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Phase properties of self-squeezed states generated by the anharmonic oscillator

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Abstract. The phase properties of the self-squeezed states generated during the evolution of the anharmonic oscillator are discussed from the point of view of the new phase formalism of Pegg and Barnett. The phase distribution, the expectation values and the variances of the Hermitian phase operator are obtained and their evolution illustrated graphically. The mean values for the phase cosine and sine functions as well as their variances are also calculated. The results are compared to the Susskind–Glogower formalism results and the results based on the measured phase concept. The relation between squeezing and the phase properties of the field is discussed briefly.

1. Introduction

Recently, Pegg and Barnett [1–3] have shown that an Hermitian optical phase operator $\hat{\phi}_\theta$ exists. It can be constructed from the phase states. This result contradicts the long-lasting common belief that no such operator can be constructed. With the use of this operator unitary operators $\exp(\pm i\hat{\phi}_\theta)$ can be constructed and a polar decomposition of the photon annihilation operator can be performed. This new formalism makes it possible to describe the quantum properties of optical phases in a direct way within quantum mechanics. There is no need for semi-classical or phenomenological methods.

In quantum optics, special attention has been paid in recent years to a class of optical field states that are called squeezed states [4]. These are non-classical states with phase sensitive noise, and it is very interesting to study their phase properties on the basis of the new phase formalism.

Phase properties of the ideal squeezed states, or the two-photon coherent states of Yuen [5], have been examined by Sanders *et al.* [6], Yao [7] and Fan *et al.* [8] with the use of the Susskind–Glogower [9] phase formalism with the non-unitary phase operators. Recently, Vaccaro and Pegg [10] have re-examined phase properties of such states from the point of view of the new Pegg–Barnett formalism. The ideal squeezed states are a special class of squeezed states with the minimum uncertainty. They are not, however, the only squeezed states.

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Quite different squeezed states can be produced during the evolution of the anharmonic oscillator. The high degree of squeezing that can be obtained from the model was shown by Tanaś and Kielich [11] for the two-mode version, and by Tanaś [12] for the one-mode version of the anharmonic oscillator. A physical situation in which the model can be applied is propagation of laser light in a nonlinear Kerr medium. Since squeezing that can occur in such a process depends on the intensity of light propagating in the medium, Tanaś and Kielich [11] referred to such squeezing as *self-squeezing*. This is the field itself that squeezes its own quantum fluctuations due to self-interaction *via* the nonlinear Kerr medium. Consequently, the states of the field produced in such a process can be referred to as *self-squeezed states*. Squeezing in the same process was later considered by Kitagawa and Yamamoto [13] who used the name crescent squeezing because of the crescent shape of the quasi-probability distribution contours obtained in the process. The evolution of the quasi-probability distribution $Q(\alpha, \alpha^*)$ in the anharmonic oscillator model has been considered by Milburn [14], Milburn and Holmes [15], Peřinova and Lukš [16] and Daniel and Milburn [17]. Recently, the two-mode version of the model considered by Tanaś and Kielich [11] was re-examined by Agarwal and Puri [18].

Phase properties of the states produced in the course of the evolution of the anharmonic oscillator have been studied by Gerry [19] within the framework of the Susskind–Glogower formalism and by Lynch [20] who applied the concept of measured phase operators introduced by Barnett and Pegg [21]. Recently, Gerry [22] has considered the phase fluctuations in the anharmonic oscillator model from the point of view of the Hermitian phase formalism of Pegg and Barnett [1–3] discussing two limiting cases of very high and very low intensities of the field.

In this paper, we re-examine the phase properties of the field generated by the anharmonic oscillator using the new phase formalism of Pegg and Barnett [1–3]. The phase probability density is obtained for such states, and the expectation values of the Hermitian phase operator as well as the variances of the phase operator are calculated as functions of the evolution time. These phase characteristics cannot be obtained in earlier phase formalisms. The cosine and sine functions of the phase operator as well as their variances are calculated and compared with the corresponding results of the Susskind–Glogower formalism and the measured phase concept.

2. The Hermitian phase formalism

The new formalism recently introduced by Pegg and Barnett [1–3] to describe the phase properties of a single mode field has successfully overcome the difficulties associated with the existence of the Hermitian phase operator. Pegg and Barnett [1–3] have shown that the Hermitian phase operator can be constructed from the phase states. As the Hermitian phase operator is constructed, quantities like expectation values of the phase operator and/or variances can be calculated for given state of the field. It is also possible to get the phase probability density which is a very spectacular phase characteristic of the optical field. These are new characteristics of the field accessible to investigation due to the new formalism. Of course, such phase functions as cosine and sine and their variances, which in the Pegg–Barnett formalism are actual cosine and sine functions of the Hermitian phase operator, can also be studied and compared with their counterparts in the Susskind–Glogower or measured phase formalism.

