Quantum phase fluctuations in parametric down-conversion with quantum pump

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The effect of quantum fluctuations of the pump on the quantum phase properties of the signal mode in the parametric down-conversion process is considered. The Pegg–Barnett hermitian phase formalism is used to calculate the joint phase probability distribution and the phase variances of the two modes. The time evolution of the field state is obtained by means of numerical diagonalization of the interaction hamiltonian. The results are illustrated graphically and compared to those for the ideal squeezed state.

1. Introduction

The nonclassical nature of the fields produced by parametric down-conversion from an intense pump beam into signal and idler modes that are in the vacuum state is well known [1–10]. It is essential for the quantum properties of fields generated in the process that the high-frequency pump photons are split into highly correlated pairs of lower-frequency signal and idler photons. In the simplest case of a nondepleted degenerate parametric process, the pump mode is assumed as classical and nondepleted, and the signal and idler modes become one mode of the subharmonic field with half the frequency of the pump mode. The time evolution of the subharmonic field is described by a Bogoliubov transformation that maps the initial vacuum state into an ideal squeezed state [1–6]. Properties of such ideal squeezed states have been widely discussed [1–6] (for a review see, e.g., ref. [11]). The parametric down-conversion process turned out to be very effective in producing squeezed states in practice [12–17].

Recently, Vaccaro and Pegg [18] discussed quantum phase properties of the ideal squeezed states from the point of view of the hermitian phase formalism introduced by Pegg and Barnett [19–21]. It has been shown that for very large squeezing the ideal squeezed vacuum comes close to the superposition of two phase states. The phase probability density for the weakly squeezed vacuum has also been obtained and its properties discussed.

In this paper we discuss the phase properties of the subharmonic field taking into account quantum properties of the pump mode. This mode is treated as dynamical variable and its quantum mechanical time evolution is accounted for. The Pegg–Barnett [19–21] hermitian phase formalism is applied to find the joint phase probability distribution as well as the phase variances for both modes. We use the method of numerical diagonalization of the interaction Hamiltonian to obtain the time evolution of the system. The results for the joint phase probability distribution, the phase distribution for the signal (subharmonic) mode, the mean number of photons in both modes, and the phase variances are obtained numerically and illustrated graphically. The results are compared to the results for the ideal squeezed state. The randomization of the phases owing to quantum fluctuations of the pump is shown to take place in the long-time limit. A limit is imposed by such fluctuations on the values of the squeeze parameter that can be obtained in the ideal down-con-

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verter model; or, in other words, there are limits for such a model to work well.

2. State evolution

To describe the subharmonic generation process, the following model hamiltonian is used:

$$H=H_0+H_I$$
$$=\hbar\omega a^\dagger a+2\hbar\omega b^\dagger b+\hbar g(b^\dagger a^2+ba^{\dagger 2}) , \quad (1)$$

where a ($a^\dagger$) and b ($b^\dagger$) are the annihilation (creation) operators of the subharmonic mode of frequency ω and the pump mode of frequency 2ω, respectively. The coupling constant g, which is real, describes the coupling between the two modes. The hamiltonian (1) is, in fact, the same as for second harmonic generation. It is the initial conditions that distinguish the two processes. In the case of second harmonic generation, mode b is initially in the vacuum state and mode a is populated. For the subharmonic generation process considered in this paper, mode b is initially populated, while mode a is in the vacuum state. The distinction between the two processes is far from being trivial, and the states generated in the two processes have quite different properties. The nonclassical character of the states obtained in the subharmonic generation process has been discussed by Hillery [22].

Since H_0 and H_I commute, there are two constants of motion: H_0 and H_I. H_0 determines the total energy stored in both modes, which is conserved by the interaction H_I. This enables us to factor out $\exp(-iH_0t/\hbar)$ from the evolution operator, in fact to drop it altogether, and to write the resulting state of the field as

$$|\psi(t)\rangle=\exp(-iH_It/\hbar)|\psi(0)\rangle , \quad (2)$$

where $|\psi(0)\rangle$ is the initial state of the field. If the Fock states are used as basis states, the interaction hamiltonian H_I is not diagonal in such a basis. To find the state evolution, we apply a numerical method to diagonalize H_I. Such method has been used earlier for second harmonic generation [23,24].

