (3)$$

where

$$b_n=\exp(-|\alpha|^2/2)\frac{\alpha^n}{\sqrt{n!}} \qquad (4)$$

is the Poissonian weight factor of the coherent state $|\alpha\rangle$ represented as a superposition of n-photon states, and the state $|n, 0\rangle=|n\rangle|0\rangle$ is the product of the Fock states with n photons in the fundamental mode and no photons in the second harmonic. With these initial conditions the resulting state (2) can be written as

$$|\psi(t)\rangle=\sum_{n=0}^{\infty} b_n \sum_{k=0}^{[n/2]} c_{nk}(t)|n-2k, k\rangle , \qquad (5)$$

where the coefficients $c_{nk}(t)$ are given by

$$c_{nk}(t)=\langle n-2k, k|\exp(-iH_It/\hbar)|n, 0\rangle . \qquad (6)$$

The summation over k runs up to the integer part of $n/2$, where n denotes the number of photons of the fundamental mode, whereas k is the number of photons created in the second-harmonic mode. The coefficients $c_{nk}(t)$ given by eq. (6) are calculated numerically to find the evolution of the field state (5) and, consequently, its phase properties.

3. Phase properties of the field

To study the phase properties of the field obtained in the second-harmonic generation process, we use here the new Pegg–Barnett [11–13] phase formalism which is based on introducing a finite $(s+1)$-dimensional space Ψ spanned by the number states $|0\rangle, |1\rangle, ..., |s\rangle$. The Hermitian phase operator operates on this finite space, and after all necessary expectation values have been calculated in Ψ, the value of s is allowed to tend to infinity. A complete orthonormal basis of $s+1$ states is defined on Ψ as

$$|\theta_m\rangle\equiv(s+1)^{-1/2}\sum_{n=0}^{s}\exp(in\theta_m)|n\rangle , \qquad (7)$$

where

$$\theta_m\equiv\theta_0+2\pi m/(s+1) \quad (m=0, 1, ..., s) . \qquad (8)$$

The value of θ_0 is arbitrary and defines a particular basis set of $s+1$ mutually orthogonal phase states. The Hermitian phase operator is defined as

$$\hat{\phi}_\theta\equiv\sum_{m=0}^{s}\theta_m|\theta_m\rangle\langle\theta_m| . \qquad (9)$$

The phase states (7) are eigenstates of the phase operator (9) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and $\theta_0+2\pi$. The unitary phase operator $\exp(i\hat{\phi}_\theta)$ is defined as the exponential function of the Hermitian operator $\hat{\phi}_\theta$. This operator acting on the eigenstate $|\theta_m\rangle$ gives the eigenvalue $\exp(i\theta_m)$, and can be written as [11–13]

$$\exp(\mathrm{i}\hat{\phi}_\theta) \equiv \sum_{n=0}^{s-1} |n\rangle\langle n+1| + \exp[\mathrm{i}(s+1)\theta_0]|s\rangle\langle 0| . \tag{10}$$

It is the last term in (10) that assures the unitarity of this operator. The first sum reproduces the Susskin–Glogower phase operator in the limit $s\to\infty$.

The expectation value of the phase operator (9) in a state $|\psi\rangle$ is given by

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \sum_{m=0}^{s} \theta_m |\langle\theta_m|\psi\rangle|^2 , \tag{11}$$

where $|\langle\theta_m|\psi\rangle|^2$ gives the probability of being found in the phase state $|\theta_m\rangle$. The density of phase states is $(s+1)/2\pi$, so in the continuum limit as s tends to infinity, we can write eq. (11) as

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta P(\theta)\,\mathrm{d}\theta , \tag{12}$$

where the continuum phase distribution $P(\theta)$ is introduced by

$$P(\theta) \lim_{s\to\infty} \frac{s+1}{2\pi} |\langle\theta|\psi\rangle|^2 , \tag{13}$$

where θ_m has been replaced by the continuum phase variable θ. As the phase distribution function $P(\theta)$ is known, all the quantum mechanical phase expectation values can be calculated with this function in a classical-like manner. The choice of the value of θ_0 defines the 2π range window of the phase values.

