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Phase properties of elliptically polarized light propagating in a Kerr medium

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Abstract. Phase properties of elliptically polarized light propagating through a nonlinear Kerr medium are considered within the framework of the Pegg–Barnett Hermitian phase formalism. The joint phase probability distribution function for the phases of two orthogonal modes describing elliptical polarization of the field is calculated and its evolution discussed and illustrated graphically. The marginal phase probability distribution for the individual phases are also calculated and discussed. Analytical formulae for phase expectation values and variances are derived for the individual phases as well as for the phase difference. It is shown that in the course of propagation the correlation between the phases of the two modes builds up. This correlation is responsible for lowering phase difference variance. The expressions for the sine and cosine functions and their variances of the individual phases as well as the phase difference are obtained and discussed. The effect of randomization of individual phases and the phase difference, which is a purely quantum effect, is shown to appear during propagation. The relation between phase randomization and degradation of the degree of polarization of the light is established.

1. Introduction

When strong elliptically polarized light propagates through an isotropic nonlinear medium, the medium becomes birefringent, which results in the self-induced rotation of the polarization ellipse—the nonlinear optical effect observed by Maker *et al.* [1] in 1964. Subsequently, propagation of light in a Kerr medium has been studied extensively, and it has become a standard topic in textbooks on nonlinear optics [2, 3]. To understand phenomena such as optically induced birefrigence there is no need for field quantization. However, if the quantum properties of light propagating through a Kerr medium are taken into account, some new effects, such as photon antibunching and squeezing, can occur. Photon statistics and photon antibunching have been considered by Ritze and Bandilla [4], Tanaś and Kielich [5] and Ritze [6]. Tanaś and Kielich [7] have shown that intense light propagating in a nonlinear Kerr medium can squeeze its own quantum fluctuations. They referred to this effect as self-squeezing, and have proved that as much as 98% of squeezing, can be obtained in this process. The description of the field in [7] was the two-mode

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quantized field description of the elliptically polarized light propagating in the medium. The one-mode version of the self-squeezing effect, applicable for circularly polarized light propagating in an isotropic Kerr medium, has been considered by Tanaś [8] in terms of an anharmonic oscillator model. The anharmonic oscillator model, which is very simple and exactly solvable, appeared to be very attractive, and many properties of the quantum states of the field produced in the model have been discussed recently [9–22].

To describe properly the effects associated with propagation of elliptically polarized light in a Kerr medium, the two-mode description of the field is needed. Such a description has already been used in the early studies [4–7] of the quantum field effects that appear during propagation. In those studies, the Heisenberg equations of motion for the field operators were solved and their solutions used to calculate appropriate quantities revealing photon antibunching or squeezing. Horák and Peřina [23] discussed the influence of losses and noises on the quantum effects in the coupled nonlinear oscillators. Their approach is based on the Heisenberg–Langevin equations of motion for the operators of the two coupled nonlinear oscillators. Recently, Agarwal and Puri [24] reexamined the problem of propagation of elliptically polarized light through a Kerr medium, using the two-mode description of the field. They discussed not only the Heisenberg equations of motion for the field operators but also the evolution of the field states themselves. The polarization state of the field propagating in a Kerr medium can be described by the Stokes parameters, which are the expectation values of the corresponding Stokes operators when the quantum description of the field is used. Quantum fluctuations in the Stokes parameters of light propagating in a Kerr medium have recently been discussed by Tanaś and Kielich [25].

In this paper we shall discuss phase properties of elliptically polarized light propagating through a Kerr medium, using the new Pegg–Barnett [26–28] Hermitian phase formalism. This formalism enables direct calculations of the expectation values and variances of the Hermitian phase operators for the two modes of the elliptically polarized light. Within this formalism we have also obtained the joint probability distribution $P(\theta_+, \theta_-)$ for the phases θ_+ and θ_- of the two modes as well as the marginal probability distributions $P(\theta_+)$ and $P(\theta_-)$. The evolution of these probability distributions during propagation is discussed and illustrated graphically. The variances of the individual mode phase operators are calculated and shown to be significantly affected by the coupling to the other mode. The essentially two-mode phase characteristics of the field—such as the phase difference variance, the phase correlation function for the phases of the two modes and the cosine and sine functions of the phase difference—are calculated. It is shown that two-mode phase characteristics depend strongly on the symmetry of the nonlinear susceptibility tensor of the medium. The relation between the inter-mode phase properties and the degree of polarization of the field is established.

2. Quantum description of elliptically polarized light

In quantum description of the electromagnetic field it is convenient to split the field into positive and negative frequency parts

$$E_i(\mathbf{r}, t) = E_i^{(+)}(\mathbf{r}, t) + E_i^{(-)}(\mathbf{r}, t), \tag{1}$$

where i denotes a polarization component of the field. Next, a mode decomposition of the field can be performed, which for the plane-wave decomposition of the free field propagating in a medium with (linear) refractive index $n(\omega)$ gives

$$E_i^{(+)}(\mathbf{r},t)=\sum_{\mathbf{k},\lambda} i\left(\frac{2\pi\hbar\omega_k}{n^2(\omega)V}\right)^{1/2} e_{ki}^{(\lambda)}a_{k\lambda}\exp\left[-i(\omega_k t-\mathbf{k}\cdot\mathbf{r})\right], \tag{2}$$

where $e_{ki}^{(\lambda)}$ is the i-th component of the polarization vector associated with the polarization state λ and the propagation vector $\mathbf{k}$, and V is the quantization volume. The operators $a_{k\lambda}$ and $a_{k\lambda}^{+}$ are the annihilation and creation operators of photons with propagation vector $\mathbf{k}$ and polarization λ satisfying the commutation relations

$$[a_{k'\lambda'}, a_{k,\lambda}^{+}]=\delta_{kk'}, \delta_{\lambda\lambda'}. \tag{3}$$

The polarization vectors satisfy the orthogonality conditions

$$\left.\begin{aligned}\sum_i e_{ki}^{(\lambda)*}e_{ki}^{(\lambda')}&=\delta_{\lambda\lambda'},\\ \sum_i e_{ki}^{(\lambda)}k_i&=0.\end{aligned}\right\} \tag{4}$$

For a monochromatic field of frequency ω propagating along the z-axis of the laboratory reference frame, we can drop the index k in our notation and write

$$E_i^{(+)}(z,t)=i\left(\frac{2\pi\hbar\omega}{n^2(\omega)V}\right)^{1/2}\exp\left[-\mathrm{i}(\omega t-kz)\right]\sum_{\lambda=1,2} e_i^{(\lambda)}a_\lambda, \tag{5}$$

with $k=n(\omega)\omega/c$. Since the summation over the two mutually orthogonal polarizations still remains in equation (5), we have a two-mode description of the field. If the field is a coherent superposition of these two modes, the two-mode description can be replaced by one mode of an elliptically polarized field

$$e_i a=e_i^{(1)}a_1+e_i^{(2)}a_2, \tag{6}$$

where $e_i^{(1)}$ and $e_i^{(2)}$ are the i-th components of the orthogonal unit polarization vectors $\hat{\mathbf{e}}^{(1)}$ and $\hat{\mathbf{e}}^{(2)}$ of the modes a_1 and a_2, and e_i is the i-th component of the polarization vector $\hat{\mathbf{e}}$ of the mode a. The relation (6) can also be considered in the reverse sense as a decomposition of initially elliptically polarized light into two orthogonal modes. Applying the orthogonality condition (4) for the polarization vectors, we get the formula

$$a=e_1^*a_1+e_2^*a_2, \tag{7}$$

where

$$e_1^*=\hat{\mathbf{e}}^*\cdot\hat{\mathbf{e}}^{(1)}, \qquad e^*=\hat{\mathbf{e}}^*\cdot\hat{\mathbf{e}}^{(2)}.$$

So far the decomposition (6) (or, equivalently (7)) is quite general and can be further specified either for two modes with mutually perpendicular linear polarizations or for right- and left-circularly polarized modes.

If a Cartesian basis is chosen, the unit polarization vectors are $\hat{\mathbf{e}}^{(1)}=\hat{\mathbf{x}}$, $\hat{\mathbf{e}}^{(2)}=\hat{\mathbf{y}}$, whereas in a circular basis we have $\hat{\mathbf{e}}^{(1)}=\hat{\mathbf{e}}^{(+)}=(\hat{\mathbf{x}}+i\hat{\mathbf{y}})/\sqrt{2}$, $\hat{\mathbf{e}}^{(2)}=\hat{\mathbf{e}}^{(-)}=(\mathbf{x}-i\hat{\mathbf{y}})/\sqrt{2}$ with $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ being the unit vectors along x and y, respectively. The unit vector $\hat{\mathbf{e}}$ of the elliptically polarized light can be written in either a Cartesian or a circular basis as

$$\hat{\mathbf{e}}=e_x\hat{\mathbf{x}}+e_y\hat{\mathbf{y}}=e_+\hat{\mathbf{e}}^{(+)}+e_-\hat{\mathbf{e}}^{(-)} \tag{8}$$

with e_x and e_y given by [29]

$$\left.\begin{aligned} e_x &= \cos\eta\cos\vartheta - i\sin\eta\sin\vartheta, \\ e_y &= \cos\eta\sin\vartheta + i\sin\eta\cos\vartheta, \end{aligned}\right\} \tag{9}$$

and

$$e_\pm = (e_x \mp i e_y)/\sqrt{2} = \frac{1}{\sqrt{2}}(\cos\eta \pm \sin\eta)e^{\mp i\vartheta}. \tag{10}$$

The parameters ϑ and η define the polarization ellipse of the field—ϑ is the azimuth of the ellipse denoting the angle between the major axis of the ellipse and the x-axis measured positive from the $+x$-axis towards the $+y$-axis, and η is the ellipticity parameter, $-\pi/4 \leqslant \eta \leqslant \pi/4$, where $\tan\eta$ describes the ratio of the minor and major axes of the ellipse with the sign defining its handedness (plus means right-handed polarization in the helicity convention).