In this paper, we consider the subharmonic generation process, which may be considered as a generalization of the parametric down-conversion process by accounting for the quantum properties of the pump mode. Thus, the initial state of the field is here taken as

$$|\psi(0)\rangle=\sum_{n=0}^{\infty} b_n|0,n\rangle , \quad (3)$$

where

$$b_n=\exp(-|\beta|^2/2)\ \beta^n/\sqrt{n!} \quad (4)$$

is the Poissonian weight factor of the coherent state $|\beta\rangle$ of the pump mode represented as a superposition of n-photon states. The state $|0,n\rangle=|0\rangle|n\rangle$ is the product of the Fock states with n photons in the pump mode and no photons in the subharmonic mode. That is, we assume the pump mode as being initially in a coherent state $|\beta\rangle$, and the subharmonic mode as being initially in the vacuum. With these initial conditions the resulting state (2) can be written as

$$|\psi(t)\rangle=\sum_{n=0}^{\infty} b_n \sum_{k=0}^{n} c_{2n,k}(t)|2k,n-k\rangle , \quad (5)$$

where the coefficients $c_{2n,k}(t)$ are given by

$$c_{2n,k}(t)=\langle 2k,n-k|\exp(-iH_It/\hbar)|0,n\rangle . \quad (6)$$

The coefficients $c_{2n,k}(t)$ given by eq. (6) are calculated numerically by diagonalizing the interaction hamiltonian H_I. This allows us to find the evolution of the state (5) and, in effect, its phase properties.

3. Phase properties of the field

In this section we will study the phase properties of the field produced in the course of subharmonic generation. To this end we use here the new Pegg–Barnett [19–21] phase formalism, which is based on introducing a finite $(s+1)$-dimensional subspace Ψ spanned by the number states $|0\rangle, |1\rangle, ..., |s\rangle$. The hermitian phase operator operates on this finite subspace, and after all necessary expectation values have been calculated in Ψ, the value of s is allowed to tend to infinity. A complete orthonormal basis of $(s+1)$ states is defined on Ψ as

$$|\theta_m\rangle\equiv(s+1)^{-1/2}\sum_{n=0}^{s}\exp(in\theta_m)|n\rangle , \quad (7)$$

where

$$\theta_m \equiv \theta_0 + 2\pi m/(s+1) \quad (m=0, 1, ..., s) . \tag{8}$$

The value of θ_0 is arbitrary and defines a particular basis set of $(s+1)$ mutually orthogonal phase states. The hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m| . \tag{9}$$

The phase states (7) are eigenstates of the phase operator (9) with the eigenvalues θ_m restricted to lie within the phase window between θ_0 and $\theta_0+2\pi$. The unitary phase operator $\exp(i\hat{\phi}_\theta)$ is defined as the exponential function of the hermitian operator $\hat{\phi}_\theta$. This operator acting on the eigenstate $|\theta_m\rangle$ gives the eigenvalue $\exp(i\theta_m)$, and can be written as [19–21]

$$\exp(i\hat{\phi}_\theta) \equiv \sum_{n=0}^{s-1} |n\rangle\langle n+1| + \exp[i(s+1)\theta_0]\,|s\rangle\langle 0| . \tag{10}$$

It is the last term in (10) that assures the unitarity of this operator. The first sum reporduces the Susskind–Glogower phase operator in the limit $s\to\infty$.

The expectation value of the phase operator (9) in a state $|\psi\rangle$ is given by

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \sum_{m=0}^{s} \theta_m |\langle\theta_m|\psi\rangle|^2 , \tag{11}$$

where $|\langle\theta_m|\psi\rangle|^2$ gives the probability of finding the phase state $|\theta_m\rangle$. The density of phase states is $(s+1)/2\pi$, so in the continuum limit, as s tends to infinity, we can write eq. (11) as

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta P(\theta)\,d\theta , \tag{12}$$

where the continuum phase distribution $P(\theta)$ is introduced by

$$P(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} |\langle\theta|\psi\rangle|^2 , \tag{13}$$

where θ_m has been replaced by the continuous phase variable θ. As the phase distribution function $P(\theta)$ is known, all the quantum mechanical phase expectation values can be calculated with this function in a classical-like manner. The choice of the value of θ_0 defines the 2π wide window of phase values.