In our case of second-harmonic generation, the state of the field (5) is in fact a two-mode state, and the phase formalism must be generalized to the two-mode case. This is straightforward and, for the state (5), we obtain

$$\langle\theta_{m_\mathrm{a}}|\langle\theta_{m_\mathrm{b}}|\psi(t)\rangle = (s_\mathrm{a}+1)^{-1/2}(s_\mathrm{b}+1)^{-1/2} \times \sum_{n=0}^{s_\mathrm{a}} b_n \sum_{k=0}^{[n/2]} \exp\{-\mathrm{i}[(n-2k)\theta_{m_\mathrm{a}} + k\theta_{m_\mathrm{b}}]\} c_{nk}(t) . \tag{14}$$

We use the indices a and b to distinguish between the fundamental (a) and second-harmonic (b) modes. There is still a freedom of choice in (14) of the values of $\theta_0^{\mathrm{a,b}}$, which define the phase value window. We can choose these values at will, so we take them as

$$\theta_0^{\mathrm{a,b}} = \varphi_{\mathrm{a,b}} - \frac{\pi s_{\mathrm{a,b}}}{s_{\mathrm{a,b}}+1} , \tag{15}$$

and we introduce the new phase values

$$\theta_{\mu_{\mathrm{a,b}}} = \theta_{m_{\mathrm{a,b}}} - \varphi_{\mathrm{a,b}} , \tag{16}$$

where the new phase labels $\mu_{\mathrm{a,b}}$ run in unit steps between the values $-s_{\mathrm{a,b}}/2$ and $s_{\mathrm{a,b}}/2$. This means that we have symmetrized the phase windows for the fundamental and second-harmonic modes with respect to the phases φ_a and φ_b, respectively. On inserting (15) and (16) into (14), taking the modulus square of (14) and performing the continuum limit transition by making the replacements

$$\sum_{\mu_{\mathrm{a,b}}=-s_{\mathrm{a,b}}/2}^{s_{\mathrm{a,b}}/2} \frac{2\pi}{s_{\mathrm{a,b}}+1} \to \int_{-\pi}^{\pi} \mathrm{d}\theta_{\mathrm{a,b}} , \tag{17}$$

we arrive at the continuous joint probability distribution for the continuous variables θ_a and θ_b, which has the form

$$P(\theta_\mathrm{a},\theta_\mathrm{b}) = \frac{1}{(2\pi)^2} \left| \sum_{n=0}^{\infty} b_n \sum_{k=0}^{[n/2]} \exp\{-\mathrm{i}[(n-2k)\theta_\mathrm{a} + k\theta_\mathrm{b} - k(2\varphi_\mathrm{a}-\varphi_\mathrm{b})]\} c_{nk}(t) \right|^2 . \tag{18}$$

Distribution (18) is normalized such that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P(\theta_\mathrm{a},\theta_\mathrm{b})\,\mathrm{d}\theta_\mathrm{a}\,\mathrm{d}\theta_\mathrm{b} = 1 . \tag{19}$$

To fix the phase windows for θ_a and θ_b, we have to assign to φ_a and φ_b particular values. It is interesting to notice that formula (18) depends, in fact, on the difference $2\varphi_\mathrm{a}-\varphi_\mathrm{b}$, which reproduces the classical phase relation for the second-harmonic generation. Classically, for the initial conditions we have chosen here, this phase difference takes the value $2\varphi_\mathrm{a}-\varphi_\mathrm{b}=\pi/2$. It turns out that this is also a good choice to fix the phase windows in the quantum description, so we make it here. Since we assume the phase of the initial coherent state of the fundamental mode to be zero (α real), we additionally assume $\varphi_\mathrm{a}=0$ without loss of generality.

The joint phase probability distribution (18) can be evaluated numerically if the mean number of photons $N_a = |\alpha|^2$ of the fundamental mode is not too large. The results are illustrated in fig. 1, where the function $P(\theta_a, \theta_b)$ is plotted in three-dimensional format for various values of the dimensionless evolution time gt. Initially, the distribution is peaked at $\theta_a = 0$ in the θ_a direction reproducing the phase distribution of the coherent state of the fundamental mode, and it is completely flat in the θ_b direction representing the uniform phase distribution of the vacuum of the second-harmonic mode. As the evolution goes on the distribution in the θ_a direction is broadened, while for the second-harmonic mode a peak starts to grow. This means that the second harmonic at the initial stage of the evolution becomes "phased", and the appearance of the peak at $\theta_b = 0$ confirms the classical relation $2\varphi_a - \varphi_b = \pi/2$, which we have applied to fix the phase window. The phase distribution in the θ_b direction is narrowing at the beginning of the evolution, meaning less uncertainty in the phase of the second harmonic. However, for later times the distribution $P(\theta_a, \theta_b)$ splits into two, and still later even more, peaks. This splitting suggests that the state of the field evolves during the evolution into a superposition of two (or more) distinguishable states with definite mean phase. The splitting of the function $Q(\alpha, \alpha^*)$ of the fundamental mode into two distinguishable peaks has been found by Nikitin and Masalov [6]. As the number of peaks in the distribution $P(\theta_a, \theta_b)$ increases, the distribution becomes more and more uniform, which means the randomization of phase. The route to the random phase distribution, however, goes through a sequence of increasing numbers of peaks.

