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# Dynamical properties of the field phase in the Jaynes–Cummings model

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**Abstract.** Phase properties of a coherent field interacting with a two-level atom in an ideal cavity are studied using the new hermitian phase formalism of Pegg and Barnett. It is shown in particular that phase properties of the field reflect the collapse and revival phenomena. The effects of finite detuning and atomic coherences are treated. The results for the variance of the phase cosine are obtained and compared with those based on earlier approaches.

## 1. Introduction

The Jaynes–Cummings model (JCM) [1] of a two-level atom interacting with a single mode of the electromagnetic field is one of the simplest non-trivial models of quantum optical resonance. While under the rotating wave approximation, the model is exactly soluble, and its behaviour is far from being simple. Many interesting features have been predicted for both the atomic variables and the statistical properties of the field (for a review see [2, 3]). Among them one finds the remarkable collapses and revivals of the Rabi oscillations [4]. New developments in cold cavity techniques with Rydberg state atoms have made laboratory realizations of the JCM possible and collapses and revivals of atomic inversion have been experimentally observed [5].

In this paper we examine another characteristic of the system—phase properties of the field. Our treatment is largely based on the recent works of Pegg and Barnett [6–8], in which they presented an elegant new theory for the phase operator of the radiation field. The theory of Pegg and Barnett is distinguished from previous theories in considering the phase operator itself rather than the conventional exponential or trigonometric function operators. It enables us to examine phase properties of optical fields in a fully quantum-mechanical manner without recourse to semiclassical or phenomenological methods.

In section 2 we outline what is essential for our present discussion from the new phase formalism of Pegg and Barnett. For the complete and thorough exposition of the theory the reader is referred to the literature [6–8]. In section 3 we use the Pegg–Barnett (PB) new formalism to study phase properties of a field resulting from interaction with an atom. We find that the phase distribution as well as the variance of the phase reflect the collapses and the revivals of atomic inversion. Next, we

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calculate the expectation values and variances for the cosine and sine functions of the phase operator. The results obtained are compared with those using the earlier Susskind–Glogower (SG) approach [9] and the measured phase (MP) concept [10]. In section 4 we investigate the effect of atomic coherence on phase properties of the field. The idea of injecting an atom initially prepared in a coherent superposition of its states into a cavity has become quite popular. Agarwal *et al.* [11, 12] and Zaheer and Zubairy [13] have shown that for a certain choice of atomic phase, the amplitude of the Rabi oscillations is strongly suppressed. In the case of spectra, for the same choice of phase, we have an asymmetric two-peaked spectrum instead of a three-peaked symmetric spectrum [13]. Squeezing in single-mode spontaneous emission from a suitably phased atom has been demonstrated by Knight and co-workers [14, 15]. It is therefore natural to expect that atomic coherence also alters the phase properties of the interacting field.

## 2. The hermitian phase formalism

The problem of optical field phase has been the subject of considerable study for many years [10, 16]. Despite this, many questions still remain unanswered. One of them concerns the construction of a hermitian phase operator. Difficulties in constructing a phase operator have even led to the belief that no such operator exists. Recently, Pegg and Barnett have shown a way out of this difficulty. Instead of utilizing the usual mathematical model of the single-mode electromagnetic field, they introduced a finite but arbitrarily large state space of $s+1$ dimensions. This space is spanned by the first $s+1$ number states, and the set of orthogonal phase states is defined by

$$|\theta_m\rangle=\frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}\exp(\mathrm{i}n\theta_m)|n\rangle, \tag{1}$$

with phases

$$\theta_m=\theta_0+\frac{2\pi m}{s+1}, \qquad m=0,1,2,\ldots,s. \tag{2}$$

All expectation values of phase variables are first calculated in the finite dimensional space before $s$ is allowed to tend to infinity. In this finite space, the hermitian phase operator is given by

$$\hat{\Phi}_\theta=\sum_{m=0}^{s}\theta_m|\theta_m\rangle\langle\theta_m|, \tag{3}$$

so that $\hat{\Phi}_\theta|\theta_m\rangle=\theta_m|\theta_m\rangle$. The hermiticity of $\hat{\Phi}_\theta$ guarantees the unitarity of the exponential phase operators $\exp(\pm\mathrm{i}\hat{\Phi}_\theta)$, and the cosine and sine combinations formed from the unitary phase operators then have properties coincident with those normally associated with phase. In the number state representation it is not difficult to show that

$$\exp(\mathrm{i}\hat{\Phi}_\theta)=|0\rangle\langle 1|+|1\rangle\langle 2|+\ldots+|s-1\rangle\langle s|+\exp[\mathrm{i}(s+1)\theta_0]|s\rangle\langle 0|. \tag{4}$$

We will be mainly concerned with physical states [8], for which energy is finite. Hence, the expectation value of the unitary phase operator $\exp(\mathrm{i}m\hat{\Phi}_\theta)$ takes the following simplified form [17]

$$\begin{aligned}\langle\exp(\mathrm{i}m\hat{\Phi}_\theta)\rangle_{\mathrm{p}}&=\left\langle\sum_{n=0}^{\infty}|n\rangle\langle n+m|\right\rangle_{\mathrm{p}},\\&=\langle\exp(\mathrm{i}m\hat{\Phi}_{\mathrm{SG}})\rangle_{\mathrm{p}},\end{aligned} \tag{5}$$

where the subscript p refers to a physical state expectation value and the abbreviation SG denotes the Susskind–Glogower approach. We can also easily obtain the relations between SG and hermitian-phase sine and cosine operators

$$\begin{aligned}\langle \cos \hat{\Phi}_\theta \rangle_p &= \tfrac{1}{2} \langle \exp(i\hat{\Phi}_\theta) + \exp(-i\hat{\Phi}_\theta) \rangle_p, \\ &= \langle \cos \hat{\Phi}_{SG} \rangle_p,\end{aligned} \tag{6}$$

$$\begin{aligned}\langle \sin \hat{\Phi}_\theta \rangle_p &= \frac{1}{2i} \langle \exp(i\hat{\Phi}_\theta) - \exp(-i\hat{\Phi}_\theta) \rangle_p, \\ &= \langle \sin \hat{\Phi}_{SG} \rangle_p,\end{aligned} \tag{7}$$

$$\begin{aligned}\langle \cos^2 \hat{\Phi}_\theta \rangle_p &= \tfrac{1}{4} \langle \exp(2i\hat{\Phi}_\theta) + \exp(-2i\hat{\Phi}_\theta) + 2 \rangle_p, \\ &= \langle \cos^2 \hat{\Phi}_{SG} \rangle_p + \tfrac{1}{4} \langle (|0\rangle\langle 0|) \rangle_p,\end{aligned} \tag{8}$$

$$\begin{aligned}\langle \sin^2 \hat{\Phi}_\theta \rangle_p &= -\tfrac{1}{4} \langle \exp(2i\hat{\Phi}_\theta) + \exp(-2i\hat{\Phi}_\theta) - 2 \rangle_p, \\ &= \langle \sin^2 \hat{\Phi}_{SG} \rangle_p + \tfrac{1}{4} \langle (|0\rangle\langle 0|) \rangle_p.\end{aligned} \tag{9}$$

