Full length article

Phase properties of second harmonics generated by different initial fields

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The joint probability distribution $P(\theta_a, \theta_b)$ for the phases θ_a and θ_b of the fundamental and second-harmonic modes is calculated, according to the Pegg-Barnett hermitian phase formalism, for different initial states of the fundamental mode undergoing second harmonic generation. The evolution of $P(\theta_a, \theta_b)$ is shown for four different initial states of the field (coherent, squeezed vacuum, Fock, and chaotic). Phase properties of the resulting fields are shortly discussed.

1. Introduction

A second harmonic generation is a nonlinear optical process that was observed in the early days of lasers [1]. In the quantum picture of the process we deal here with a nonlinear process in which two photons are annihilated and one photon with doubled frequency is created. The quantum states of the field generated in the process have a number of unique quantum features such as photon antibunching [2] and squeezing [3,4] for both fundamental and second-harmonic modes (for a review and extensive literature see ref. [5]). Recently, Ekert and Rzążewski [6] have discussed the dependence of the second harmonic intensity upon the statistical properties of the fundamental mode. They have considered four cases of the initial state of the fundamental mode: Fock state, coherent state, chaotic state, and squeezed vacuum state. The squeezed vacuum state appeared to be the most efficient in the production of the second harmonic light under assumption that the mean number of initial photons is the same for all cases.

When the intensity of the second-harmonic mode is studied, the phase information carried by the state of the initial field is irrelevant, and it is sufficient to know its photon number distribution. However, the second harmonic generation is a phase-sensitive process, and the initial phase information is nonlinearly transformed during the process, leading to definite phase properties of the outgoing field. The hermitian phase formalism introduced recently by Pegg and Barnett [7–9] allows for tracing such nonlinear phase transformations and getting quantitative information about the phase properties of the outgoing field.

In this paper we shall study the phase properties of the field produced in the second harmonic generation process taking into account the phase properties of the initial state of the field. The joint phase probability distribution of the outgoing field is calculated and illustrated graphically for all four cases of the initial state of the fundamental mode considered in ref. [6]. It is shown that the phase of the second-harmonic mode is locked to the phase of the fundamental mode by the classical phase-matching condition, but in quantum description the phases are subject to quantum fluctuations and their phase distributions have finite widths. Owing to the quantum fluctuations the phase distribution is eventually randomized. It is also shown that in the course of the evolution a degree of correlation builds up between the phases of the two modes, which leads to the appearance of the modulation structure even for initially completely flat phase distribution of the Fock

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state. Our numerical calculations are based on the method of numerical diagonalization of the interaction hamiltonian [6,10,11], which enables us to find the evolution of the system.

2. Quantum evolution of the field

The second harmonic generation can be described by the following model hamiltonian

$$H=H_0+H_I$$
$$=\hbar\omega a^\dagger a+2\hbar\omega b^\dagger b+\hbar g(b^\dagger a^2+ba^{\dagger 2}) , \quad (1)$$

where a ($a^\dagger$) and b ($b^\dagger$) are the annihilation (creation) operators of the fundamental mode of frequency ω and the second-harmonic mode of frequency 2ω, respectively. The coupling constant g, which is assumed as real and positive, describes the coupling between the two modes. Since H_0 and H_I commute, there are two constants of motion: H_0 and H_I. H_0 determines the total energy stored in both modes, which is conserved by the interaction H_I. Thus, the free evolution operator $\exp(-iH_0t/\hbar)$ can be factored out from the evolution operator, and in fact dropped. The resulting state of the field is then given by

$$|\Psi(t)\rangle=\exp(-iH_It/\hbar)\;|\Psi(0)\rangle , \quad (2)$$

where $|\Psi(0)\rangle$ is the initial state of the field. The interaction hamiltonian H_I is not diagonal in the Fock-state basis, and to find the state evolution we apply here the method of numerical diagonalization of H_I, which has already been used several times in the context of second harmonic generation [6,10,11].