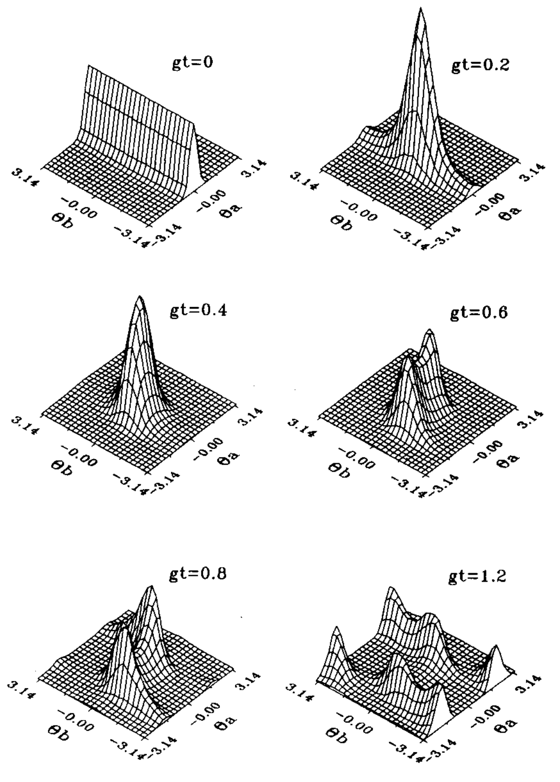

Fig. 1. Plot of the joint probability distribution $P(\theta_a, \theta_b)$, for various values of gt and $N_a = 4$.

When integration of $P(\theta_a, \theta_b)$ over one of the phases is performed, the marginal phase distribution $P(\theta_a)$ or $P(\theta_b)$ for the phase θ_a or θ_b is obtained. For example

$$P(\theta_a) = \int_{-\pi}^{\pi} P(\theta_a, \theta_b)\, d\theta_b . \tag{20}$$

The marginal distributions $P(\theta_a)$ and $P(\theta_b)$ allow one to calculate the mean values and variances of the phase operators, defined by (9), of the fundamental and second-harmonic modes. We have for example

$$\langle \hat{\phi}_{\theta_a} \rangle = \varphi_a + \int_{-\pi}^{\pi} \theta_a P(\theta_a)\, d\theta_a = \varphi_a , \tag{21}$$

$$\langle \hat{\phi}_{\theta_b} \rangle = \varphi_b + \int_{-\pi}^{\pi} \theta_b P(\theta_b)\, d\theta_b = \varphi_b . \tag{22}$$

Since the functions $P(\theta_a)$ and $P(\theta_b)$ are even functions of their arguments the integrals in (21) and (22) are zero. This is true under the assumption we have made that $2\varphi_a - \varphi_b = \pi/2$. The mean values of the phases, given by (21) and (22), reproduce the classical phase relations in the second-harmonic generation.

Quantum mechanically, however, there are definite uncertainties in the measurements of the phases, which are given by the variances of the phase operators,

$$\langle (\Delta\hat{\phi}_{\theta_a})^2 \rangle = \langle \hat{\phi}^2_{\theta_a} \rangle - \langle \hat{\phi}_{\theta_a} \rangle^2$$

$$= \int_{-\pi}^{\pi} \theta_a^2 P(\theta_a)\, d\theta_a . \tag{23}$$

The formula for the variance of the phase operator for the second-harmonic mode is the same as (23) after changing the index a into b.

The integrations in (20) and (23) can be performed explicitly, giving the formulas for $P(\theta_a)$ and $P(\theta_b)$ as well as for the phase variances in terms of the coefficients $c_{nk}(t)$. Numerical evaluation of the resulting formulas allows us to find the evolution of the phase variances. The results are presented in fig. 2, where the variances for the phases of the fundamental and second-harmonic modes are plotted against gt. It is seen that the variance for the fundamental mode starts from its minimum value for the coherent state, increases rapidly, and after several oscillations becomes close to $\pi^2/3$ – the value for the uniformly distributed phase. So the fundamental mode phase is rapidly randomized. The situation is different with the variance of the second-harmonic mode phase, which starts from the value $\pi^2/3$ of the vacuum state, goes down, reaches its absolute minimum, goes up, and after a few oscillations again becomes close to $\pi^2/3$. This shows that at the beginning of the evolution the second harmonic generated from the vacuum acquires a definite phase, which is randomized again at later stages of the evolution. There is a time at which the second-harmonic phase is defined best.

In our numerical calculations we have assumed for the mean number of photons of the initial coherent state of the fundamental mode the value $N_a = |\alpha|^2 = 4$. Since this number is greater than unity it leads to solutions that have already some characteristic features of the solutions for $N_a \gg 1$ (classical limit), on the other hand, this number is small enough to make the quantum effects clearly visible and save computing time.