According to equation (7), the annihilation operator of the elliptically polarized field can be written as

$$a = e_x^* a_x + e_y^* a_y = e_+^* a_+ + e_-^* a_-, \tag{11}$$

where e_x, e_y and $e_\pm$ are given by equations (9) and (10), and the operators $a_\pm$ are

$$a_\pm = \frac{1}{\sqrt{2}}(a_x \mp i a_y). \tag{12}$$

Hence, the annihilation operator a of the elliptically polarized light is a superposition of two orthogonal modes in either a Cartesian or a circular basis.

Defining a coherent state of the field with respect to the operator a by the relation

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{13}$$

we have, simultaneously,

$$|\alpha\rangle = |\alpha_x\rangle|\alpha_y\rangle = |\alpha_+\rangle|\alpha_-\rangle, \tag{14}$$

where $|\alpha_x\rangle$, $|\alpha_y\rangle$ and $|\alpha_+\rangle$, $|\alpha_-\rangle$ are the coherent states defined with respect to the annihilation operators a_x, a_y and a_+, a_-. According to equations (11), (13) and (14), the following relations hold

$$\alpha = e_x^* \alpha_x + e_y^* \alpha_y = e_+^* \alpha_+ + e_-^* \alpha_-, \tag{15}$$

and, due to the orthogonality relations

$$e_x^* e_x + e_y^* e_y = 1, \qquad e_+^* e_+ + e_-^* e_- = 1,$$

one obtains

$$\left.\begin{aligned} \alpha_x &= e_x \alpha \\ \alpha_y &= e_y \alpha \end{aligned}\right\} \tag{16}$$

$$\alpha_\pm = e_\pm \alpha, \tag{17}$$

where e_x, e_y and $e_\pm$ are given by equations (9) and (10), and

$$|\alpha_x|^2 + |\alpha_y|^2 = |\alpha_+|^2 + |\alpha_-|^2 = |\alpha|^2.$$

Thus, Cartesian or circular bases can be used alternatively to describe the propagation of elliptically polarized light in a nonlinear Kerr medium. In isotropic media, however, the circular basis is much more advantageous than the Cartesian basis.

Relations (15)–(17) and (8)–(10) facilitate the decomposition of a coherent state of elliptically polarized light—with the polarization ellipse described by the azimuth ϑ and the ellipticity η—into two orthogonal modes that are also in coherent states, and vice versa. However, if the nonlinear interaction between the field and the medium takes place, the resulting state may no longer be a coherent state, even if it were such a state initially. In this case, the relations (13)–(17) for the coherent state amplitudes are valid only for the initial coherent states. Quantum evolution of the field propagating through a nonlinear Kerr medium requires the two-mode description of the field. The corresponding equations of motion will be given in the following section.

3. Quantum evolution of elliptically polarized light propagating in a Kerr medium

The classical description of light propagating through a nonlinear Kerr medium involves third-order nonlinear polarization of the medium. A monochromatic light field of frequency ω propagating in the medium induces third-order polarization of this medium at frequency ω, which can be written as follows [2, 3]:

$$P_i^{(+)}(\omega)=\sum_{jkl}\chi_{ijkl}(-\omega,-\omega,\omega,\omega)E_j^{(-)}(\omega)E_k^{(+)}(\omega)E_l^{(+)}(\omega), \tag{18}$$

where $\chi_{ijkl}(-\omega,-\omega,\omega,\omega)$ is the third-order nonlinear susceptibility tensor of the medium, and the decomposition of the field into positive- and negative-frequency parts (as in equation (1)) has been used; albeit, in the classical description, the field amplitudes $E_i^{(\pm)}(\omega)$ are classical quantities. With such decomposition of the field, the intensity of the light beam is given by

$$I(\omega)=\frac{cn(\omega)}{2\pi}\sum_i E_i^{(-)}(\omega)E_i^{(+)}(\omega), \tag{19}$$

where $n(\omega)$ is the refractive index of the medium at frequency ω determined by the linear (first-order) polarization of the medium.

For an isotropic medium with a centre of inversion, the nonlinear susceptibility tensor, $\chi_{ijkl}(\omega)=\chi_{ijkl}(-\omega,-\omega,\omega,\omega)$, can be written as [2, 3]

$$\chi_{ijkl}(\omega)=\chi_{xxyy}(\omega)\delta_{ij}\delta_{kl}+\chi_{xyxy}(\omega)\delta_{ik}\delta_{jl}+\chi_{xyyx}(\omega)\delta_{il}\delta_{jk}, \tag{20}$$

with the additional relation

$$\chi_{xxxx}(\omega)=\chi_{yyyy}(\omega)=\chi_{xxyy}(\omega)+\chi_{xyxy}(\omega)+\chi_{xyyx}(\omega). \tag{21}$$

Taking into account the permutation symmetry of the tensor $\chi_{ijkl}(\omega)$ with respect to the first and the second pairs of indices, we have, moreover, $\chi_{xyxy}(\omega)=\chi_{xyyx}(\omega)$. The light beam is assumed to propagate along the z-axis of the laboratory reference frame.

Inserting the polarization (18) into the Maxwell equations and applying the slowly varying amplitude approximation, one obtains the following equation for the amplitudes of the field [3]:

$$\frac{\mathrm{d}E_i^{(+)}(\omega)}{\mathrm{d}z}=\frac{i2\pi\omega}{n(\omega)c}P_i^{(+)}(\omega), \tag{22}$$

where the slowly-varying amplitudes $E_i^{(+)}(\omega)$ are assumed to be dependent on z.

By equation (22) we have, for example,

$$\frac{\mathrm{d}E_x^{(+)}(\omega)}{\mathrm{d}z}=\frac{i2\pi\omega}{n(\omega)c}\{\chi_{xxyy}(\omega)E_x^{(-)}(\omega)[E_x^{(+)2}(\omega)+E_y^{(+)2}(\omega)]$$
$$+2\chi_{xyxy}(\omega)[E_x^{(-)}(\omega)E_x^{(+)}(\omega)+E_y^{(-)}(\omega)E_y^{(+)}(\omega)]E_x^{(+)}(\omega)\}. \tag{23}$$

If the circular basis is introduced, which is the natural basis for isotropic media, with the circular components of the field

$$E_{\pm}^{(+)}(\omega)=\frac{1}{\sqrt{2}}[E_x^{(+)}(\omega)\mp iE_y^{(+)}(\omega)], \tag{24}$$

the nonlinear polarization becomes

$$P_{\pm}^{(+)}(\omega)=2\chi_{xyxy}(\omega)|E_{\pm}^{(+)}(\omega)|^2E_{\pm}^{(+)}(\omega)+2[\chi_{xxyy}(\omega)+\chi_{xyxy}(\omega)]|E_{\mp}^{(+)}(\omega)|^2E_{\pm}^{(+)}(\omega) \tag{25}$$

which gives the equations of motion for the circular component of the field

$$\frac{\mathrm{d}E_{\pm}^{(+)}(\omega)}{\mathrm{d}z}=\frac{i4\pi\omega}{n(\omega)c}\{\chi_{xyxy}(\omega)|E_{\pm}^{(+)}(\omega)|^2+[\chi_{xxyy}(\omega)+\chi_{xyxy}(\omega)]|E_{\mp}^{(+)}(\omega)|^2\}E_{\pm}^{(+)}(\omega). \tag{26}$$