If the initial state of the second-harmonic mode is the vacuum state, the initial state $|\Psi(0)\rangle$ can be written as

$$|\Psi(0)\rangle=\sum_{n=0}^{\infty} b_n|n,0\rangle , \quad (3)$$

where the state $|n,0\rangle=|n\rangle|0\rangle$ is the Fock state with n photons in the fundamental mode and no photons in the second harmonic. The amplitudes $b_n=\langle n|\Psi_a\rangle$ in the decomposition of the initial state $|\Psi_a\rangle$ of the fundamental mode are so far not specified, and will be specified later on for some particular states $|\Psi_a\rangle$ we are going to consider. With the initial state of the field given by (3) the resulting state (2) takes the form

$$|\Psi(t)\rangle=\sum_{n=0}^{\infty} b_n\sum_{k=0}^{[n/2]} c_{nk}(t)\;|n-2k,k\rangle , \quad (4)$$

where the coefficients $c_{nk}(t)$ are given by

$$c_{nk}(t)=\langle n-2k,k|\;\exp(-iH_It/\hbar)\;|n,0\rangle . \quad (5)$$

The summation over k runs up to the integer part of $n/2$, where n denotes the number of photons of the fundamental mode, while k is the number of photons created in the second harmonic mode. The coefficients $c_{nk}(t)$, given by eq. (5), are calculated numerically with the method of diagonalization of H_I. This enables us to find the field evolution and, consequently, its phase properties.

3. Phase properties of the field

Here, we use the hermitian phase formalism of Pegg and Barnett [7–9] to describe the phase properties of the field produced in the second harmonic generation process. This formalism is based on introducing a finite $(s+1)$-dimensional space Ψ spanned by the number states $|0\rangle$, $|1\rangle$, ..., $|s\rangle$. The hermitian phase operator operates on this finite space, and after all necessary expectation values have been calculated in Ψ, the value of s is allowed to tend to infinity. A complete orthonormal basis of $(s+1)$ states is defined on Ψ as

$$|\theta_m\rangle\equiv\frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}\exp(in\theta_m)|n\rangle , \quad (6)$$

where

$$\theta_m\equiv\theta_0+\frac{2\pi m}{s+1}, \quad (m=0,1,...,s) . \quad (7)$$

The value of θ_0 is arbitrary and defines a particular basis set of $(s+1)$ mutually orthogonal phase states. The hermitian phase operator is defined as

$$\hat{\phi}_\theta\equiv\sum_{m=0}^{s}\theta_m|\theta_m\rangle\langle\theta_m| . \quad (8)$$

The phase states (6) are eigenstates of the phase operator (8) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and $\theta_0+2\pi$. The unitary phase operator $\exp(\mathrm{i}\hat{\phi}_\theta)$ is defined as the exponential function of the hermitian operator $\hat{\phi}_\theta$. This operator acting on the eigenstate $|\theta_m\rangle$ gives the eigenvalue $\exp(\mathrm{i}\theta_m)$, and can be written as [7–9]

$$\exp(\mathrm{i}\hat{\phi}_\theta) \equiv \sum_{n=0}^{s-1} |n\rangle\langle n+1| + \exp[\mathrm{i}(s+1)\theta_0]\,|s\rangle\langle 0| \,. \tag{9}$$

This is the last term in (9) that assures the unitarity of this operator. The first sum reproduces the Susskind–Glogower phase operator in the limit $s\to\infty$.

The expectation value of the phase operator (8) in a state $|\psi\rangle$ is given by

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \sum_{m=0}^{s} \theta_m\,|\langle\theta_m|\psi\rangle|^2 \,, \tag{10}$$

where $|\langle\theta_m|\psi\rangle|^2$ gives the probability of being found in the phase state $|\theta_m\rangle$. The density of phase states is $(s+1)/2\pi$, so in the continuum limit as s tends to infinity, we can write eq. (10) as

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta P(\theta)\,\mathrm{d}\theta \,, \tag{11}$$

where the continuum phase distribution $P(\theta)$ is introduced by

$$P(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi}\,|\langle\theta|\psi\rangle|^2 \,, \tag{12}$$

where θ_m has been replaced by the continuous phase variable θ. As the phase distribution function $P(\theta)$ is known, all the quantum mechanical phase expectation values can be calculated with this function in a classical-like manner. The choice of the value of θ_0 defines the 2π range window of the phase values.