Equations (8) and (9) indicate that the hermitian-phase and SG approaches give strongly different results only for fields in the quantum regime, where the contributions from the vacuum state are appreciable.

## 3. Phase properties of the cavity field

Jaynes and Cummings [1] considered a simple-model Hamiltonian characterizing the interaction of a two-level atom with a single mode quantized radiation field inside an ideal cavity

$$H = \hbar\omega_0 R^z + \hbar\omega a^+ a + \hbar g (R^+ a + R^- a^+), \tag{10}$$

where the two-level system has been represented by the spin $\frac{1}{2}$ operators $R^\pm$ and $R^z$, $g$ is the atom-field coupling constant, $a^+$ and $a$ are the usual creation and annihilation operators of a photon in the cavity eigenmode with frequency $\omega$. The Hamiltonian equation (10) has two obvious constants of motion [2, 4]

$$H_I = \hbar\omega (a^+ a + R^z), \tag{11}$$

$$H_{II} = \hbar g (R^+ a + R^- a^+) + \hbar \Delta\omega R^z, \tag{12}$$

with $\Delta\omega$ being the field–atom frequency difference

$$\Delta\omega = \omega_0 - \omega \equiv \Delta, \tag{13}$$

and we will refer to $\Delta$ as the detuning parameter. One can verify directly that

$$H = H_I + H_{II}, \tag{14}$$

$$[H_I, H_{II}] = 0. \tag{15}$$

The mutual commutability of $H_I$ and $H_{II}$ leads to the factorization for the time translation operator $U(t)$

$$U(t) = \exp(-iHt/\hbar) = U_I(t)\, U_{II}(t), \tag{16}$$

where

$$U_I(t) = \exp(-iH_I t/\hbar), \qquad U_{II}(t) = \exp(-iH_{II} t/\hbar). \tag{17}$$

As has been pointed out in [2], we can interpret the operator $U_{\rm I}(t)$ as the quantized field version of the semiclassical unitary transformation operator from the laboratory frame to the rotating frame of reference and the operator $U_{\rm II}(t)$ describes the time development of the system in the latter frame. We therefore drop $U_{\rm I}(t)$ altogether later on and will work then in the intermediate II-picture introduced by Yoo and Eberly [2]. This picture coincides with the usual interaction picture when the exact resonance condition is met. Now, one can diagonalize the Hamiltonian $H_{\rm II}$ with the results

$$
\begin{aligned}
&H_{\rm II}|0;g\rangle = -\frac{\hbar\Delta}{2}|0;g\rangle,\\
&H_{\rm II}|\phi_n^{\pm}\rangle = \pm\hbar\lambda_n|\phi_n^{\pm}\rangle,\\
&\qquad \lambda_n^2 = g^2 n + \Delta^2/4,\\
&|\phi_n^{\pm}\rangle = \begin{pmatrix}\cos\varphi_n\\ -\sin\varphi_n\end{pmatrix}|n;g\rangle + \begin{pmatrix}\sin\varphi_n\\ \cos\varphi_n\end{pmatrix}|n-1;e\rangle,\\
&\qquad n = 1, 2, \ldots, \infty,\\
&\qquad tg\varphi_n = g\sqrt{n}/(\lambda_n - \Delta/2).
\end{aligned}
\tag{18}
$$

Since the eigenvalues and the eigenfunctions are known in closed form, all the dynamical questions can be answered. Our further discussion is restricted to the case of an initially coherent field

$$
|\alpha\rangle = \exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{19}
$$

the phase properties of which are well known [8]. We consider in this section the atom to be initially in the ground state; the initial state of the total system is then

$$
|\phi(0)\rangle = \exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n;g\rangle. \tag{20}
$$

From equations (17), (18) and (20) one readily finds the state of the system at time $t$

$$
\begin{aligned}
|\phi(t)\rangle &= U_{\rm II}(t)|\phi(0)\rangle\\
&= \sum_{n=0}^{\infty} b_n e^{in\beta}\big[(\cos^2\varphi_n e^{-{\rm i}\lambda_n t} + \sin^2\varphi_n e^{{\rm i}\lambda_n t})|n;g\rangle\\
&\quad + \sin\varphi_n\cos\varphi_n(e^{-{\rm i}\lambda_n t} - e^{{\rm i}\lambda_n t})|n-1;e\rangle\big],
\end{aligned}
\tag{21}
$$

where we have introduced the notations

$$
\begin{aligned}
\alpha &= (\bar{n})^{1/2}e^{{\rm i}\beta},\\
b_n &= \exp(-\bar{n}/2)\frac{(\bar{n})^{n/2}}{\sqrt{n!}}.
\end{aligned}
\tag{22}
$$

With these preliminary steps in hand, it is now fairly easy to arrive at the following expression for the phase distribution

$$
\begin{aligned}
|\langle\theta_m|\phi(t)\rangle|^2 = \frac{1}{s+1}\Big\{1 &+ 2\sum_{n>k} b_n b_k A_{n,k}(t)\cos[(n-k)(\theta_m-\beta)]\\
&+ 2\sum_{n>k} b_n b_k B_{n,k}(t)\sin[(n-k)(\theta_m-\beta)]\Big\},
\end{aligned}
\tag{23}
$$

with $A_{n,k}(t)$ and $B_{n,k}(t)$ given by

$$A_{n,k}(t)=\cos(\lambda_n t)\cos(\lambda_k t)+\cos[2(\varphi_n-\varphi_k)]\sin(\lambda_n t)\sin(\lambda_k t), \tag{24}$$

$$B_{n,k}(t)=\cos(2\varphi_k)\sin(\lambda_k t)\cos(\lambda_n t)-\cos(2\varphi_n)\sin(\lambda_n t)\cos(\lambda_k t), \tag{25}$$

where

$$\cos(2\varphi_n)=\frac{-\Delta/2}{(g^2 n+\Delta^2/4)^{1/2}}, \qquad \sin(2\varphi_n)=\frac{g\sqrt{n}}{(g^2 n+\Delta^2/4)^{1/2}}. \tag{26}$$