Equations (26) immediately show the advantage of the circular basis over the Cartesian basis used in equation (23). One easily checks that $(\mathrm{d}/\mathrm{d}z)|E_{\pm}^{(+)}(\omega)|^2=0$, i.e. the intensities $|E_{\pm}^{(+)}(\omega)|^2$ of both circular components are constants of motion. This is not the case for the Cartesian components. Since the intensities $|E_{\pm}^{(+)}(\omega)|^2$ do not depend on z, equation (26) has the following simple exponential solution [30]

$$E_{\pm}^{(+)}(\omega;z)=\exp(\mathrm{i}\Phi_{\pm}z)E_{\pm}^{(+)}(\omega;z=0), \tag{27}$$

where

$$\Phi_{\pm}=\frac{4\pi\omega}{n(\omega)c}\{\chi_{xyxy}(\omega)|E_{\pm}^{(+)}(\omega)|^2+[\chi_{xxyy}(\omega)+\chi_{xyxy}(\omega)]|E_{\mp}^{(+)}|^2\} \tag{28}$$

determines the light-intensity-dependent phase of the field (self-phase-modulation or intensity-dependent refractive index). These classical nonlinear effects are well known [2, 3], and are not the subject of our interest in this paper. We are interested in quantum phase properties of the field propagating in a Kerr medium, and we need quantum equations of motion for the field. Such equations—the Heisenberg equations of motion for the field operators—can be obtained from the following effective interaction Hamiltonian [7]

$$H_{\mathrm{I}}=\tfrac{1}{2}\hbar\kappa\{a_+^{+2}a_+^2+a_-^{+2}a_-^2+4da_+^{+}a_-^{+}a_-a_+\}, \tag{29}$$

where the nonlinear coupling constant κ is real and is given by

$$\kappa=\frac{V}{\hbar}\left[\frac{2\pi\hbar\omega}{n^2(\omega)V}\right]^2 2\chi_{xyxy}(\omega), \tag{30}$$

with V denoting the quantization volume, and we have introduced a nonlinear asymmetry parameter d, defined as

$$2d=1+\frac{\chi_{xxyy}(\omega)}{\chi_{xyxy}(\omega)}. \tag{31}$$

If the nonlinear susceptibility tensor χ is symmetric with respect to all its indices, the asymmetry parameter d is equal to unity. Otherwise $d\neq1$ and describes the asymmetry of the nonlinear properties of the medium. When the medium is composed of identical molecules the asymmetry parameter d is related to the hyperpolarizability of individual molecules [7]. Ritze [6] has calculated this asymmetry parameter for atoms with a degenerate one-photon transition, obtaining the results

$$d=\begin{cases}(2J-1)(2J+3)/[2(2J^2+2J+1)] & \text{for } J\leftrightarrow J \text{ transitions,}\\ (2J^2+3)/[2(6J^2-1)] & \text{for } J\leftrightarrow J-1 \text{ transitions.}\end{cases} \tag{32}$$

The operators $a_\pm$ in the Hamiltonian (29) are the annihilation operators for the circularly right- and left-polarized modes.

Using the interaction Hamiltonian (29) and the commutation rules (3), one can easily write the Heisenberg equations of motion describing the time evolution of the field operators. In the travelling wave case, the time t is replaced by $-n(\omega)z/c$, and we obtain the following equation:

$$\frac{\mathrm{d}a_\pm(z)}{\mathrm{d}z}=i\frac{n(\omega)}{c}\kappa[a_\pm^+(z)a_\pm(z)+2da_\mp^+(z)a_\mp(z)]a_\pm(z). \tag{33}$$

When the relation, obtained from equation (5),

$$E_\pm^{(+)}(\omega)=i\left[\frac{2\pi\hbar\omega}{n^2(\omega)V}\right]^{1/2}a_\pm \tag{34}$$

is applied, equation (33) reverts to the form (26), which makes the quantum-classical correspondence quite transparent; but now we deal with the quantum field.

Since the number of photons in the two modes $a_\pm^+a_\pm$ are constants of motion (they commute with the Hamiltonian (29)), equation (33) has the simple exponential solution [6, 7]

$$a_\pm(\tau)=\exp\{i\tau[a_\pm^+(0)a_\pm(0)+2da_\mp^+(0)a_\mp(0)]\}a_\pm(0), \tag{35}$$

where we have introduced the notation

$$\tau=n(\omega)\kappa z/c. \tag{36}$$

The solutions (35) are exact operator solutions for the field operators of light propagating through a nonlinear, isotropic Kerr medium. These equations were used for calculations of such quantum effects as photon antibunching [6] and squeezing [7].

To describe the evolution of the field states we can use the evolution operator $U(\tau)$, which according to equations (29) and (36), and after replacement $t=-n(\omega)z/c$, has the form

$$\begin{aligned}U(\tau)&=\exp\left\{i\frac{\tau}{2}[a_+^{+2}a_+^2+a_-^{+2}a_-^2+4da_+^+a_-^+a_-a_+]\right\}\\ &=\exp\left\{i\frac{\tau}{2}[\hat{n}_+(\hat{n}_+-1)+\hat{n}_-(\hat{n}_--1)+4d\hat{n}_+\hat{n}_-]\right\},\end{aligned} \tag{37}$$

where we have introduced the number operators $\hat{n}_\pm=a_\pm^+a_\pm$ for the two circularly polarized modes. The resulting state of the field is thus given by

$$|\psi(\tau)\rangle=U(\tau)|\psi(0)\rangle, \tag{38}$$

where $|\psi(0)\rangle$ is the initial state of the field. If the initial state of the field is a coherent state of elliptically polarized light, one obtains [24]

$$|\psi(\tau)\rangle = U(\tau)|\alpha_+, \alpha_-\rangle = \sum_{n_+, n_-} b_{n_+} b_{n_-} \exp\left\{ i(n_+\varphi_+ + n_-\varphi_-) + i\frac{\tau}{2}[n_+(n_+-1) + n_-(n_--1) + 4dn_+n_-] \right\} |n_+, n_-\rangle, \tag{39}$$

where

$$b_{n_\pm} = \exp(-|\alpha_\pm|^2/2)\frac{|\alpha_\pm|^{n_\pm}}{\sqrt{n_\pm!}} \tag{40}$$

and the state $|n_+, n_-\rangle = |n_+\rangle|n_-\rangle$ is the Fock state. Here we have used $\alpha_\pm = |\alpha_\pm|\exp(i\varphi_\pm)$.

Properties of the states (39) have recently been discussed by Agarwal and Puri [24]. Tanaś and Kielich [25] have considered quantum fluctuations in the Stokes parameter defining the polarization of the field. In this paper, we examine phase properties of the states (39) using the Pegg–Barnett [26–28] Hermitian phase formalism.

4. Phase properties of elliptically polarized light propagating in a Kerr medium

The new Hermitian phase formalism introduced by Pegg and Barnett [26–28] is a way off the difficulties associated with the existence of the Hermitian phase operator. They have shown that the Hermitian phase operator can be constructed from the phase states [31]. As the Hermitian phase operator is constructed, quantities like expectation values and variances of the phase operator can be calculated for a given state of the field. The phase probability density, which is a very spectacular phase characteristic of optical fields, can also be obtained within this formalism. These are new characteristics of optical fields, open for investigation because of the new formalism. Of course, there are 'old' phase characteristics such as phase cosine and sine and their variances that were available for investigation in the Susskind–Glogower [32] phase formalism or the measured phase formalism [33], and they also can be investigated in the new formalism. This time, however, the phase cosine and sine are actual cosine and sine functions of the Hermitian phase operator.

Here, we reproduce some basic formulae of Pegg and Barnett [26–28], which we will use in this paper to study the phase properties of elliptically polarized light propagating in a Kerr medium. The idea of Pegg and Barnett [26–28] is based on introducing, for one mode of the field, a finite $(s+1)$ dimensional space Ψ spanned by the number states $|0\rangle, |1\rangle, \ldots, |s\rangle$. The Hermitian phase operator operates on this finite space, and after all necessary expectation values have been calculated in Ψ, the value of s is allowed to tend to infinity. A complete orthonormal basis of $(s+1)$ states is defined on Ψ as

$$|\theta_m\rangle \equiv (s+1)^{-1/2} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle, \tag{41}$$

where

$$\theta_m \equiv \theta_0 + 2\pi m/(s+1), \qquad (m = 0, 1, \ldots, s). \tag{42}$$

The value of θ_0 is arbitrary and defines a particular basis set of $(s+1)$ mutually orthogonal phase states. The Hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m|. \tag{43}$$

Of course, the phase states (41) are eigenstates of the phase operator (43), with the eigenvalues θ_m restricted to lie within a phase window θ_0 and $\theta_0+2\pi$. The unitary phase operator $\exp(i\hat{\phi}_\theta)$ can be defined as the exponential function of the Hermitian operator $\hat{\phi}_\theta$. This operator when acting on the eigenstate $|\theta_m\rangle$ gives the eigenvalue $\exp(i\theta_m)$, and can be written as [26–28]

$$\exp(i\hat{\phi}_\theta) \equiv |0\rangle\langle 1| + |1\rangle\langle 2| + \ldots + |s-1\rangle\langle s| + \exp[i(s+1)\theta_0]|s\rangle\langle 0|, \tag{44}$$

and its Hermitian conjugate is

$$[\exp(i\hat{\phi}_\theta)]^+ = \exp(-i\hat{\phi}_\theta), \tag{45}$$

with the same set of eigenstates $|\theta_m\rangle$ but with eigenvalues $\exp(-i\theta_m)$.