In our case of second harmonic generation, the state of the field (4) is in fact a two-mode state, and the generalization of the phase formalism into the two-mode case gives, for the state (4), the result

$$\langle\theta_{m_a}|\langle\theta_{m_b}|\psi(t)\rangle = [(s_a+1)(s_b+1)]^{-1/2} \times \sum_{n=0}^{s_a} b_n \sum_{k=0}^{[n/2]} \exp\{-\mathrm{i}[(n-2k)\theta_{m_a}+k\theta_{m_b}]\}\,c_{nk}(t) \,, \tag{13}$$

where we have used the indices a and b to distinguish between the fundamental (a) and second-harmonic (b) mode. There is still a freedom of choice in (13) of the values of $\theta_0^{a,b}$, which define the phase values window. We have chosen these values as

$$\theta_0^{a,b} = \varphi_{a,b} - \pi s_{a,b}/(s_{a,b}+1) \,, \tag{14}$$

and we have introduced the new phase values

$$\theta_{\mu_{a,b}} = \theta_{m_{a,b}} - \varphi_{a,b} \,, \tag{15}$$

where the new phase labels $\mu_{a,b}$ run in unit step between the values $-s_{a,b}/2$ and $s_{a,b}/2$. This means that we have symmetrized the phase windows for the fundamental and second-harmonic modes with respect to the phases φ_a and φ_b, respectively. On inserting (14) and (15) into (13), taking the modulus square of (13), and performing the continuum limit transition, we arrive at the continuous joint probability distribution for the continuous phase variables θ_a and θ_b, which has the form

$$P(\theta_a,\theta_b) = \frac{1}{(2\pi)^2}\left|\sum_{n=0}^{\infty} b_n \exp(-in\varphi_a) \times \sum_{k=0}^{[n/2]} \exp\{-\mathrm{i}[(n-2k)\theta_a + k\theta_b - k(2\varphi_a-\varphi_b)]\}\,c_{nk}(t)\right|^2 . \tag{16}$$

The distribution (16) is normalized such that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P(\theta_a,\theta_b)\,\mathrm{d}\theta_a\,\mathrm{d}\theta_b = 1 \,. \tag{17}$$

To choose the phase windows for θ_a and θ_b, we have to assign to φ_a and φ_b particular values. It is interesting to notice that formula (16) depends, in fact, on the difference $2\varphi_a-\varphi_b$, which reproduces the classical phase synchronism relation for the second harmonic generation. Classically, if there is no second harmonic initially, this quantity must be $2\varphi_a-\varphi_b=\pm\pi/2$. This means that the phase of the second harmonic is locked to the phase of the fundamental mode by this relation. It turns out that this is also a good choice to fix the phase windows in the quantum description. If the initial phase φ_a of the fundamental mode is zero then $\varphi_b=\pm\pi/2$, i.e., the second harmonic is shifted in phase by $\pi/2$ or $-\pi/2$ with respect to the fundamental mode.

The joint probability distribution (16) is plotted in figs. 1–4 to show the evolution of the phase properties of light during the second harmonic generation with various initial states of the fundamental mode. The initial states for the fundamental mode are chosen as

(a) Coherent state, for which we take

$$b_n = \exp(-|\alpha|^2/2)\,\alpha^n/\sqrt{n!} \qquad (18)$$

with α real ($\varphi_a = 0$) and $2\varphi_a - \varphi_b = \pi/2$.