Now we make the particular choice of $\theta_0$ as

$$\theta_0=\beta-\frac{\pi s}{s+1}, \tag{27}$$

so that from equation (2)

$$\theta_m=\beta+\frac{2\pi\mu}{s+1}, \tag{28}$$

where $\mu\equiv m-s/2$ is a new phase label which goes in integer steps from $-s/2$ to $s/2$. The phase distribution (equation (23)) then becomes symmetric in $\mu$. In the limit, when $s$ tends to infinity, the continuous phase variable can be introduced replacing $\mu 2\pi/(s+1)$ by $\theta$ and $2\pi/(s+1)$ by $\mathrm{d}\theta$. This leads to the continuous phase probability distribution

$$P(\theta,t)=\frac{1}{2\pi}\left\{1+2\sum_{n>k} b_n b_k A_{n,k}(t)\cos[(n-k)\theta] + 2\sum_{n>k} b_n b_k B_{n,k}(t)\sin[(n-k)\theta]\right\}, \tag{29}$$

which is normalized so that

$$\int_{-\pi}^{\pi} P(\theta,t)\,\mathrm{d}\theta=1. \tag{30}$$

It does not appear possible to express the sums in equation (29) in closed form. But for not too large $\bar{n}$, the direct numerical evaluations can be performed. The results are shown in figure 1, where we have plotted $P(\theta,t)$ against $\theta$ in the polar coordinate system for various times and for different values of the detuning parameter $\Delta$. At time $t=0$, $P(\theta,t)$ has a lengthened leaf shape corresponding to the initial coherent state of the field. As the interaction is turned on, this lengthened leaf gradually splits into two satellite distributions rotating in opposite directions. This can be seen clearly for the case of exact resonance (figure 1 (*a*)) and for cases with relatively small $\Delta$ (figure 1 (*b*)). After a certain interval of time, the two satellite distributions overlap and then split again. This time the two distributions are on their way to the right side, where they again overlap, and so forth. It is even more interesting that the overlappings take place regularly with the period coinciding with that of the revivals of the atomic inversion. In figure 1 we have presented only the first overlapping. To make the comparison easier we have scaled the time by the factor

$$T_{\mathrm{R}}=\frac{2\pi}{g}[\bar{n}+\Delta^2/(4g^2)]^{1/2}, \tag{31}$$

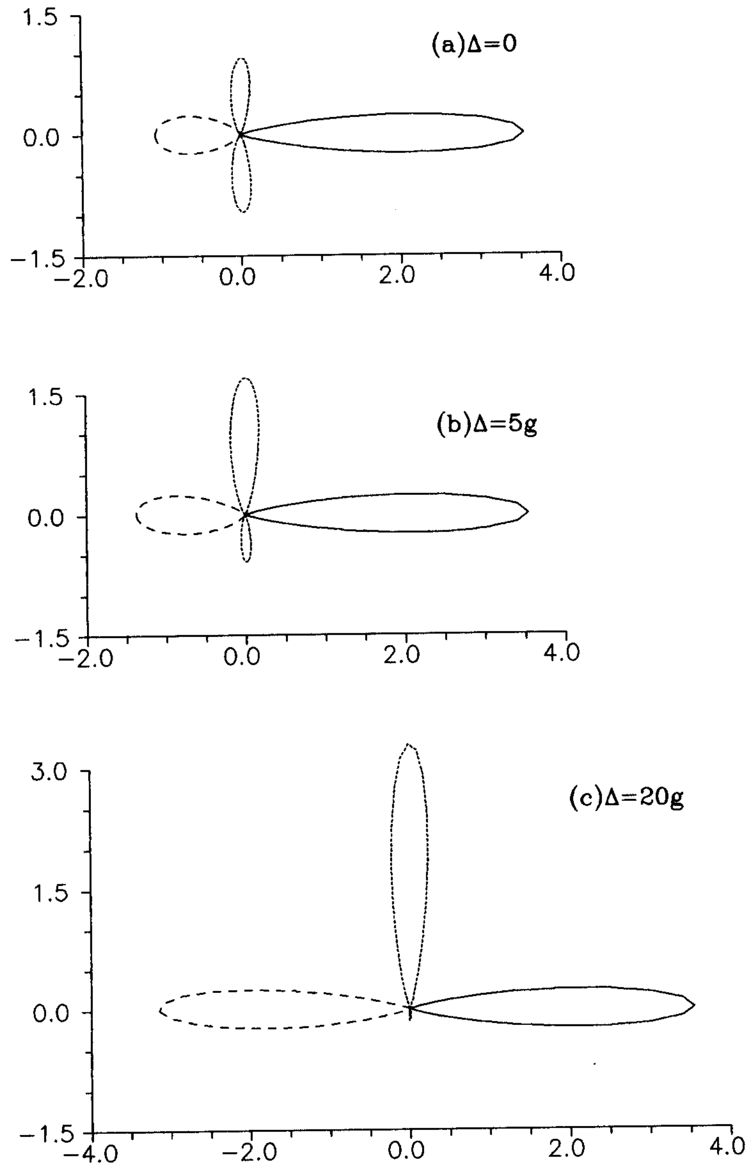


Figure 1. Phase probability distribution $P(\theta, t)$ plotted in polar form for various detunings, $\bar{n}=20$, $\beta=0$. The scaled time $T=gt/[2\pi(\bar{n}+\Delta^2/4g^2)^{1/2}]$. The curves shown are $T=0$ (solid), $T=0{\cdot}5$ (dotted) and $T=1$ (dashed).

so that the atomic inversion revives [4] (and together with it the satellite distributions overlap) at $T=1,2,3,\ldots$. The mean photon number $\bar{n}$ has been taken as 20. For smaller values of $\bar{n}$, the stochastic regime sets in early and the splitting of the phase distribution into two well-separated peaks as well as the revivals of the atomic inversion become difficult to be seen.