To make further comparisons easier, it is useful to relate this new operator to the Susskind–Glogower phase operator, which is given by [34]:

$$\begin{aligned}\langle \exp(im\hat{\phi}_\theta)\rangle &= \langle [\exp(i\hat{\phi}_\theta)]^m\rangle = \lim_{s\to\infty} \left\langle \left\{ \sum_{n=0}^{s-m} |n\rangle\langle n+m| \right.\right. \\ &\quad \left.\left. + \exp[i(s+1)\theta_0] \sum_{n=0}^{m-1} |s-n\rangle\langle m-1-n| \right\}\right\rangle \\ &= \langle \hat{\exp}(im\phi_{SG})\rangle + \lim_{s\to\infty} \left\langle \left\{ \exp[i(s+1)\theta_0] \sum_{n=0}^{m-1} |s-n\rangle\langle m-1-n| \right\}\right\rangle,\end{aligned} \tag{46}$$

where the Susskind–Glogower phase operator is given by

$$\hat{\exp}(im\phi_{SG}) \equiv \sum_{n=0}^{\infty} |n\rangle\langle n+m|. \tag{47}$$

From the definition (47) and the definition

$$\hat{\exp}(-im\phi_{SG}) \equiv [\hat{\exp}(im\phi_{SG})]^+, \tag{48}$$

one gets for $m=1$

$$\begin{aligned}\hat{\exp}(i\phi_{SG})\,\hat{\exp}(-i\phi_{SG}) &= 1, \\ \hat{\exp}(-i\phi_{SG})\,\hat{\exp}(i\phi_{SG}) &= 1 - |0\rangle\langle 0|,\end{aligned} \tag{49}$$

which explicitly shows the non-unitary character of the Susskind–Glogower phase operator.

If the expectation values are calculated in the 'physical states', according to their definition by Pegg and Barnett [27, 28], the last term in equation (46) becomes negligible and some additional useful relations between expectation values in such states of the Pegg–Barnett phase operators and of the Susskind–Glogower phase operators can be obtained. For example, the following relations hold [34]:

$$\langle \exp(im\hat{\phi}_\theta)\rangle_p = \langle \hat{\exp}(im\phi_{SG})\rangle_p, \tag{50}$$

$$\langle \cos\hat{\phi}_\theta\rangle_p = \tfrac{1}{2}\langle \exp(i\hat{\phi}_\theta) + \exp(-i\hat{\phi}_\theta)\rangle_p = \langle \hat{\cos}\,\phi_{SG}\rangle_p, \tag{51}$$

$$\langle \sin\hat{\phi}_\theta\rangle_p = \frac{1}{2i}\langle \exp(i\hat{\phi}_\theta) - \exp(-i\hat{\phi}_\theta)\rangle_p = \langle \hat{\sin}\,\phi_{SG}\rangle_p, \tag{52}$$

$$\langle \cos^2 \hat{\phi}_\theta \rangle_p = \tfrac{1}{4} \langle \exp(i2\hat{\phi}_\theta) + \exp(-i2\hat{\phi}_\theta) + 2 \rangle_p$$
$$= \langle \widehat{\cos}^2 \phi_{SG} \rangle_p + \tfrac{1}{4} \langle (|0\rangle\langle 0|) \rangle_p, \tag{53}$$

$$\langle \sin^2 \hat{\phi}_\theta \rangle_p = -\tfrac{1}{4} \langle \exp(i2\hat{\phi}_\theta) + \exp(-i2\hat{\phi}_\theta) - 2 \rangle_p$$
$$= \langle \widehat{\sin}^2 \phi_{SG} \rangle_p + \tfrac{1}{4} \langle (|0\rangle\langle 0|) \rangle_p, \tag{54}$$

where the subscript p refers to a physical state expectation value.

In the two-mode case considered in this paper, we can discuss, besides the phase operators for individual modes, the two-mode phase characteristics such as the phase difference between the two modes. In the Pegg–Barnett formalism, the phase difference operator is simply the difference of the phase operators for the two modes. For the physical states we have the following relations

$$\langle \exp[im(\hat{\phi}_+ - \hat{\phi}_-)] \rangle_p = \langle \exp(im\hat{\phi}_+) \exp(-im\hat{\phi}_-) \rangle_p$$
$$= \left\langle \sum_{n=0}^{\infty} (|n\rangle\langle n+m|)_+ \sum_{k=0}^{\infty} (|k+m\rangle\langle k|)_- \right\rangle_p$$
$$= \left\langle \widehat{\exp}_+(im\phi_{SG}) \, \widehat{\exp}_-(-im\phi_{SG}) \right\rangle_p, \tag{55}$$

$$\langle \cos(\hat{\phi}_+ - \hat{\phi}_-) \rangle_p = \langle \widehat{\cos}(\phi_+ - \phi_-)_{SG} \rangle_p \tag{56}$$

$$\langle \cos^2(\hat{\phi}_+ - \hat{\phi}_-) \rangle_p = \langle [\widehat{\cos}(\phi_+ - \phi_-)_{SG}]^2 \rangle_p$$
$$+ \tfrac{1}{4} \langle [(\hat{1})_+ (|0\rangle\langle 0|)_- + (|0\rangle\langle 0|)_+ (\hat{1})_-] \rangle_p, \tag{57}$$

$$\langle [\Delta \cos(\hat{\phi}_+ - \hat{\phi}_-)]^2 \rangle_p = \langle [\Delta \widehat{\cos}(\phi_+ - \phi_-)_{SG}]^2 \rangle_p$$
$$+ \tfrac{1}{4} \langle [(\hat{1})_+ (|0\rangle\langle 0|)_- + (|0\rangle\langle 0|)_+ (\hat{1})_-] \rangle_p, \tag{58}$$

where the operators $\hat{\phi}_\pm$ are the phase operators for the two circularly polarized modes of the field defined as in equation (43), and the subscripts $\pm$ denote the mode for which the corresponding operators are defined. Relations (50)–(58) will be used in the paper to describe phase properties of the field propagating in a Kerr medium. Since the state (39) of light propagating in a Kerr medium is a 'physical state'

$$\left(\lim_{n_\pm \to \infty} b_{n_\pm} = 0 \right),$$

the above relations can be applied when the expectation values in this state are calculated.

Generalization of the Pegg–Barnett formalism to the two-mode case studied in this paper is straightforward, and the joint phase probability amplitude for the field being in state (39) is given by

$$\langle \theta_{m_-} | \langle \theta_{m_+} | \psi(\tau) \rangle = (s_+ + 1)^{-1/2} (s_- + 1)^{-1/2}$$
$$\times \sum_{n_+=0}^{s_+} \sum_{n_-=0}^{s_-} b_{n_+} b_{n_-} \exp\left\{ in_+(\varphi_+ - \theta_{m_+}) + in_-(\varphi_- - \theta_{m_-}) \right.$$
$$\left. + i\frac{\tau}{2} [n_+(n_+ - 1) + n_-(n_- - 1) + 4dn_+n_-] \right\}. \tag{59}$$

Since the initial states of the two modes are coherent states, i.e. partial phase states, it is convenient to choose phase values windows symmetrical to the phases $\varphi_\pm$ of the coherent states. This means

$$\theta_0^\pm = \varphi_\pm - \frac{\pi s_\pm}{s_\pm + 1} \tag{60}$$

and

$$\varphi_\pm - \theta_{m_\pm} = -\theta_{\mu_\pm}, \tag{61}$$

where the new phase labels $\mu_\pm$ run in unit step between the values $-s_\pm/2$ and $s_\pm/2$. By taking the modulus square of equation (59), after taking into account equations (60) and (61), and performing the continuum limit transition by making the replacements

$$\sum_{\mu_\pm = -s_\pm/2}^{s_\pm/2} \frac{2\pi}{s_\pm + 1} \to \int_{-\pi}^{\pi} \mathrm{d}\theta_\pm, \tag{62}$$

we arrive at the continuous, joint phase distribution function given by

$$P(\theta_+, \theta_-) = \frac{1}{(2\pi)^2} \left| \sum_{n_+=0}^{\infty} \sum_{n_-=0}^{\infty} b_{n_+} b_{n_-} \times \exp\left\{ -in_+\theta_+ - in_-\theta_- + i\frac{\tau}{2}[n_+(n_+ - 1) + n_-(n_- - 1) + 4dn_+n_-] \right\} \right|^2, \tag{63}$$

with the normalization

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P(\theta_+, \theta_-)\, \mathrm{d}\theta_+\, \mathrm{d}\theta_- = 1. \tag{64}$$

The phase distribution function $P(\theta_+, \theta_-)$ given by equation (63) describes the phase properties of elliptically polarized light propagating through a Kerr medium. This function depends on τ, which means the evolution of phase properties of light propagating in the medium. Because of the double summation in formula (63), it is not easy to predict how the form of the distribution function $P(\theta_+, \theta_-)$ changes during the evolution. However, due to Poissonian factors $b_{n_\pm}$, the summations in equation (63) can be evaluated numerically if the mean numbers of photons in the two modes, $N_\pm = |\alpha_\pm|^2$, are not too great. Examples of the evolution of the phase distribution function $P(\theta_+, \theta_-)$ are shown in figures 1–3, where this function is plotted in a three-dimensional format for various sets of parameters.