(b) Squeezed vacuum state, for which

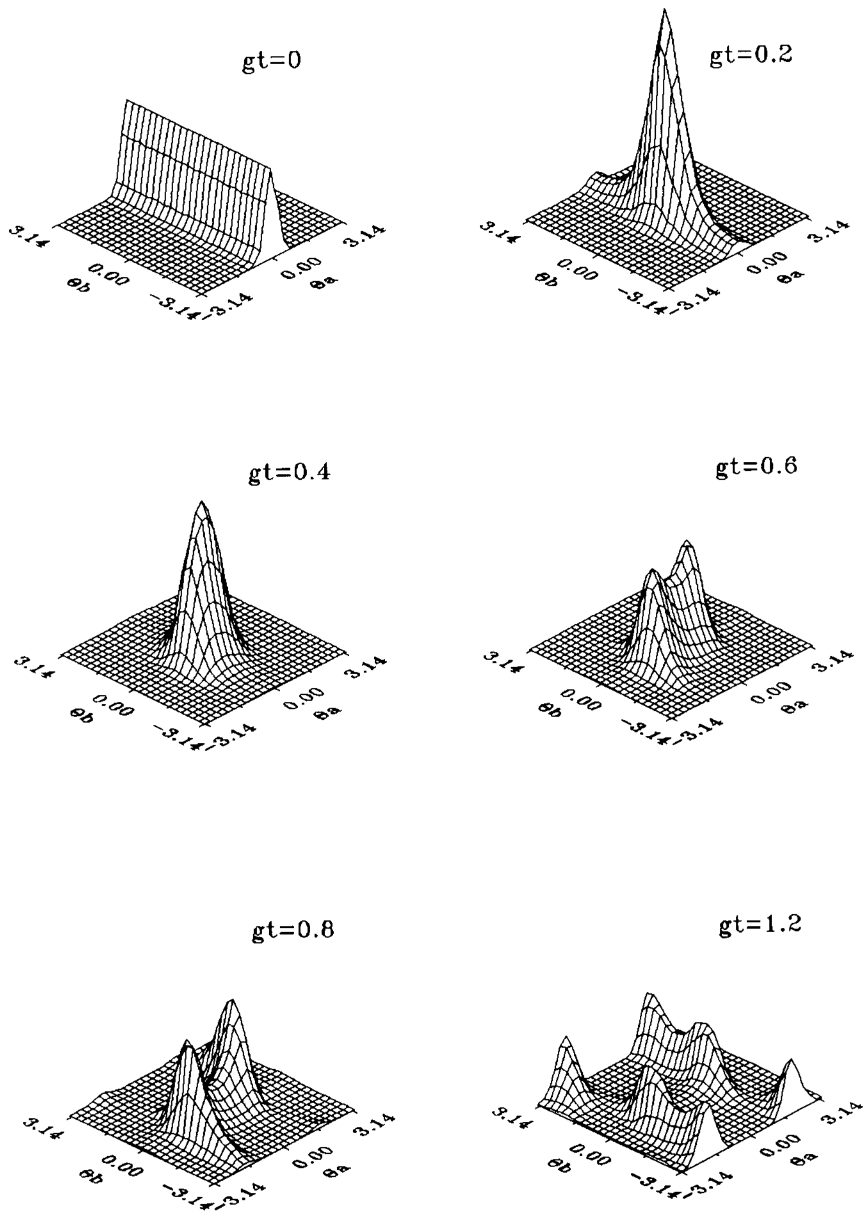

Fig. 1. The evolution of the joint probability distribution $P(\theta_a, \theta_b)$ for the initial state of the fundamental mode being the coherent state with the mean number of photons $N_a = 4$.