We should note that the above described property of phase distribution is also characteristic for the $Q$-function. In a recent work [18], Eiselt and Risken have shown that collapses and revivals can be understood in terms of interferences of quasi-probabilities in phase space. Thus the phase distribution of the field can be considered as an alternative description with respect to the quasiprobability distribution $Q(\alpha, \alpha^*)$. The bifurcation of the phase distribution also appears in the models closely related to the JCM, such as the Raman-coupled model or the two-photon JCM [19].

The dynamical behaviour of $P(\theta, t)$ can be predicted easily from the dressed-state viewpoint. Indeed, the state vector given by equation (21) may be rewritten in terms of eigenstates of the Hamiltonian $H_{\mathrm{II}}$

$$|\phi(t)\rangle = \sum_{n=0}^{\infty} b_n(\cos\varphi_n e^{\mathrm{i}(n\beta-\lambda_n t)}|\phi_n^+\rangle - \sin\varphi_n e^{\mathrm{i}(n\beta+\lambda_n t)}|\phi_n^-\rangle). \tag{32}$$

Equation (32) shows clearly that, if the initial distribution of the phase state is narrow enough, the interaction causes it to split into two phase states rotating in opposite directions in the polar coordinate system. The relation between the weights of the peaks rotating counter-clockwise and clockwise is approximately equal to $(\sin^2\varphi_{\bar{n}}/\cos^2\varphi_{\bar{n}})$ or

$$\frac{(g^2\bar{n}+\Delta^2/4)^{1/2}+\Delta/2}{(g^2\bar{n}+\Delta^2/4)^{1/2}-\Delta/2}. \tag{33}$$

Here we have used the fact that the Poisson photon distribution is sharply peaked around $\bar{n}$. At exact resonance $\Delta=0$, the two phase states become symmetric with respect to $\beta$. As the detuning parameter $\Delta$ increases, the intensity of one peak increases at the expense of decreased intensity in the other peak (figure 1 ($b$)). In the far-off-resonance limit ($\Delta^2 \gg g^2\bar{n}$), the phase distribution $P(\theta, t)$ is found from equation (29) to be

$$P(\theta, t) = \frac{1}{2\pi}\left\{1+2\sum_{n>k} b_n b_k \cos[(n-k)(\theta-g^2t/\Delta)]\right\} + O(\varepsilon_\Delta^2), \tag{34}$$

where

$$\varepsilon_\Delta^2 = 4g^2\bar{n}/\Delta^2 \ll 1, \tag{35}$$

and $O(\varepsilon_\Delta^2)$ is a set ofoscillatingterms with their amplitudes of order $\varepsilon_\Delta^2$ or smaller. The phase state remains almost coherent with the phase $\theta$ replaced by $(\theta-g^2t/\Delta)$ (figure 1 ($c$)). Our results are in complete accord with those of Yoo and Eberly [2]. In particular, they show that far-off the resonance, the field at time $t$ is roughly described by a pure state

$$|\alpha(t)\rangle = \sum_{n=0}^{\infty} b_n \exp[\mathrm{i}n(\beta+g^2t/\Delta)]|n\rangle, \tag{36}$$

which develops in time from the initial coherent state $|\alpha\rangle$ by an effective Hamiltonian given by

$$H_{\text{eff}} \equiv -(g^2/\Delta)a^+a. \tag{37}$$

It is clear that equation (34) can be obtained immediately from the Hamiltonian (37).

We now proceed to calculate the expectation value and the variance of the hermitian phase operator. They are described by the summations

$$\langle\hat{\Phi}_\theta\rangle = \sum_m \theta_m |\langle\theta_m|\phi(t)\rangle|^2, \tag{38}$$

$$\langle\Delta\Phi_\theta^2\rangle = \sum_m (\theta_m - \langle\hat{\Phi}_\theta\rangle)^2 |\langle\theta_m|\phi(t)\rangle|^2, \tag{39}$$

which may be transformed into integrals over the variable $2\pi\mu/(s+1)=\theta$, over the range $-\pi$ to $\pi$. The integrals encountered are elementary, and one finds

$$\langle\hat{\Phi}_\theta\rangle = \beta - 2\sum_{n>k} b_n b_k B_{n,k}(t) \frac{(-1)^{n-k}}{n-k}, \tag{40}$$

$$\langle\Delta\Phi_\theta^2\rangle = \frac{\pi^2}{3} + 4\sum_{n>k} b_n b_k A_{n,k}(t) \frac{(-1)^{n-k}}{(n-k)^2} - 4\left[\sum_{n>k} b_n b_k B_{n,k}(t) \frac{(-1)^{n-k}}{n-k}\right]^2, \tag{41}$$

where the coefficients $A_{n,k}(t)$ and $B_{n,k}(t)$ are given by equations (24)–(26). At exact resonance $B_{n,k}(t)=0$, the second term in equation (40) vanishes and the average value of the phase $\langle\hat{\Phi}_\theta\rangle$ reduces to the constant $\beta$. This may be understood from the fact that although in this case the phase distribution evolves, it maintains all the time the symmetry with respect to the initial value of the average phase $\beta$. For nonzero detunings, $\langle\hat{\Phi}_\theta\rangle$ oscillates around $\beta$ (we have put $\beta$ equal to zero everywhere) with increasing amplitude when $\Delta$ increases (figure 2). At far-off-resonance, the amplitude of the oscillations of $\langle\hat{\Phi}_\theta\rangle$ does not depend on $\Delta$ any more and the time behaviour of $\langle\hat{\Phi}_\theta\rangle$ becomes nearly periodic

$$\langle\hat{\Phi}_\theta\rangle = \beta - 2\sum_{n>k} b_n b_k \sin[(n-k)(g^2t/\Delta)] \frac{(-1)^{n-k}}{n-k} + O(\varepsilon_\Delta^2). \tag{42}$$

Note that the period of $\langle\hat{\Phi}_\theta\rangle$, which is equal to $2\pi\Delta/g^2$, is dependent on $\Delta$ only and is the same for all $\bar{n}$ obeying equation (35).