In figure 1, one mode is initially assumed as vacuum ($N_+ = 0$), which means that the distribution is flat along the θ_+ direction initially, and it remains flat all the time. Along the direction θ_-, where $N_- = 4$, the distribution is peaked initially at $\theta_- = 0$, i.e. at the phase φ_- of the coherent state $|\alpha_-\rangle$, and as the evolution proceeds the peak of the distribution is shifted and the distribution becomes broader. This behaviour corresponds to the behaviour of the phase distribution function in the one-mode case of the anharmonic oscillator [35]. If the mean number of photons is different from zero in both modes, the distribution $P(\theta_+, \theta_-)$ is peaked initially for $\theta_+ = \theta_- = 0$, and this peak is shifted and broadened along both directions.

The shape of the distribution depends on the values of the mean number of photons $N_\pm$ in the two modes, which is clearly seen in figures 2 and 3. It is also seen that the initially chosen window of the phase values, symmetrized with respect to the phases $\varphi_\pm$ of the initial coherent states of the field, is no longer well suited to describe the phase distribution. Because of the shift of the maximum of the distribution, the peak splits into pieces. To minimize the phase variance, the window should be shifted dynamically during the evolution. In figures 1–3 we have assumed the asymmetry parameter d is unity.

Integrating the distribution function $P(\theta_+, \theta_-)$ over one of the phases θ_+ or θ_- leads to the marginal distribution $P(\theta_-)$ or $P(\theta_+)$ for the individual phases. We have for example,

$$P(\theta_+) = \int_{-\pi}^{\pi} P(\theta_+, \theta_-)\, d\theta_- = \frac{1}{2\pi} \sum_{n_-=0}^{\infty} b_{n_-}^2 \left\{ 1 + 2 \sum_{n_+ > n'_+} b_{n_+} b_{n'_+} \right.$$
$$\left. \times \cos\left\{ (n_+ - n'_+) \left[\theta_+ - \frac{\tau}{2}(n_+ + n'_+ - 1 + 4dn_-) \right] \right\} \right\}, \tag{65}$$

and the expression for $P(\theta_-)$ can be obtained from (65) by interchanging the subscripts plus and minus. If there is no coupling between the two modes in the medium, i.e. $d=0$, the summation over n_- gives unity and the distribution $P(\theta_+)$ becomes the same as in the one-mode case [35]. According to equation (32), it can happen only for $\frac{1}{2} \leftrightarrow \frac{1}{2}$ transitions contributing to the coupling constant. Otherwise,

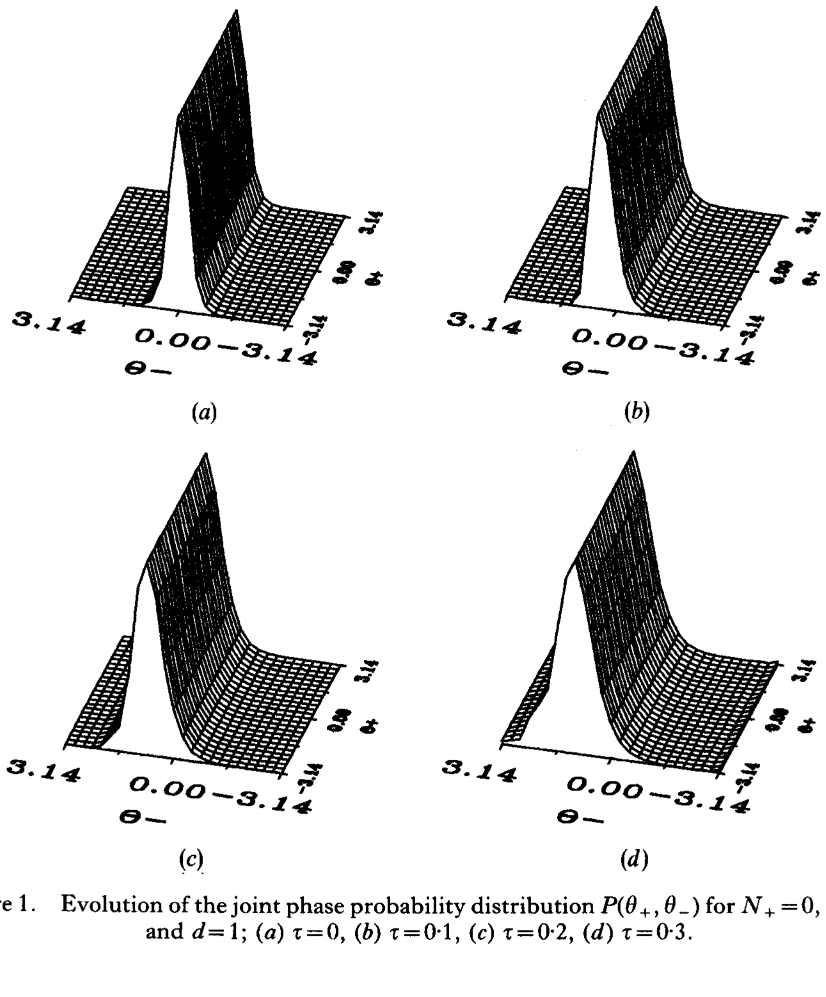

Figure 1. Evolution of the joint phase probability distribution $P(\theta_+, \theta_-)$ for $N_+=0$, $N_-=4$ and $d=1$; (*a*) $\tau=0$, (*b*) $\tau=0{\cdot}1$, (*c*) $\tau=0{\cdot}2$, (*d*) $\tau=0{\cdot}3$.

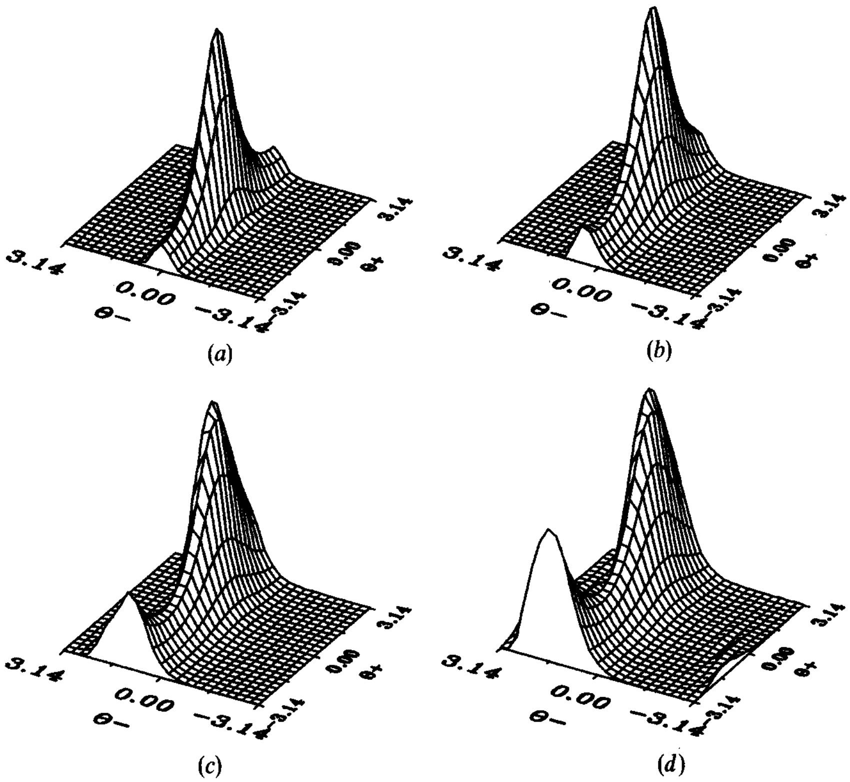

Figure 2. Same as figure 1, but for $N_+ = 0{\cdot}25$ and other values unchanged.

the phase distribution $P(\theta_+)$ for one mode depends on the intensity of the other mode and the asymmetry parameter d of the nonlinear medium. In figure 4 we illustrate the dependence of $P(\theta_+)$ on the mean number of photons N_- of the other mode, assuming $N_+ = 0{\cdot}25$, $d = 1$ and $\tau = 0{\cdot}1$. It is clear from figure 4 that the distribution $P(\theta_+)$ is not only shifted but also broadened as the intensity of the other mode increases.