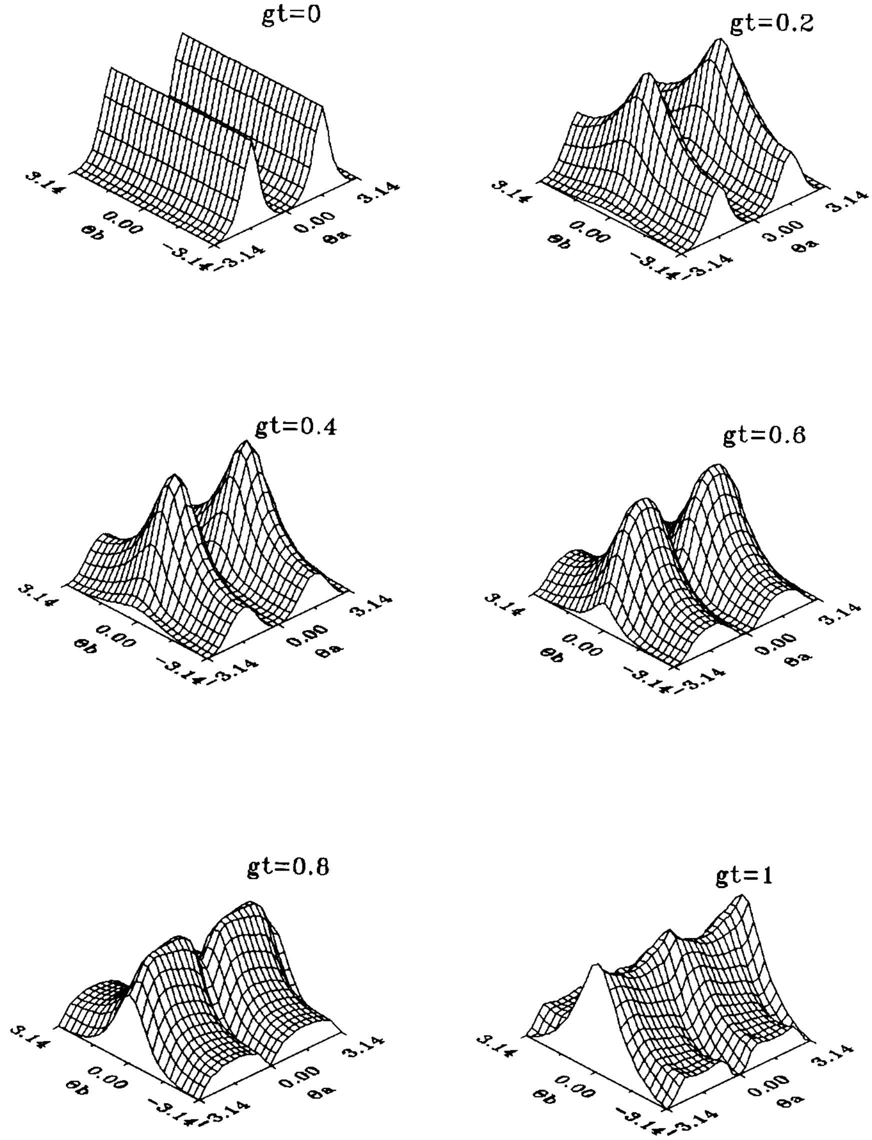

Fig. 2. Same as fig. 1, but for the squeezed vacuum state with $r=0.5$ initially.

$$b_n = \frac{(-1)^{n/2}}{\sqrt{\cosh r}} \frac{\sqrt{n!}}{(n/2)!} \times (\tfrac{1}{2}\tanh r)^{n/2} \exp(\mathrm{i}n\eta), \quad (n\text{ even}) \qquad (19)$$
$$= 0, \qquad (n\text{ odd})$$

with $\eta = \varphi_a = 0$ and $2\varphi_a - \varphi_b = -\pi/2$.

(c) Fock state, for which

$$b_n = \delta_{n,n'}, \qquad (20)$$

and $2\varphi_a - \varphi_b = \pi/2$.

(d) Chaotic light, which is described by the density matrix diagonal in the n-photon states with the photon number distribution given by