The variance of phase given by equation (41) is depicted as a function of the scaled time $T$ in figure 3 (*a*) for exact resonance and in figure 3 (*b*) for far-off-resonance. Here, care must be taken in interpreting the results obtained. At $t=0$ we have chosen and fixed the reference phase $\theta_0$ as in (27). This particular choice of $\theta_0$ determines a phase window that totally encloses the peak and yields the mean phase $\beta$ and the minimal variance. At scaled time $T=1$, the phase distribution shifts by $\pi$ from its initial state (figure 1), and we have a situation in which the $2\pi$ phase window has one peak at $\beta-\pi$ and another at $\beta+\pi$. Such a phase window maximizes the variance of the phase. The influence of a chosen reference phase on phase properties of the field has been explored in detail by Barnett and Pegg [7]. Here, we may conclude that both maxima and minima of the variance of the phase correspond to the revivals of the atomic inversion. This, however, concerns only the first maxima and minima. For longer time, the variance of phase shows quick and small oscillations near $\pi^2/3$ (figure 3 (*a*))—the phase variance of a field with randomly

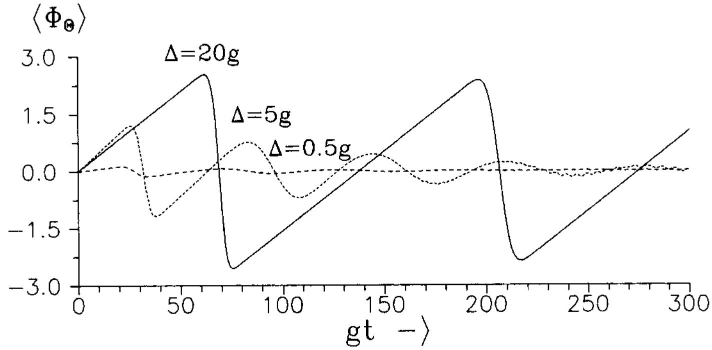


Figure 2. Plot of the mean value of the phase operator as a function of dimensionless time $gt$ for various $\Delta$, $\bar{n}=20$, $\beta=0$.

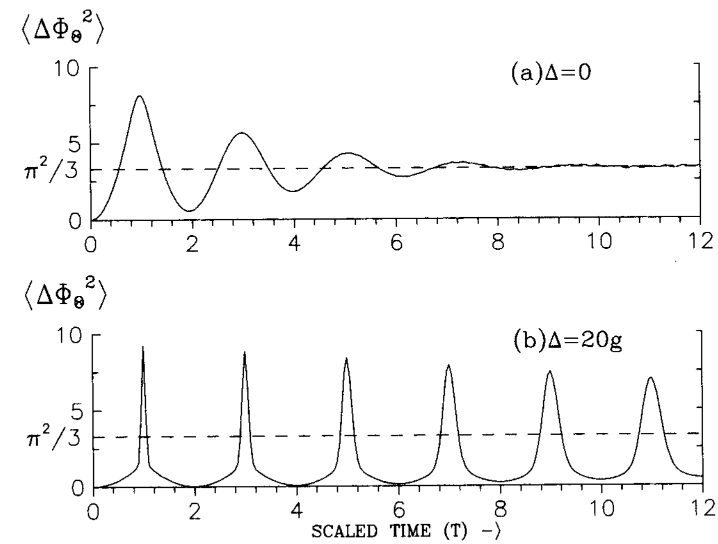


Figure 3. The variance of the phase as a function of the scaled time $T=gt/[2\pi(\bar{n}+\Delta^2/4g^2)^{1/2}]$ for (*a*) exact resonance, (*b*) far-off-resonance. The mean photon number $\bar{n}=20$, $\beta=0$. Note that for $t=0\langle\Delta\Phi_\theta^2\rangle\approx 1/(4\bar{n})>0$.

distributed phase. Figure 3 (*b*) illustrates the time behaviour of the variance of the phase in the far-off-resonance limit. In this situation the extrema of the phase variance have no clear connection to the revivals of the atomic inversion because the latter are less visible as $\Delta$ increases [4]. They reflect only the rotation of the phase state on the polar diagram (figure 1 (*c*)).

Phase characteristics of the field, such as the phase distribution, the expectation value of the phase operator and its variance, can be obtained within the Pegg–Barnett formalism only. They are either lacking or have no consistent description in other approaches so far. However, there are phase characteristics, such as the cosine and sine functions of the phase and their variance, that have their counterparts in other formalisms. To calculate the expectation values of the cosine and sine functions of the phase we take into account that the state of the field in the JCM is a physical state, for which the relation (5) holds. Then, we obtain

$$\langle\cos\hat{\Phi}_\theta\rangle=\sum_n b_n b_{n+1}[\cos\beta A_{n+1,n}(t)-\sin\beta B_{n+1,n}(t)], \tag{43}$$

$$\langle \sin \hat{\Phi}_\theta \rangle = \sum_n b_n b_{n+1}[\sin \beta A_{n+1,n}(t) + \cos \beta B_{n+1,n}(t)]. \tag{44}$$

Similarly, from equations (5), (8) and (9) we obtain

$$\langle \cos^2 \hat{\Phi}_\theta \rangle = \frac{1}{2} + \frac{1}{2} \sum_n b_n b_{n+2}[\cos 2\beta A_{n+2,n}(t) - \sin 2\beta B_{n+2,n}(t)], \tag{45}$$

$$\langle \sin^2 \hat{\Phi}_\theta \rangle = \frac{1}{2} - \frac{1}{2} \sum_n b_n b_{n+2}[\cos 2\beta A_{n+2,n}(t) - \sin 2\beta B_{n+2,n}(t)]. \tag{46}$$

Our formulae (45), (46) show that $\langle \cos^2 \hat{\Phi}_\theta \rangle + \langle \sin^2 \hat{\Phi}_\theta \rangle = 1$, which is not the case with the SG formalism. Two approaches give the same results for the sine and cosine functions, but the SG squares of the phase sine and cosine differ from those obtained using PB formalism by the amount

$$\tfrac{1}{4}|\langle \phi(t)|0\rangle|^2 = \tfrac{1}{4}\exp(-\bar{n})\left(1 + \frac{\sin^2(gt)}{1 + \Delta^2/(4g^2)}\right), \tag{47}$$

as anticipated from equations (8) and (9). This difference is proportional to the probability of finding the field at time $t$ in the vacuum state. It is negligible for $\bar{n} \gg 1$, but is essential when $\bar{n}$ is small. Before comparing the PB variance of the phase cosine with the SG and MP results, we recall that the measured phase concept is based on analysing actual techniques used in phase measurements, such as homodyne detection and prepared atom experiments [10]. The phase cosine is defined in this case as appropriately normalized field quadrature

$$\cos \hat{\Phi}_{\mathrm{MP}} = \frac{a + a^+}{2(\langle \hat{n} \rangle + \frac{1}{2})^{1/2}}. \tag{48}$$

The cosine variance is then simply equal to the variance of the quadrature field component divided by a factor which contains the expectation value of the photon number. These quantities have been evaluated more than once by many authors [1–4, 14, 15, 20–23], so we will not repeat them here. The results for the variance of the phase cosine based on three different definitions are shown in figure 4 for $\Delta = 0$. When $\bar{n}$ is small (figure 4 (*a*)), the variance of the SG phase cosine behaves in quite a different way compared with the PB and MP cases. It is shifted from the other two and decreases in the short-time region while the PB and MP quantities increase. Difficulties associated with the SG formalism in the few-photon regime [8] allow us to assume that in this case the PB and MP curves describe the behaviour of the noise in the phase cosine more adequately.