In the case of subharmonic generation considered here, the state of the field (5) is in fact a two-mode state, and the phase formalism must be generalized to the two-mode case. This is straightforward and, for the state (5) we obtain

$$\langle\theta_{m_a}|\langle\theta_{m_b}|\psi(t)\rangle = (s_a+1)^{-1/2}(s_b+1)^{-1/2} \times \sum_{n=0}^{s_b} b_n \sum_{k=0}^{n} \exp\{-i[2k\theta_{m_a} + (n-k)\theta_{m_b}]\} c_{2n,k}(t) . \tag{14}$$

We use the indices a and b to distinguish between the subharmonic (a) and pump (b) modes. There is still a freedom of choice in (14) of the values of $\theta_0^{a,b}$, which define the phase window. We can choose these values at will, so we take them as

$$\theta_0^{a,b} = \varphi_{a,b} - \pi s_{a,b}/(s_{a,b}+1) , \tag{15}$$

and we introduce the new phase values

$$\theta_{\mu_{a,b}} = \theta_{m_{a,b}} - \varphi_{a,b} , \tag{16}$$

where the new phase labels $\mu_{a,b}$ run between the values $-s_{a,b}/2$ and $s_{a,b}/2$ with unit step. This means that we symmetrized the phase windows for the subharmonic and pump modes with respect to the phases φ_a and φ_b, respectively.

On inserting (15) and (16) into (14), taking the modulus squared of (14) and taking the continuum limit by making the replacement

$$\sum_{\mu_{a,b}=-s_{a,b}/2}^{s_{a,b}/2} \frac{2\pi}{s_{a,b}+1} \to \int_{-\pi}^{\pi} d\theta_{a,b} , \tag{17}$$

we arrive at the continuous joint probability distribution for the continuous variables θ_a and θ_b, which has the form

$$P(\theta_a, \theta_b) = \frac{1}{(2\pi)^2} \left| \sum_{n=0}^{\infty} b_n \sum_{k=0}^{n} c_{2n,k}(t) \times \exp\{-i[2k\theta_a + (n-k)\theta_b + k(2\varphi_a - \varphi_b)]\} \right|^2 . \tag{18}$$

The distribution (18) is normalized such that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P(\theta_a, \theta_b)\,d\theta_a\,d\theta_b = 1 . \tag{19}$$

To fix the phase windows for θ_a and θ_b, we have to assign to φ_a and φ_b particular values. Formula (18) depends on the phase difference $2\varphi_a - \varphi_b$ only. This is the classical phase relation for the parametric amplifier, and we take this value equal to $2\varphi_a - \varphi_b = \pi/2$. Since we assume the phase of the initial coherent state of the pump mode as zero (β real), we additionally assume $\varphi_b = 0$.

The joint phase probability distribution (18) can be evaluated numerically if the mean number of photons $N_b = |\beta|^2$ of the pump mode is not too great. The results are presented in fig. 1, where the function $P(\theta_a, \theta_b)$ is plotted in three-dimensional format for various values of the dimensionless evolution time gt. Initially, the distribution is peaked at $\theta_b = 0$ in the θ_b direction, reproducing the phase distribution of the coherent state of the pump mode, and it is completely flat in the θ_a direction, representing the uniform phase distribution of the vacuum for the subharmonic mode. As time elapses two phase peaks start to grow in the θ_a direction, suggesting the appearance of a superposition of two states in the resulting field. For later times the number of peaks in the distribution increases, and for very long times the number of peaks becomes large, making the phase distribution practically uniform.