$$p(n) = \frac{1}{N_a + 1} \left(\frac{N_a}{N_a + 1} \right)^n, \qquad (21)$$

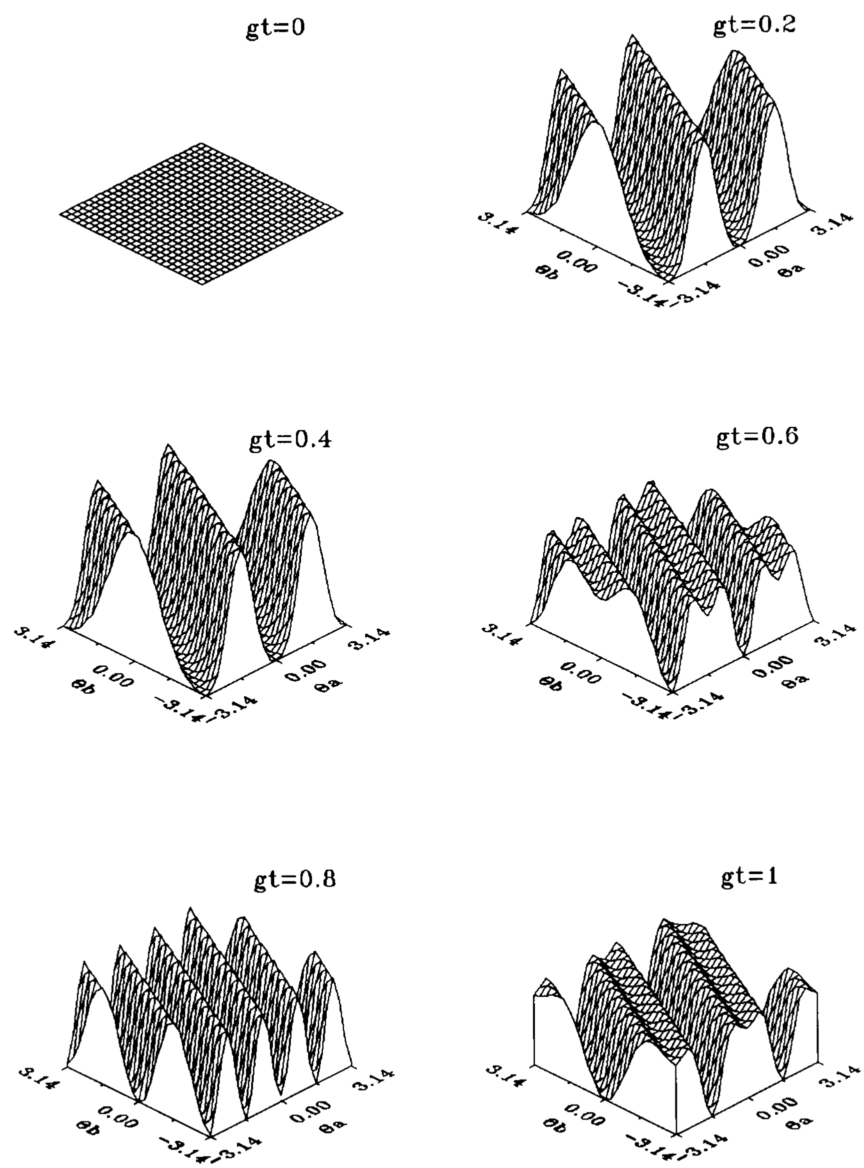

Fig. 3. Same as fig. 1, but for the Fock state with $n=4$ photons initially (the scaling factor in z-axis is 80).

where N_a is the mean number of photons, and $2\varphi_a-\varphi_b=\pi/2$.

In fig. 1 the evolution of the joint probability distribution $P(\theta_a, \theta_b)$ is shown for the initially coherent state $|\alpha\rangle$ of the fundamental mode with the mean number of photons $N_a=|\alpha|^2=4$. It is seen that a maximum in the distribution along the θ_b axis builds up at initial stages of the evolution. This means that the second harmonic is generated with definite mean phase. In fact this phase is $\varphi_b=-\pi/2$, i.e., it is shifted by $-\pi/2$ with respect to the phase $\varphi_a=0$ of the fundamental mode. Such phase relation is in accordance with a classical picture of second harmonic generation. In the quantum picture, however, the phase distribution has a finite width. The narrowing of the distribution along the θ_b axis is evident at early stages