In the Jaynes–Cummings model, $\langle \hat{n} \rangle$ is modulated with the maximal amplitude equal to unity, since the atom absorbs and emits only a single photon. This modulation is essential for small $\bar{n}$ and leads to drastic changes in the time behaviour of $\langle \Delta \cos^2 \hat{\Phi}_{\mathrm{MP}} \rangle$ compared with the corresponding quadrature phase variance. In figure 4 (*a*) we see that the variance of the measured phase cosine increases as time goes on, while the first quadrature of the field in this situation exhibits squeezing [21, 22]. For large $\bar{n}$ the modulation of $\langle \hat{n} \rangle$ becomes negligible and the MP cosine variance curve now reproduces the main features of the squeezing curve [20] (figure 4 (*b*)).

In figure 4 (*b*) we also see that when $\bar{n}$ is large the PB and SG results are almost the same but, they still show some discrepancies with the MP results. The PB and SG

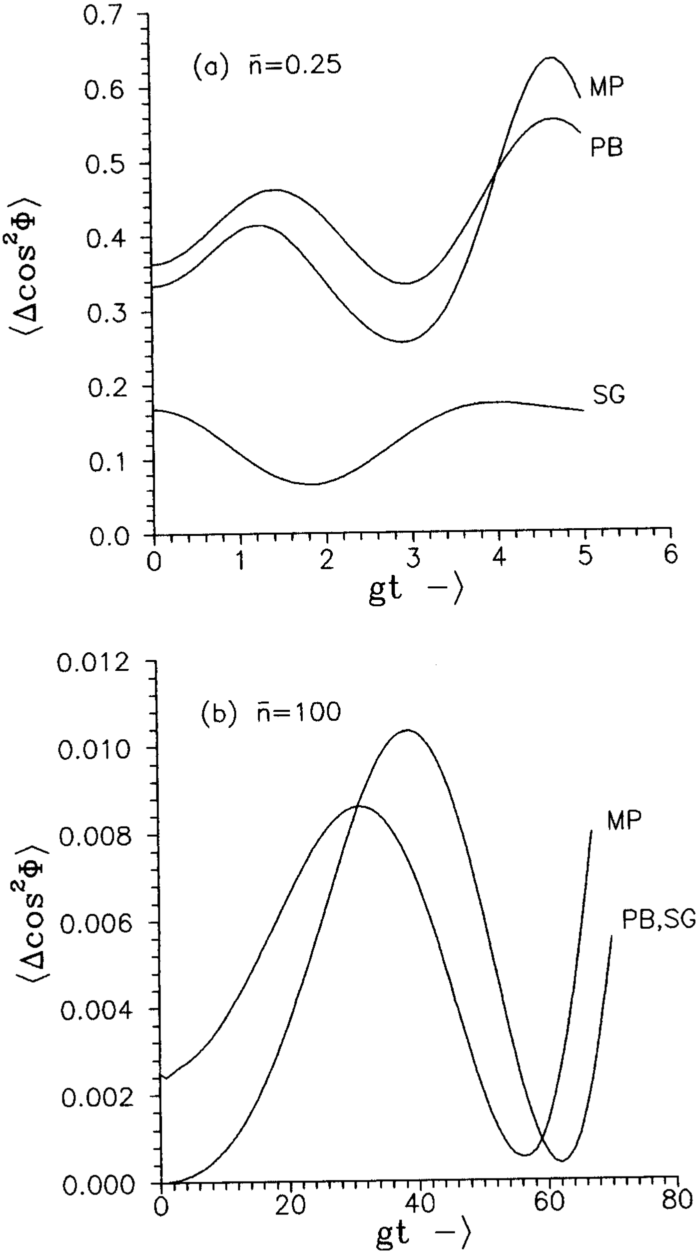


Figure 4. Time evolution of the variance of the phase cosine. Results for different approaches are compared: Pegg–Barnett (PB), Susskind–Glogower (SG) and measured phase (MP). (*a*) Weak field, $\bar{n}=0{\cdot}25$; (*b*) strong field, $\bar{n}=100$; $\beta=0$.

curves do not fall below their initial values and have their first minimums shifted slightly forward compared with the MP curve. Thus, although the simplified structure of the MP operators can be attractive, as in the case with the ideal squeezed state [24], they should be used with extreme care for investigating phase properties of the field.

## 4. Effect of atomic coherence on phase properties of the field

So far we have assumed the atom to be initially in the ground state. However, the atom can be prepared in a coherent superposition of the upper and lower levels by an external coherent field [12] and then be sent into the cavity. Let us consider the initial atomic state

$$|\phi_a(0)\rangle = \cos(\xi/2)|e\rangle + e^{i\eta}\sin(\xi/2)|g\rangle, \tag{49}$$

and assume the field to be initially in a coherent state as before. To isolate the effects of atomic coherence from that of finite detuning, we restrict ourselves to the exact resonance, $\Delta=0$. The expression for the state vector of the total atom–field system is then found to be

$$|\phi(t)\rangle = \sum_{n=0}^{\infty} b_n e^{in\beta}\{\cos(\xi/2)[-i\sin(g\sqrt{n+1}t)|n+1;g\rangle + \cos(g\sqrt{n+1}t)|n;e\rangle]$$
$$+ e^{i\eta}\sin(\xi/2)[\cos(g\sqrt{n}t)|n;g\rangle - i\sin(g\sqrt{n}t)|n-1;e\rangle]\}. \tag{50}$$