As the phase distribution function $P(\theta_+, \theta_-)$ (or $P(\theta_+)$) is known, the expectation values and the variances of the Hermitian phase operators defined by equation (43) can be calculated in a classical-like manner by performing appropriate integrations. We have

$$\begin{aligned}\langle\psi(\tau)|\hat{\phi}_+|\psi(\tau)\rangle &= \varphi_+ + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \theta_+ P(\theta_+, \theta_-)\,\mathrm{d}\theta_+\,\mathrm{d}\theta_- \\ &= \varphi_+ + \int_{-\pi}^{\pi} \theta_+ P(\theta_+)\,\mathrm{d}\theta_+ \\ &= \varphi_+ - 2\sum_{n_-=0}^{\infty} b_{n_-}^2 \sum_{n_+ > n'_+} b_{n_+} b_{n'_+} \frac{(-1)^{n_+ - n'_+}}{n_+ - n'_+} \\ &\quad \times \sin\left\{\frac{\tau}{2}(n_+ - n'_+)(n_+ + n'_+ - 1 + 4dn_-)\right\},\end{aligned} \tag{66}$$

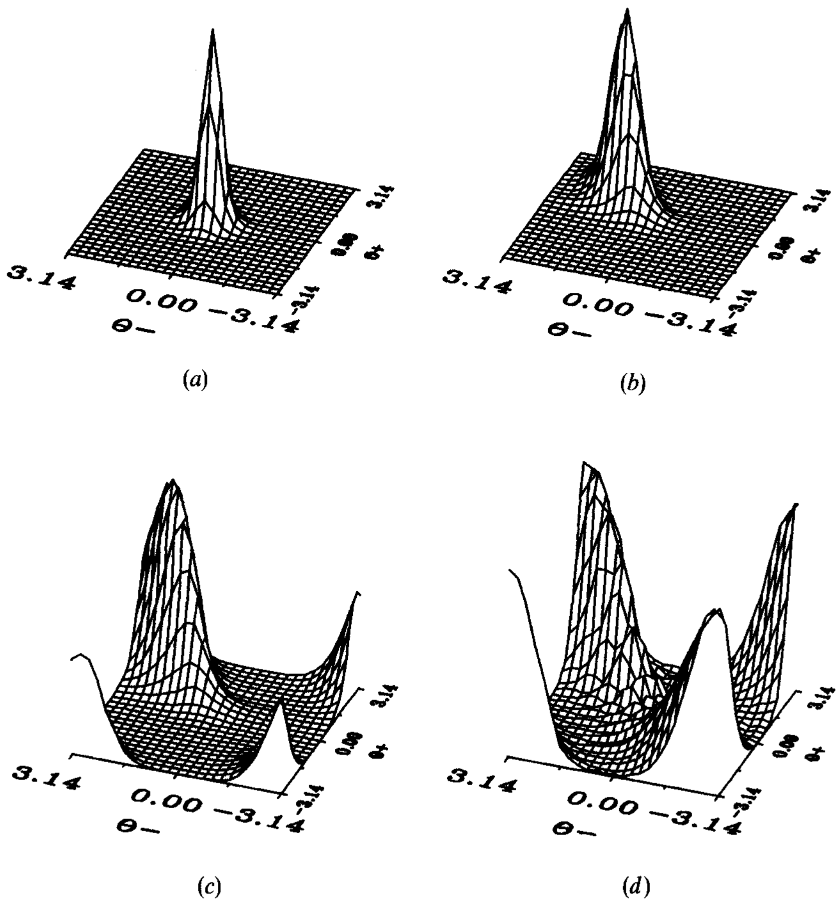

Figure 3. Same as figure 1, but for $N_+ = 4$ and other values unchanged.

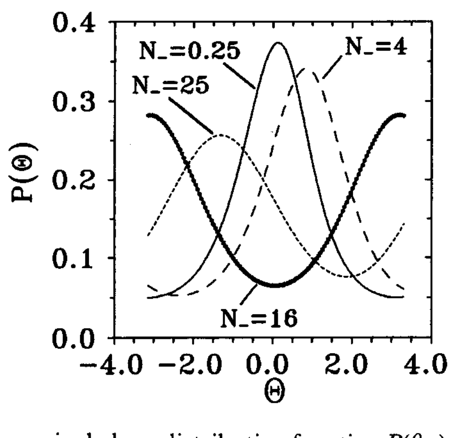

Figure 4. Plot of the marginal phase distribution function $P(\theta_+)$ for $N_+ = 0{\cdot}25$, $d = 1$, $\tau = 0{\cdot}1$ and various values of N_-.

and for the variance

$$\begin{aligned}\langle\psi(\tau)|(\Delta\hat{\phi}_+)^2|\psi(\tau)\rangle &= \langle\psi(\tau)|\hat{\phi}_+^2|\psi(\tau)\rangle - \langle\psi(\tau)|\hat{\phi}_+|\psi(\tau)\rangle^2\\ &= \int_{-\pi}^{\pi} \theta_+^2 P(\theta_+)\,\mathrm{d}\theta_+ - \left[\int_{-\pi}^{\pi} \theta_+ P(\theta_+)\,\mathrm{d}\theta_+\right]^2\\ &= \frac{\pi^2}{3} + 4\sum_{n_-=0}^{\infty} b_{n_-}^2 \sum_{n_+>n'_+} b_{n_+} b_{n'_+} \frac{(-1)^{n_+-n'_+}}{(n_+-n'_+)^2}\\ &\quad\times\cos\left\{\frac{\tau}{2}(n_+-n'_+)(n_++n'_+-1+4dn_-)\right\}\\ &\quad-\left\{2\sum_{n_-=0}^{\infty} b_{n_-}^2 \sum_{n_+>n'_+} b_{n_+} b_{n'_+} \frac{(-1)^{n_+-n'_+}}{n_+-n'_+}\right.\\ &\quad\left.\times\sin\left\{\frac{\tau}{2}(n_+-n'_+)(n_++n'_+-1+4dn_-)\right\}\right\}^2. \qquad (67)\end{aligned}$$

Again, if $d=0$, formulae (66) and (67) go over into the corresponding formulae for the anharmonic oscillator model [35]. It is clear from the distribution function (63) as well as from formulae (66) and (67) that, for $2d$ being an integer, phase properties of light propagating in the medium are periodic with the period $T=2\pi$, the same as in the one-mode case. For $2d$ being a fraction of integers, the periodic behaviour is still preserved but with different period. Generally, however, if $2d$ cannot be expressed as a fraction of integers the periodicity of phase properties is lost. This means that periodic behaviour appears for definite symmetry of the non-linear properties of the medium only.

Formula (66), which describes the evolution of the mean phase of one mode, shows that due to the coupling between the two modes there is a shift in phase of the 'plus' mode that depends on the intensity of the 'minus' mode. This is shown in figure 5, where the mean phase is drawn, according to formula (66), for various values of the mean number of photons N_-.

One can also expect from the broadening of the phase distribution $P(\theta_+)$ shown in figure 4 that the variance of the phase operator $\hat{\phi}_+$ will increase as the number of

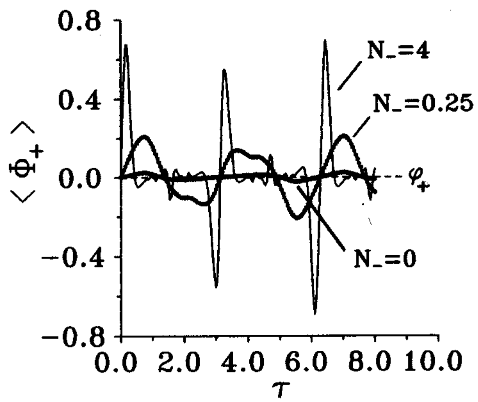

Figure 5. Evolution of the mean value of the phase $\langle\hat{\phi}_+\rangle$ for $N_+=0{\cdot}25$, $d=1$ and various values of N_-.

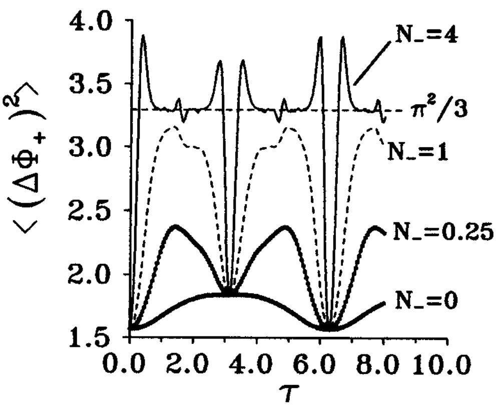

Figure 6. Evolution of the phase variance $\langle(\Delta\hat{\phi}_+)^2\rangle$ for $N_+=0{\cdot}25$, $d=1$ and various values of N_-.

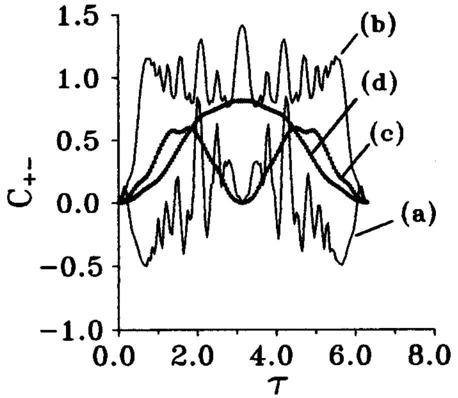

Figure 7. Evolution of the intermode phase correlation function $C_{+-}(\tau)$: (*a*) $N_+=0{\cdot}25$, $N_-=4$ and $d=1$; (*b*) $N_+=0{\cdot}25$, $N_-=4$ and $d=1/2$; (*c*) $N_+=N_-=0{\cdot}25$ and $d=1$; (*d*) $N_+=N_-=0{\cdot}25$ and $d=1/2$.

photons N_- in the other mode increases. This is shown in figure 6, where the variance (67) is plotted for various N_-. For $N_->1$ the variance rapidly increases to the values around $\pi^2/3$—the value for the uniformly distributed phase. This means that the phase of the 'plus' mode is randomized because of interaction with the other mode. Of course, there is also the randomization effect of the 'minus' mode from interaction with the 'plus' mode.