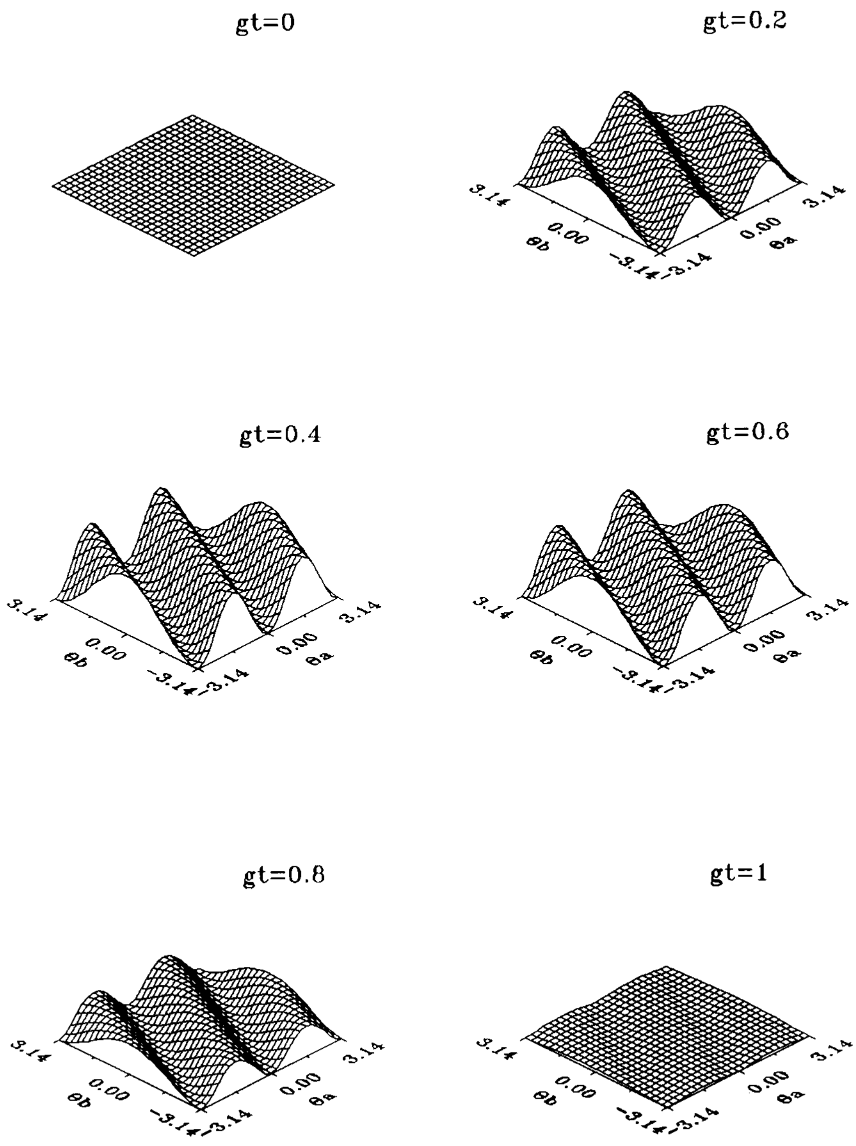

Fig. 4. Same as fig. 1, but for chaotic light with $N_a = 0.27$ initially (the scaling factor in z-axis is 1000).

of the evolution, but at later times the phase distribution splits into two, and later even more, peaks. Although the distribution becomes more complex at later times, there is the symmetry with respect to the point $\theta_a = \theta_b = 0$, which is preserved all the time. Eventually, owing to quantum fluctuations, both phases are randomized, i.e., the distribution $P(\theta_a, \theta_b)$ becomes more and more uniform after appearance of more and more peaks. It is important to notice that there is a limit for the reduction of quantum phase fluctuations in the second-harmonic mode. Quantitatively such fluctuations can be described by the phase variance. More detailed discussion of the second harmonic generation by coherent light will be given elsewhere, and here we want to compare the joint phase distributions obtained for different initial states.

In fig. 2 the evolution of the distribution $P(\theta_a, \theta_b)$

is illustrated assuming the initial state of the fundamental mode being a squeezed vacuum state. The amplitudes of the squeezed vacuum state are given by (19) with the phase $\eta=\varphi_a=0$, and the phase window is fixed by the relation $2\varphi_a-\varphi_b=-\pi/2$. This means that the second harmonic generated by such state is shifted by $\varphi_b=\pi/2$ with respect to $\eta=\varphi_a=0$. An immediate result seen from fig. 2, as compared to fig. 1, is that the number of peaks in the distribution is doublet. This is a consequence of the fact that the squeezed vacuum state has itself a two-peak structure, which is close to the superposition of two phase states [12]. The two-peak structure of the squeezed vacuum state is preserved during the evolution, and is reflected in the symmetry of the distribution with respect to the line $\theta_a=0$. This symmetry is different from that for the coherent initial state. The narrowing of the distribution along the θ_b axis is also visible at early stages of the evolution. We have assumed the squeeze parameter $r=0.5$ for this figure, which means the mean number of photons $N_a=\sinh^2 r=0.27$.

Both coherent and squeezed vacuum states are states that carry a definite phase information, which is transferred into the second-harmonic mode. It may be even more interesting to look at the phase distribution $P(\theta_a, \theta_b)$ of the resulting field, when the initial state is a Fock state or chaotic light. For such light we have initially completely flat phase distribution (uniform distribution). In fig. 3 the results are shown for the Fock state with the number of photons $n=4$. The phase window is chosen such that $2\varphi_a-\varphi_b=\pi/2$. A well resolved modulation structure of $P(\theta_a, \theta_b)$ appears at early times of the evolution and evolves into more complicated (but more uniform) structure at later times. This structure of the joint probability distribution is the result of the correlation between the phases of the two modes that builds up in the course of evolution. Initially uncorrelated modes become correlated during the second harmonic generation. This fact can be easily explained for the simplest possible case of second harmonic generation by the field in the Fock state with two photons. In this case there are only two different from zero coefficients $c_{nk}(t)$: $c_{20}(t)=\cos\sqrt{2}gt$, and $c_{21}(t)=-\mathrm{i}\sin\sqrt{2}gt$. This allows us to write down the analytical formula for the distribution $P(\theta_a, \theta_b)$. According to (16) and (20) we have