Following the same lines as in section 3, one obtains the phase distribution in the form

$$P(\theta,t) = \cos^2(\xi/2)P_e(\theta,t) + \sin^2(\xi/2)P_g(\theta,t) + \frac{\sin\xi}{4\pi}\sum_{n,k} b_n b_k$$
$$\times\{\sin[g(\sqrt{n}-\sqrt{k})t]\sin[(n-k-1)\theta-(\beta+\eta)] + n\Leftrightarrow k\}, \tag{51}$$

where

$$P_g(\theta,t) = \frac{1}{2\pi}\left\{1 + 2\sum_{n>k} b_n b_k \cos[(n-k)\theta]\cos[g(\sqrt{n}-\sqrt{k})t]\right\}, \tag{52}$$

$$P_e(\theta,t) = \frac{1}{2\pi}\left\{1 + 2\sum_{n>k} b_n b_k \cos[(n-k)\theta]\cos[g(\sqrt{n+1}-\sqrt{k+1})t]\right\}, \tag{53}$$

are the phase distributions for the atom being initially in the ground and excited states, respectively. Equation (52) for $P_g(\theta,t)$ can be derived immediately from equation (29), putting $\Delta=0$.

We also find for the average value of the phase

$$\langle\hat{\Phi}_\theta\rangle = \beta + \tfrac{1}{2}\sin\xi\cos(\beta+\eta)\left\{\sum_{\substack{n,k\\ n\neq k+1}} b_n b_k \sin[g(\sqrt{k+1}-\sqrt{n})t]\frac{(-1)^{n-k}}{n-k-1} + n\Leftrightarrow k\right\}, \tag{54}$$

and the variance

$$\langle \Delta\Phi_\theta^2\rangle = \cos^2(\xi/2)\langle \Delta\Phi_\theta^2\rangle_{\rm e} + \sin^2(\xi/2)\langle \Delta\Phi_\theta^2\rangle_{\rm g}$$
$$+\sin\xi\sin(\beta+\eta)\left\{\sum_{\substack{n,k\\ n\neq k+1}} b_n b_k \sin[g(\sqrt{k+1}-\sqrt{n})t]\frac{(-1)^{n-k}}{(n-k-1)^2}+n\Leftrightarrow k\right\}$$
$$-\tfrac{1}{4}\sin^2\xi\cos^2(\beta+\eta)\left\{\sum_{\substack{n,k\\ n\neq k+1}} b_n b_k \sin[g(\sqrt{k+1}-\sqrt{n})t]\frac{(-1)^{n-k}}{n-k-1}+n\Leftrightarrow k\right\}^2, \tag{55}$$

where $\langle \Delta\Phi_\theta^2\rangle_{\rm e}$ and $\langle \Delta\Phi_\theta^2\rangle_{\rm g}$, the variances calculated with the phase distributions (52) and (53), are

$$\langle \Delta\Phi_\theta^2\rangle_{\rm g} = \frac{\pi^2}{3}+4\sum_{n>k} b_n b_k \cos[g(\sqrt{n}-\sqrt{k})t]\frac{(-1)^{n-k}}{(n-k)^2}, \tag{56}$$

$$\langle \Delta\Phi_\theta^2\rangle_{\rm e} = \frac{\pi^2}{3}+4\sum_{n>k} b_n b_k \cos[g(\sqrt{n+1}-\sqrt{k+1})t]\frac{(-1)^{n-k}}{(n-k)^2}. \tag{57}$$

For $\cos(\xi/2)=0$ or $\sin(\xi/2)=0$, no atomic phase is present and equations (51)–(57) reduce to those of the initially purely excited or de-excited atom.

The phase distribution (51) is graphically illustrated in figure 5 for $\beta=0$, and for various values of the atomic phase $\eta$. The case of particular interest is when the two levels are equally populated, to $\cos(\xi/2)=\sin(\xi/2)$ $(\xi=\pm\pi/2)$. For $\eta=\pi/2$ (figure 5 (*a*)) the time behaviour of the phase distribution is exactly the same as in the case of an initially de-excited atom without detuning. This also holds true for the time behaviour of the atomic inversion [11–13]. As before, the phase distribution splits into two counter-rotating distributions when the evolution proceeds. The two satellite distributions overlap at the scaled time $T=1,2,3\ldots$. This is the time when the revivals of the atomic inversion occur. The initial atomic state with phase $\eta=\pi/3$ gives rise to asymmetry between the two satellite distributions (figure 5 (*b*)), which is similar to the asymmetry caused by nonvanishing detuning. For $\eta=0$, one peak is completely quenched and we see the phase distribution rotating and slowly changing its shape on the polar diagram (figure 5 (*c*)). It is known that [11–13] the state with $\xi=\pi/2$, $\beta+\eta=0$ is an eigenstate of the semiclassical Hamiltonian of the atom–field system. If the atom and the field are initially prepared in this state, atomic inversion will essentially remain unaffected despite the large field intensity.

From the similarity of the phase distributions, it is not difficult to predict the similarity of the average values and the variances of the phase, so we will not discuss them here. We may conclude the effects of atomic coherence on phase properties of the field as follows. First, the time behaviour of the phase distribution preserves its synchronization with the collapses and revivals of the atomic inversion. Secondly, the effect of the atomic coherence on phase properties of the field when the relative phase $\beta+\eta$ ranges from $\pi/2$ to 0 (or from $\pi/2$ to $\pi$) resembles very much that of finite detuning as $\Delta$ increases from 0 to the far-off-resonance limit. This resemblance can be explained as a consequence of the fact that in both cases the atom and the field become obviously de-coupled.