Except for the phase properties of the individual modes, it is interesting in the two-mode case to study the behaviour of the phase difference between the two modes. In the Pegg–Barnett formalism the phase-difference operator is simply the difference of the phase operators for the two modes. The mean value of the phase-difference operator is given by

$$\langle\psi(\tau)|(\hat{\phi}_+-\hat{\phi}_-)|\psi(\tau)\rangle=\langle\psi(\tau)|\hat{\phi}_+|\psi(\tau)\rangle-\langle\psi(\tau)|\hat{\phi}_-|\psi(\tau)\rangle, \tag{68}$$

and can be calculated according to equation (66) and the corresponding equation for $\langle\hat{\phi}_-\rangle$ (obtained by interchanging '+' and '−').

To calculate the variance of the phase-difference operator we can use

$$\langle[\Delta(\hat{\phi}_+-\hat{\phi}_-)]^2\rangle=\langle(\Delta\hat{\phi}_+)^2\rangle+\langle(\Delta\hat{\phi}_-)^2\rangle-2\{\langle\hat{\phi}_+\hat{\phi}_-\rangle-\langle\hat{\phi}_+\rangle\langle\hat{\phi}_-\rangle\}. \tag{69}$$

The variance $\langle(\Delta\hat{\phi}_+)^2\rangle$ and $\langle(\Delta\hat{\phi}_-)^2\rangle$ can be calculated according to equation (67) and its counterpart for the 'minus' mode obtained by interchanging '+' and '−'. The last term describing the correlation between the phases of the two modes can be calculated as

$$\begin{aligned}
\mathbf{C}_{+-}(\tau)&=\langle\psi(\tau)|\hat{\phi}_+\hat{\phi}_-|\psi(\tau)\rangle-\langle\psi(\tau)|\hat{\phi}_+|\psi(\tau)\rangle\langle\psi(\tau)|\hat{\phi}_-|\psi(\tau)\rangle\\
&=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\theta_+\theta_-P(\theta_+,\theta_-)\,\mathrm{d}\theta_+\,\mathrm{d}\theta_-\\
&\quad-\int_{-\pi}^{\pi}\theta_+P(\theta_+)\,\mathrm{d}\theta_+\int_{-\pi}^{\pi}\theta_-P(\theta_-)\,\mathrm{d}\theta_-\\
&=4\sum_{n_+>n'_+}\sum_{n_->n'_-}b_{n_+}b_{n'_+}b_{n_-}b_{n'_-}\frac{(-1)^{n_+-n'_+}}{n_+-n'_+}\frac{(-1)^{n_--n'_-}}{n_--n'_-}\\
&\qquad\times\sin\left\{\frac{\tau}{2}(n_+-n'_+)[n_++n'_+-1+2d(n_-+n'_-)]\right\}\\
&\qquad\times\sin\left\{\frac{\tau}{2}(n_--n'_-)[n_-+n'_--1+2d(n_++n'_+)]\right\}\\
&\quad-4\left\{\sum_{n_-=0}^{\infty}b_{n_-}^2\sum_{n_+>n'_+}b_{n_+}b_{n'_+}\frac{(-1)^{n_+-n'_+}}{n_+-n'_+}\sin\left\{\frac{\tau}{2}(n_+-n'_+)(n_++n'_+-1+4\mathrm{d}n_-)\right\}\right\}\\
&\quad\times\left\{\sum_{n_+=0}^{\infty}b_{n_+}^2\sum_{n_->n'_-}b_{n_-}b_{n'_-}\frac{(-1)^{n_--n'_-}}{n_--n'_-}\sin\left\{\frac{\tau}{2}(n_--n'_-)(n_-+n'_--1+4\mathrm{d}n_+)\right\}\right\}
\end{aligned} \tag{70}$$

From equation (70) it is evident that the phase correlation $\mathbf{C}_{+-}(\tau)$ is equal to zero if the two phases are uncorrelated, i.e. when $d=0$. Of course, there are no correlations between the two phases for $\tau=0$ because the initial state of the field is a product state. During the evolution, the correlation between the two phases arises. This correlation, calculated from formula (70), is plotted against τ in figure 7 for various mean photon numbers $N_\pm$ of the two modes and two values of the asymmetry parameter d. The strengths of the correlation depends crucially on the value of the asymmetry parameter d. The highest values of the correlation are obtained for $d=1/2$. This means that the minimum of the phase difference variance, in view of equation (69), is obtained for $d=1/2$. The phase difference variance is shown in figure 8.

The phase correlation function defined by equation (70) is an essentially two-mode characteristic of the field propagating in a Kerr medium that can be calculated using the Pegg–Barnett phase formalism. The phases of the two modes become correlated in the course of propagation.

Except for the phases themselves and their variances, the sine and cosine functions of the individual phases and of the phase difference can be calculated with the Pegg–Barnett formalism. These phase characteristics of the field can be compared with the corresponding results of the Susskind–Glogower formalism. The

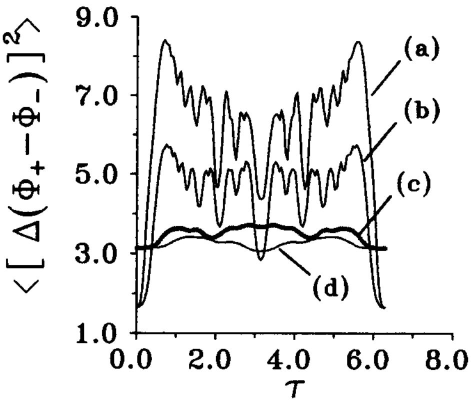

Figure 8. Evolution of the phase-difference variance $\langle[\Delta(\hat{\phi}_+-\hat{\phi}_-)]^2\rangle$; the parameters describing curves (*a*)–(*d*) are the same as in figure 7.

sine and cosine functions of the individual mode phases and their variances can be calculated and compared with the Susskind–Glogower results using formulae (50)–(54). This gives, for example,

$$\langle\psi(\tau)|\exp(im\hat{\phi}_+)|\psi(\tau)\rangle=\sum_{n=0}^{\infty}\langle\psi(\tau)(|n\rangle\langle n+m|)_+\psi(\tau)\rangle$$

$$=N_+^{m/2}\sum_{n_+=0}^{\infty}\sum_{n_-=0}^{\infty}b_{n_+}^2b_{n_-}^2\frac{\exp\left\{im\left[\varphi_++\frac{\tau}{2}(2n_++m-1+4dn_-)\right]\right\}}{[(n_++1)(n_++2)\dots(n_++m)]^{1/2}},\tag{71}$$

and according to (51) and (52)

$$\langle\psi(\tau)|\cos\hat{\phi}_+|\psi(\tau)\rangle=N_+^{1/2}\sum_{n_+=0}^{\infty}\sum_{n_-=0}^{\infty}b_{n_+}^2b_{n_-}^2\frac{\cos[\varphi_++\tau(n_++2dn_-)]}{(n_++1)^{1/2}},\tag{72}$$

$$\langle\psi(\tau)|\sin\hat{\phi}_+|\psi(\tau)\rangle=N_+^{1/2}\sum_{n_+=0}^{\infty}\sum_{n_-=0}^{\infty}b_{n_+}^2b_{n_-}^2\frac{\sin[\varphi_++\tau(n_++2dn_-)]}{(n_++1)^{1/2}},\tag{73}$$

and from (53) and (54)

$$\left.\begin{array}{l}\langle\psi(\tau)|\cos^2\hat{\phi}_+|\psi(\tau)\rangle\\ \langle\psi(\tau)|\sin^2\hat{\phi}_+|\psi(\tau)\rangle\end{array}\right\}=\frac{1}{2}\pm\frac{1}{2}N_+\sum_{n_+=0}^{\infty}\sum_{n_-=0}^{\infty}b_{n_+}^2b_{n_-}^2\frac{\cos[2\varphi_++\tau(2n_++1+4dn_-)]}{[(n_++1)(n_++2)]^{1/2}},\tag{74}$$

The results for the phase operator $\hat{\phi}_-$ can be obtained from (71)–(74) by interchanging the indexes '+' and '−'. For $d=0$, the summation over n_- gives unity and formulae (71)–(74) go over into corresponding formulae for the one-mode anharmonic oscillator model [35, 36]. It is clear from (71)–(74) that the interaction with the other mode changes the expectation values of the sine and cosine functions of the phase $\hat{\phi}_+$. However, there is no big difference in behaviour of the quantities given by (71)–(74) and their counterparts for the one-mode case, which we have already discussed in detail [35], and we refer rather to those results instead of discussing (71)–(74).