$$P(\theta_a, \theta_b) = (2\pi)^{-2}[1+\sin 2\sqrt{2}\,gt\cos(2\theta_a-\theta_b)]\,, \qquad (22)$$

where we have put $2\varphi_a-\varphi_b=\pi/2$. Formula (22) depends on the difference $2\theta_a-\theta_b$ rather than on the two phases separately. This is generally true for Fock states, and this is just the reason for which the structure of $P(\theta_a, \theta_b)$ has edges along the lines $2\theta_a-\theta_b=\text{const}$. From the point of view of the time evolution formula (22) is periodic, but the periodicity is lost for the states with $n>2$ (e.g. $n=4$ as in fig. 3). The phase correlation function can be calculated according to

$$C_{\theta_a\theta_b} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \theta_a\theta_b P(\theta_a, \theta_b)\,\mathrm{d}\theta_a\,\mathrm{d}\theta_b\,, \qquad (23)$$

which for $n=2$ and $2\varphi_a-\varphi_b=\pi/2$ gives

$$C_{\theta_a\theta_b} = -\tfrac{1}{2}\sin 2\sqrt{2}gt\,. \qquad (24)$$

It is now clear that this is the correlation between the phases θ_a and θ_b that is responsible for the appearance of the structure in $P(\theta_a, \theta_b)$ for Fock states. It is, moreover, worth to notice that the marginal distributions $P(\theta_a)$ and $P(\theta_b)$ obtained from $P(\theta_a, \theta_b)$ by integration over one of the phases are equal $1/(2\pi)$ i.e., either phase is uniformly distributed, and all single-mode phase properties do not change during the evolution.

In fig. 4 the results for the initial chaotic light are shown. The distribution $P(\theta_a, \theta_b)$ in this case is obtained by summing the n-photon results with the weight factor given by (21) with the mean number of photons $N_a=0.27$. This number corresponds to that taken for the squeezed vacuum state (fig. 2). Because of poor convergence of the series when $N_a>1$ in both cases, we take this number small for numerical reasons. The modulation of the phase distribution, which is smoother than for Fock states, is still maintained in this case. For small mean numbers of photons the behaviour is almost periodic for some time and, for $gt=1$, we have again almost uniform phase distribution. For very long times the randomization of the phases eventually takes place.

4. Conclusions

In this paper we have studied phase properties of light produced in the second harmonic generation process for different initial states of the field. The joint probability distribution for phases θ_a and θ_b of the fundamental and the second-harmonic modes has been calculated according to the hermitian phase formalism of Pegg and Barnett [7–9]. The method of numerical diagonalization of the interaction hamiltonian has been used for numerical evaluation of the phase distribution function. The results have been illustrated graphically for four different initial states of the field. It has been shown that the classical phase synchronism relation is also valid in the quantum case. However, quantum fluctuations lead to a finite width of the phase distribution, i.e., they impose a quantum limit on the uncertainty in measuring the phase. It has been shown that for the initial states with a definite phase information (coherent, or squeezed vacuum) this information is transferred into the second-harmonic mode. The differences between the phase distributions for coherent and squeezed vacuum states are clearly visible from the pictures. We have also shown that even for the second harmonic generation by the light with uniformly distributed phase there is a structure in $P(\theta_a, \theta_b)$, which arises as a result of correlation between the two phases appearing during the evolution. The long time limit effect of quantum fluctuations is the randomization of the phases.

The second harmonic generation discussed here is one of the nonlinear optical processes that transform the incoming field into the resulting field in a nonlinear way. Such transformation affects also the phase properties of the field. Other nonlinear processes lead to other phase properties of the resulting field, and some of them have already been discussed [12–15].

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