In order to set properly the time scale on which the essential changes of phase properties considered in this paper take place, we have plotted in fig. 3 the evolution of the mean numbers of photons for the fundamental ($\langle a^+a \rangle$) and second-harmonic ($\langle b^+b \rangle$) modes. Since our calculations are fully quantum mechanical, the flow of power back and forth between the fundamental and the second-harmonic mode is clearly visible, in contrast to the classical solutions that are monotonic in time. Of course, the conservation law $\langle a^+a \rangle + 2\langle b^+b \rangle = N_a$ is satisfied.

3. Conclusions

We have discussed the quantum phase properties of the field generated in the process of second-harmonic generation. The Pegg–Barnett phase formalism has been applied to find the joint probability distribution for the two modes of the field as well as the variances of the phase operators for the individual modes. The method of numerical diagonalization of the interaction Hamiltonian has been used to get the evolution of the field state. The joint phase probability distribution for the two modes of the field shows some interesting features. It shows the creation of a definite phase of the second-harmonic mode at the beginning of the evolution. The phase of the fundamental mode is degraded at the same time. This

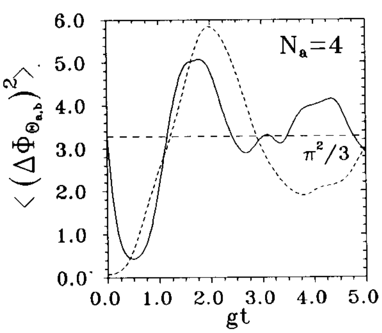

Fig. 2. Evolution of the phase variances for the fundamental mode (dashed line) and the second-harmonic mode (solid line), for $N_a=4$.

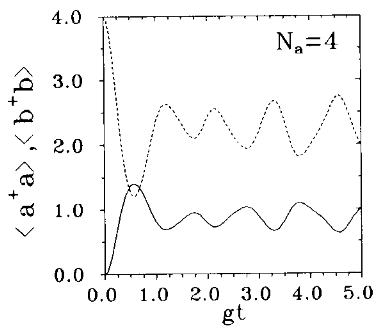

Fig. 3. Evolution of the mean number of photons for the fundamental (dashed line) and the second-harmonic (solid line) mode, for $N_a=4$.

behaviour is confirmed by the evolution of the phase variances of the two modes. Another interesting feature is the splitting of the joint phase probability distribution into separate peaks. This suggests that the quantum state of the field develops into superpositions of distinguishable states with a definite phase (or approaching close to such superpositions). At later stages of the evolution the phases of the two modes are randomized, but this process goes through the appearance of more and more peaks in the distribution, which eventually leads to the uniform phase distribution. The competition between induced and spontaneous processes in the second-harmonic generation manifests itself clearly in the phase properties of the field.

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References

[1] P.A. Franken, A.E. Hill, C.W. Peters and G. Weinreich, Phys. Rev. Lett. 7 (1961) 118.
[2] M. Kozierowski and R. Tanaś, Opt. Commun. 21 (1977) 229.
[3] L. Mandel, Opt. Commun. 42 (1982) 437.
[4] L. Wu, H.J. Kimble, J.L. Hall and H. Wu, Phys. Rev. Lett. 57 (1986) 2520.
[5] S. Kielich, M. Kozierowski and R. Tanaś, Opt. Acta 32 (1985) 1023.
[6] S.P. Nikitin and A.V. Masalov, unpublished.
[7] B. Yurke and D. Stoler, Phys. Rev. Lett. 57 (1986) 13.
[8] A. Miranowicz, R. Tanaś and S. Kielich, Quantum Opt. 2 (1990) 253.
[9] Ts. Gantsog and R. Tanaś, J. Mod. Opt., in press.
[10] Ts. Gantsog and R. Tanaś, Quantum Opt., in press.
[11] D.T. Pegg and S.M. Barnett, Europhys. Lett. 6 (1988) 483.
[12] S.M. Barnett and D.T. Pegg, J. Mod. Opt. 36 (1989) 7.
[13] D.T. Pegg and S.M Barnett, Phys. Rev. A 39 (1989) 1665.
[14] J.A. Vaccaro and D.T. Pegg, Opt. Commun. 70 (1989) 529.
[15] C.C. Gerry, Opt. Commun. 75 (1990) 168.
[16] D.F. Walls and C.T. Tindle, Nuovo Cimento Lett. 2 (1971) 915; J. Phys. A 8 (1972) 534.
[17] J. Mostowski and K. Rzążewski, Phys. Lett. A 66 (1978) 275.
[18] A. Ekert and K. Rzążewski, Opt. Commun. 65 (1988) 225.