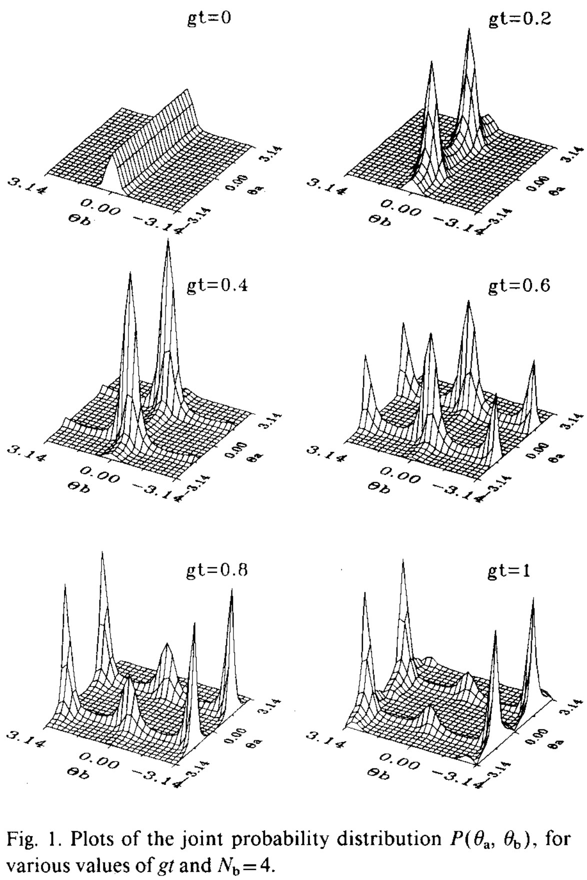

Fig. 1. Plots of the joint probability distribution $P(\theta_a, \theta_b)$, for various values of gt and $N_b = 4$.

The two peaks that appear at the beginning of the evolution correspond, in fact, to the two-peak phase distribution of the ideal squeezed states, which was indicated by Vaccaro and Pegg [18]. Here, however, we deal with the joint probability distribution for the phases θ_a and θ_b of the signal and pump mode, rather than with the phase distribution of the signal mode alone. A broadening of the phase distribution in the pump mode is clearly visible from the pictures. Our approach explicitly takes into account the quantum phase properties of the pump mode. It is interesting that in the course of the evolution the number of peaks increases in the direction θ_b (the pump mode phase), but the two-peak symmetry is preserved in the θ_a direction (the signal mode phase).

When integration of $P(\theta_a, \theta_b)$ over one of the phases is performed, the marginal phase distributions $P(\theta_a)$ or $P(\theta_b)$ for the phase θ_a or θ_b is obtained. For example,

$$P(\theta_a) = \int_{-\pi}^{\pi} P(\theta_a, \theta_b)\, d\theta_b \,. \tag{20}$$

The marginal distribution $P(\theta_a)$ for the signal mode can be compared to the corresponding phase distribution for the ideal squeezed state. At the initial stages of the evolution the two distributions are hardly distinguishable, but at later times, when the quantum character of the pump mode becomes essential, they differ considerably. An example is shown in fig. 2, where both distributions are compared for the time $gt = 1$ (this corresponds to the squeeze parameter $r = 2\sqrt{N_b}\, gt = 1$). For numerical reasons we have taken here for the mean number of photons of the pump mode the value of $N_b = 0.25$. It is clear, however, that as r increases the two peaks of the phase distribution for the ideal squeezed state become narrower, while the quantum fluctuations of the pump mode cause a broadening of the phase distribution for the signal mode. So, the quantum fluctuations of the pump mode impose a limit on the values of the

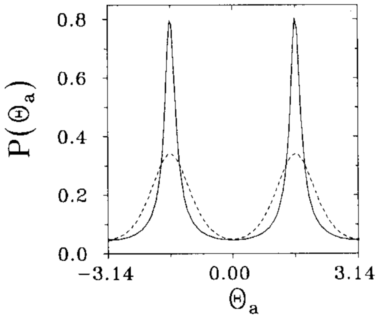

Fig. 2. The phase probability distribution $P(\theta_a)$ for the signal mode: the ideal squeezed state ($r=1$): solid line: the present case ($gt=1$) dashed line. $N_b=0.25$ in both cases.

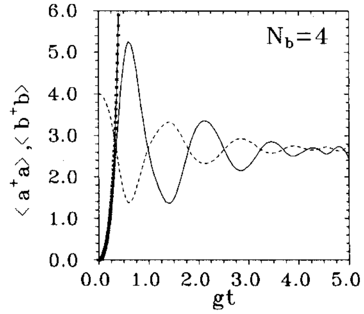

Fig. 3. The time evolution of the mean number of photons: the signal mode: solid line; the pump mode: dashed line; and the ideal down-converter: star line.

squeezing parameter r that can be obtained in real physical situations.

It is well known that for the squeezed vacuum the mean number of photons of the signal mode is equal to $\langle a^\dagger a\rangle = \sinh^2 r$, which is a monotonic function of the squeeze parameter r (or the evolution time gt). In fig. 3 we show the solutions for the mean numbers of photons for the signal and the pump modes for our case of quantized pump. The quastum solution is oscillatory, in contrast to the solution for the ideal down-converter with classical, nondepleted pump, which is shown, for comparison, by the star line. As far as the mean number of photons $\langle a^\dagger a\rangle$ increases, the two solutions are practically indistinguishable, but when the oscillations start, they become quite different. Thus, we can say that the first maximum in $\langle a^\dagger a\rangle$ sets a limit for the values of gt, or the squeeze parameter r, for which the model of ideal down-converter works well.