Here, we will concentrate on the essentially two-mode phase characteristics of light, such as the sine and cosine functions of the phase difference. Using equations (55) and (39), we arrive at

$$\langle\psi(\tau)|\exp[im(\hat{\phi}_+-\hat{\phi}_-)]|\psi(\tau)\rangle=(N_-/N_+)^{m/2}\exp[i\tau m^2(2d-1)]$$
$$\times\sum_{n_+=m}^{\infty}b_{n_+}^2[(n_+-m+1)(n_+-m+2)\ldots(n_+-1)n_+]^{1/2}$$
$$\times\exp\{im[\varphi_+-\tau(2d-1)n_+]\}\sum_{n_-=0}^{\infty}b_{n_-}^2\frac{\exp\{-im[\varphi_--\tau(2d-1)n_-]\}}{[(n_-+1)(n_-+2)\ldots(n_-+m)]^{1/2}},\tag{75}$$

which leads to

$$\langle\psi(\tau)|\begin{Bmatrix}\cos(\hat{\phi}_+-\hat{\phi}_-)\\ \sin(\hat{\phi}_+-\hat{\phi}_-)\end{Bmatrix}|\psi(\tau)\rangle=\left(\frac{N_-}{N_+}\right)^{1/2}\sum_{n_+=1}^{\infty}\sum_{n_-=0}^{\infty}b_{n_+}^2b_{n_-}^2\frac{\sqrt{n_+}}{\sqrt{(n_-+1)}}$$
$$\times\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}[\varphi_+-\varphi_--\tau(2d-1)(n_+-n_--1)],\tag{76}$$

$$\langle\psi(\tau)|\begin{Bmatrix}\cos^2(\hat{\phi}_+-\hat{\phi}_-)\\ \sin^2(\hat{\phi}_+-\hat{\phi}_-)\end{Bmatrix}|\psi(\tau)\rangle=\frac{1}{2}\pm\frac{1}{2}\frac{N_-}{N_+}\sum_{n_+=2}^{\infty}\sum_{n_-=0}^{\infty}b_{n_+}^2b_{n_-}^2\left[\frac{(n_+-1)n_+}{(n_-+1)(n_-+2)}\right]^{1/2}$$
$$\times\cos\{2(\varphi_+-\varphi_-)-2\tau(2d-1)(n_+-n_--2)\}.\tag{77}$$

An immediate result, seen clearly from formulae (75)–(77), is that the evolution of the sine and cosine functions of the phase difference depends strongly on the value of the asymmetry parameter d. For $d=1/2$, $2d-1=0$, and formulae (75)–(77) do not depend on τ, which means that the expectation value as well as the variance of the cosine (or sine) of the phase difference remain unchanged during the evolution. For $d\neq 1/2$, the cosine and the sine functions of the phase difference do change during the propagation, and examples of such evolution are shown in figures 9 and 10.

In figure 9, the variance of the phase difference cosine is plotted against τ for various values of the mean photon numbers N_+ and N_- and d values of 1/2 and 1.

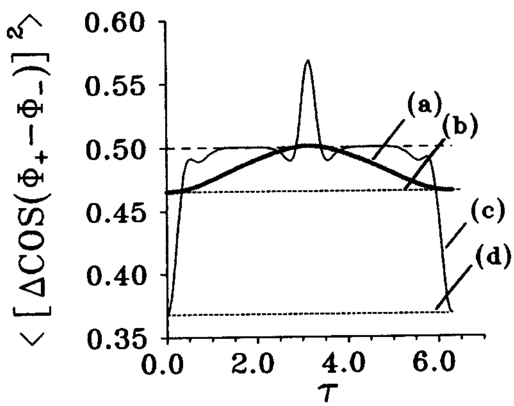

Figure 9. Evolution of the variance of the phase difference cosine function $\langle[\Delta\cos(\hat{\phi}_+-\hat{\phi}_+)]^2\rangle$; parameters of the curves (a)–(d) are the same as in figure 7. The 0·5-level corresponds to the uniformly distributed phase difference; it is also obtained for $N_-=0$.

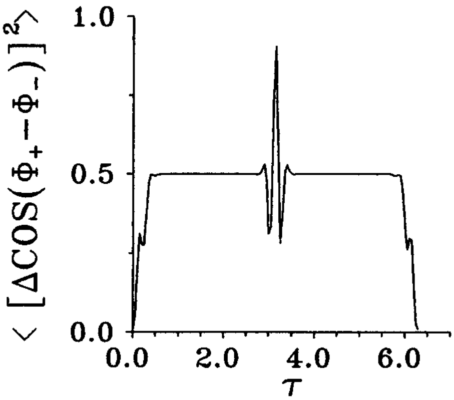

Figure 10. Same as figure 9, but for $N_+=4$, $N_-=16$ and $d=1$.

There is no τ-dependence for $d=1/2$, and the noise level lowers as the numbers of photons $N_\pm$ increase. For $d=1$, the cosine variance increases initially, becomes close to the value 0·5, and after the period goes down to its initial value. The value 0·5 of the variance corresponds to the uniformly distributed phase difference. This means the randomization of the phase difference during propagation.

This effect is even more pronounced for $N_\pm \gg 1$, as seen from figure 10, where for most of the period the variance is 0·5. The results obtained here agree with earlier results for the degree of polarization of light propagating in a Kerr medium [24, 25], and have clear interpretation. Only for $d=1/2$, light remains completely polarized, if it were initially.

Classically, uniformly distributed phase difference means unpolarized light, and this result is clearly seen also from our quantum mechanical calculations. However, it is interesting to compare the variances for the cosine function of the phase difference (figure 9) and for the phase difference itself (figure 8). Although the reduction of the phase difference variance for $d=1/2$ is quite evident, it still evolves in τ, whereas the cosine variance does not depend on τ. The cosine function of the phase difference rather than the phase difference itself defines the degree of polarization of light propagating in the medium. The Pegg–Barnett phase formalism makes it possible to distinguish between the two phase characteristics.

Using formulae (58), (76) and (77), one can easily compare the results obtained within the Pegg–Barnett formalism and the results of the Susskind–Glogower formalism. In our case of light propagating in a Kerr medium, the Susskind–Glogower results for variance of the phase difference cosine are shifted down with respect to the Pegg–Barnett results by $\frac{1}{4}(e^{-N_+}+e^{-N_-})$. This shift is essential for small values of N_+ and N_- only. For the numbers of photons in figure 10, the two curves are already indistinguishable.

Of course, for $N_\pm \to \infty$ the classical behaviour of light propagating through a Kerr medium is obtained.

Conclusions

In this paper we have considered phase properties of elliptically polarized light propagating in a nonlinear Kerr medium using the new Hermitian phase formalism introduced recently by Pegg and Barnett [26–28]. To describe elliptically polarized light, the two-mode description of the field is needed. Using such a description, we have calculated the joint distribution $P(\theta_+, \theta_-)$ for the phases θ_+ and θ_- of the two modes. Integrating this function over one of the variables, the marginal probability distributions $P(\theta_+)$ and $P(\theta_-)$ are obtained. We have discussed the evolution of these distribution functions of light propagating in a Kerr medium. We have shown that the maximum of the distribution function is shifted and the distribution broadened in the course of the propagation of light in the medium.

Knowledge of these distribution functions enables the expectation values and variances of the phase operators to be calculated classically. We have performed such calculations, obtaining results for the individual mode phase variances as well as for the phase difference variance. We have shown that the phase of one mode is randomized due to its own nonlinear interaction with the medium as well as due to the coupling to the other mode. The corresponding analytical formulae have been obtained and illustrated graphically.

Special attention has been paid to the essentially two-mode phase characteristics of the field. The correlation between the phases of the two modes has been calculated, and we have shown that such a correlation builds up during propagation. The degree of correlation depends strongly on the asymmetry parameter d of the medium; the strongest correlation arises for $d=1/2$. This correlation significantly lowers the phase difference variance.

The results for the sine and cosine functions of the phase difference and their variances have been obtained and compared with the results of the Susskind–Glogower formalism. We have shown that the expectation values and variances of the sine cosine functions of the phase differences are not affected by the propagation process when $d=1/2$. In this case, the degree of polarization of the field is also not affected. Otherwise, for $d\neq 1/2$, the variance of the phase difference cosine rapidly increases and becomes close to 1/2—the value for the randomly distributed phase difference. This means randomization of the phase difference, which corresponds to the degradation of the degree of polarization of the field. For $d\neq 1/2$ and $N_\pm \gg 1$, the light rapidly becomes unpolarized. The phase properties of elliptically polarized light propagating through a Kerr medium considered in this paper have confirmed earlier results [24, 25] predicting the degradation of the degree of polarization of light. This effect is quantum mechanical in nature and cannot be obtained if the field is treated classically.

The joint phase probability distribution discussed in this paper splits into separate peaks if the state of the field becomes a discrete superposition of coherent states [37], and it is a very spectacular way of presenting such superpositions.

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