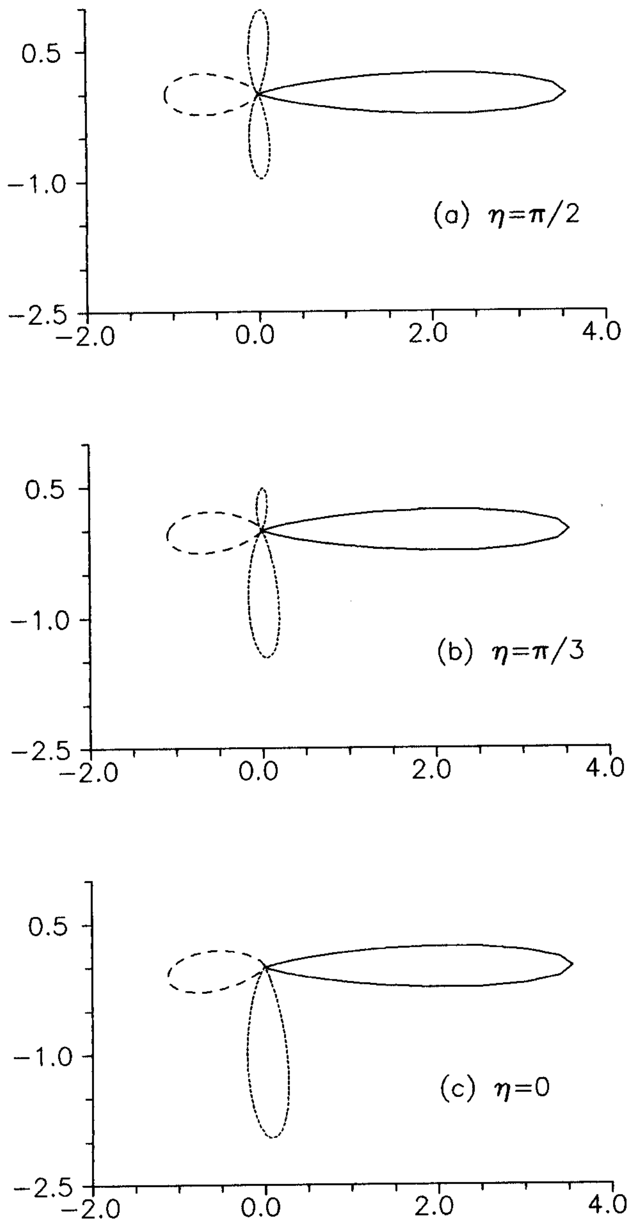


Figure 5. Phase probability distribution $P(\theta, t)$ under the influence of atomic coherence. The detuning parameter $\Delta=0$; $\bar{n}=20$; $\beta=0$. The scaled time $T=gt/[2\pi(\bar{n})^{1/2}]$, and as before the curves shown are $T=0$ (solid), $T=0{\cdot}5$ (dotted) and $T=1$ (dashed).

## 5. Conclusion

We have used Pegg–Barnett phase formalism to study phase properties of a coherent field interacting with a two-level atom in a loss-less cavity. We have found that, for large enough mean photon numbers $\bar{n}$, the phase distribution of the field splits into two counter-rotating peaks. When the two peaks are well-separated, atomic inversion shows no oscillations; when they collide, revivals occur. The collapse and revival phenomena are also reflected in the time behaviour of the variance of the field phase. Taking into account the influence of the fixed phase window, we have shown that the revivals correspond to the first maxima and minima of the phase variance. When the atomic inversion goes over into the quasi-chaotic regime, the variance of the phase oscillates around $\pi^2/3$. This means that the field is in a state close to that with random distribution of phase. The average value of the phase, which remains constant at exact resonance, begins oscillating near the initial value $\beta$ as $\Delta$ increases.

Comparing the effects of the finite detuning on phase properties of the field with those of the atomic coherence, we have pointed out that they are identical. This identity takes place for two different situations: when the field is relatively weak, so that the influence of the finite detuning is noticeable; when the field is strong, so that the semiclassical limit is valid. We have also calculated the phase cosine and sine functions and their variances. The results have been compared with those based on the Susskind–Glogower formalism and the measured phase concept. It has been established that, for states with a reasonable photon number the PB and SG definitions yield identical curves for the variances of the phase cosine, which are different from the MP curve.

## References

[1] Jaynes, E. T., and Cummings, F. W., 1963, *Proc. IEEE*, **51**, 89.
[2] Yoo, H. I., and Eberly, J. H., 1985, *Phys. Rep.*, **118**, 239.
[3] Barnett, S. M., Filipowicz, P., Javanainen, J., Knight, P. L., and Meystre, P., 1986, *Frontiers in Quantum Optics*, edited by E. R. Pike and S. Sarkar (Bristol: Hilger), p. 485.
[4] Narozhny, N. B., Sanchez-Mondragon, J. J., and Eberly, J. H., 1981, *Phys. Rev.* A, **23**, 236.
[5] Rempe, G., Walther, H., and Klein, N., 1987, *Phys. Rev. Lett.*, **58**, 353.
[6] Pegg, D. T., and Barnett, S. M., 1988, *EuroPhys. Lett.*, **6**, 483.
[7] Barnett, S. M., and Pegg, D. T., 1989, *J. mod. Opt.*, **36**, 7.
[8] Pegg, D. T., and Barnett, S. M., 1989, *Phys. Rev.* A, **39**, 1665.
[9] Susskind, L., and Glogower, J., 1964, *Physics*, **1**, 49.
[10] Barnett, S. M., and Pegg, D. T., 1986, *J. Phys.* A, **19**, 3849.
[11] Agarwal, G. S., and Puri, R. R., 1988, *J. opt. Soc. Am.* B, **5**, 1669.
[12] Puri, R. R., and Agarwal, G. S., 1987, *Phys. Rev.* A, **35**, 3433.
[13] Zaheer, K., and Zubairy, M. S., 1989, *Phys. Rev.* A, **39**, 2000.
[14] Knight, P. L., 1986, *Phys. Scr. T*, **12**, 51.
[15] Wodkiewicz, K., Knight, P. L., Buckle, S. J., and Barnett, S. M., 1987, *Phys. Rev.* A, **35**, 2567.
[16] Carruthers, P., and Nieto, M. M., 1968, *Rev. mod. Phys.*, **40**, 411.
[17] Vaccaro, J. A., and Pegg, D. T., 1989, *Optics Commun.*, **70**, 529.
[18] Eiselt, J., and Risken, H., 1989, *Optics Commun.*, **72**, 351.
[19] Phoenix, S. J. D., and Knight, P. L., 1990, *J. opt. Soc. Am.* B, **7**, 116.
[20] Meystre, P., and Zubairy, M. S., 1985, *Phys. Lett.* A, **89**, 390.
[21] Kuklinski, J. R., and Madajczyk, J. L., 1988, *Phys. Rev.* A, **37**, 3175.
[22] Butler, M., and Drummond, P. D., 1986, *Optica Acta*, **33**, 1.
[23] Shumovsky, A. S., Fam Le Kien and Aliskenderov, E. I., 1987, *Phys. Lett.* A, **124**, 351.
[24] Gronbech-Jensen, N., Christiansen, P. L., and Ramanujam, P. S., 1989, *J. opt. Soc. Am.* B, **6**, 2423.