From the point of view of the quantum phase fluctuations considered here, we can calculate the evolution of the phase variance according to the formula

$$\langle (\Delta\hat{\phi}_{\theta_a})^2\rangle = \langle \hat{\phi}^2_{\theta_a}\rangle - \langle \hat{\phi}_{\theta_a}\rangle^2$$

$$= \int_{-\pi}^{\pi} \theta_a^2 P(\theta_a)\, d\theta_a , \qquad (21)$$

where $P(\theta_a)$ is the marginal phase distribution for the signal mode given by eq. (20). After changing the index a into b in eq. (21), the formula for the phase variance of the pump mode is obtained. According to (18) and (20), the variance (21) can be expressed in terms of the coefficients $c_{2n,k}(t)$ that are calculated numerically to obtain the evolution of the phase variances. The results are illustrated in fig. 4. It is seen that the phase variance of the signal mode starts from the value $\pi^2/3$ (the vacuum state value), goes to a minimum, and after a few oscillations comes again close to $\pi^2/3$. For comparison, the phase variance for the ideal squeezed state is shown (the line with stars). The two variances are initially indistinguishable, but the phase variance for the ideal squeezed state monotonically approaches its asymptotic value $\pi^2/4$, while for the quantum pump case the phase variance of the signal (subharmonic) mode starts to oscillate at later times. This confirms the statement that there is a limit imposed by the quantum fluctuations of the pump on the applicability of the ideal down-converter model. The phase variance of the pump mode rapidly increases from its initial value for the coherent state, and also shows oscillatory behaviour approaching the value $\pi^2/3$ at the

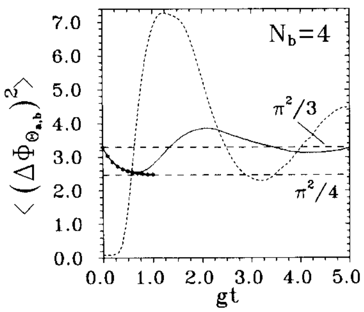

Fig. 4. Plots of the phase variances: the signal mode: solid line; the pump mode: dashed line; and the ideal squeezed state: star line. $N_b=4$.

long-time limit. Thus, the long-time effect of the quantum fluctuations of the pump mode is a randomization of the phase distribution for both signal and pump modes. The route to this randomization is through a sequence of more and more peaks in the joint distribution $P(\theta_a, \theta_b)$, as is already seen from fig. 1.

4. Conclusions

Our aim in this paper was to study the effect of quantum fluctuations of the pump mode on the quantum phase properties of light generated in the down-conversion process. We have applied the new Pegg–Barnett [19–21] phase formalism to find the joint phase distribution function $P(\theta_a, \theta_b)$ for the phases of the signal (θ_a) and the pump (θ_b) modes. It has been shown that initially this function has two peaks, and the marginal probability distribution $P(\theta_a)$ for the signal mode has also two peaks and is indistinguishable from the phase distribution for the ideal squeezed state. For longer evolution times, however, the two distributions differ essentially. The phase properties of the ideal squeezed states have recently been discussed by Vaccaro and Pegg [18], and by Grønbech-Jensen et al. [25]. Our calculations show that quantum fluctuations of the pump mode set a limit on the squeeze parameter r that can be obtained in practice. Of course, there are other factors, like damping, that we ignored in our considerations, but which will affect the results, especially in the long-time limit. We have shown here that even for a unitary evolution with the quantized pump mode the phase properties of the resulting state of the field differ considerably from those of the ideal squeezed state. The quantum fluctuations of the pump mode lead eventually to a randomization of the phases of both signal and pump modes. There is, however, an interval of time for which the quantum fluctuations of the pump do not affect essentially the properties of the signal mode, and the ideal down-converter model works well.

For numerical reasons, our calculations have been performed for mean values of photons of the pump mode that are rather small. However, the value $N_b=4$ leads to results that already have some features of the solutions for $N_b \gg 1$ (classical limit) while, on the other hand, the quantum properties are still clearly visible.

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