Here, we adduce the main formulas of Pegg and Barnett [1–3] which we will use in the paper to study the phase properties of the anharmonic oscillator states. Their

approach is based on introducing a finite $(s+1)$-dimensional space Ψ spanned by the number states $|0\rangle, |1\rangle, \ldots, |s\rangle$. The Hermitian phase operator operates on this finite space, and after all necessary expectation values have been calculated in Ψ, the value of s is allowed to tend to infinity. A complete orthonormal basis of $(s+1)$ states is defined on Ψ as

$$|\theta_m\rangle \equiv (s+1)^{-1/2} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle, \tag{1}$$

where

$$\theta_m \equiv \theta_0 + 2\pi m/(s+1), \quad (m=0,1,\ldots,s). \tag{2}$$

The value of θ_0 is arbitrary and defines a particular basis set of $(s+1)$ mutually orthogonal phase states. The Hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m|, \tag{3}$$

Of course, the phase states (1) are eigenstates of the phase operator (3) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and $\theta_0 + 2\pi$. The unitary phase operator $\exp(i\hat{\phi}_\theta)$ can be defined as the exponential function of the Hermitian operator $\hat{\phi}_\theta$. This operator acting on the eigenstate $|\theta_m\rangle$ gives the eigenvalue $\exp(i\theta_m)$, and can be written as [1–3]

$$\exp(i\hat{\phi}_\theta) \equiv |0\rangle\langle 1| + |1\rangle\langle 2| + \ldots + |s-1\rangle\langle s| + \exp[i(s+1)\theta_0]|s\rangle\langle 0|, \tag{4}$$

and its Hermitian conjugate is

$$[\exp(i\hat{\phi}_\theta)]^+ = \exp(-i\hat{\phi}_\theta), \tag{5}$$

with the same set of eigenstates $|\theta_m\rangle$ but with eigenvalues $\exp(-i\theta_m)$.

To make further comparisons easier, it is useful to relate this new operator to the Susskind–Glogower phase operator, which is given by the following relation [10]

$$\begin{aligned}\langle \exp(im\hat{\phi}_\theta)\rangle &= \langle [\exp(i\hat{\phi}_\theta)]^m\rangle \\ &= \lim_{s\to\infty} \left\langle \left\{ \sum_{n=0}^{s-m} |n\rangle\langle n+m| + \exp[i(s+1)\theta_0] \sum_{n=0}^{m-1} |s-n\rangle\langle m-1-n| \right\} \right\rangle \\ &= \langle \hat{\exp}(im\phi_{SG})\rangle \\ &\quad + \lim_{s\to\infty} \left\langle \left\{ \exp[i(s+1)\theta_0] \sum_{n=0}^{m-1} |s-n\rangle\langle m-1-n| \right\} \right\rangle,\end{aligned} \tag{6}$$

where the Susskind–Glogower phase operator is given by

$$\hat{\exp}(im\phi_{SG}) \equiv \sum_{n=0}^{\infty} |n\rangle\langle n+m|. \tag{7}$$

In contrast to the Pegg–Barnett unitary phase operator, the Susskind–Glogower exponential operator is defined as a whole and is not unitary. From the definition (7) and the definition

$$\hat{\exp}(-im\phi_{SG}) \equiv [\hat{\exp}(im\phi_{SG})]^+, \tag{8}$$

one gets for $m=1$

$$\left.\begin{aligned}\hat{\exp}(\mathrm{i}\phi_{\mathrm{SG}})\,\hat{\exp}(-\mathrm{i}\phi_{\mathrm{SG}})&=1,\\ \hat{\exp}(-\mathrm{i}\phi_{\mathrm{SG}})\,\hat{\exp}(\mathrm{i}\phi_{\mathrm{SG}})&=1-|0\rangle\langle 0|,\end{aligned}\right\}\tag{9}$$

which explicitly shows the non-unitarity of the Susskind–Glogower phase operator.

When 'physical states', according to their definition by Pegg and Barnett [2, 3], are considered, there are some additional useful relations between expectation values in such states of the Pegg–Barnett phase operators and of the Susskind–Glogower phase operators. For example, the following relations hold [10]

$$\langle\exp(\mathrm{i}m\hat{\phi}_\theta)\rangle_{\mathrm{p}}=\langle\hat{\exp}(\mathrm{i}m\phi_{\mathrm{SG}})\rangle_{\mathrm{p}},\tag{10}$$

$$\begin{aligned}\langle\cos\hat{\phi}_\theta\rangle_{\mathrm{p}}&=\tfrac{1}{2}\langle\exp(\mathrm{i}\hat{\phi}_\theta)+\exp(-\mathrm{i}\hat{\phi}_\theta)\rangle_{\mathrm{p}}\\&=\langle\hat{\cos}\,\phi_{\mathrm{SG}}\rangle_{\mathrm{p}},\end{aligned}\tag{11}$$

$$\begin{aligned}\langle\sin\hat{\phi}_\theta\rangle_{\mathrm{p}}&=\frac{1}{2\mathrm{i}}\langle\exp(\mathrm{i}\hat{\phi}_\theta)-\exp(-\mathrm{i}\hat{\phi}_\theta)\rangle_{\mathrm{p}}\\&=\langle\hat{\sin}\,\phi_{\mathrm{SG}}\rangle_{\mathrm{p}},\end{aligned}\tag{12}$$

$$\begin{aligned}\langle\cos^2\hat{\phi}_\theta\rangle_{\mathrm{p}}&=\tfrac{1}{4}\langle\exp(\mathrm{i}2\hat{\phi}_\theta)+\exp(-\mathrm{i}2\hat{\phi}_\theta)+2\rangle_{\mathrm{p}}\\&=\langle\hat{\cos}^2\,\phi_{\mathrm{SG}}\rangle_{\mathrm{p}}+\tfrac{1}{4}\langle(|0\rangle\langle 0|)\rangle_{\mathrm{p}},\end{aligned}\tag{13}$$

$$\begin{aligned}\langle\sin^2\hat{\phi}_\theta\rangle_{\mathrm{p}}&=-\tfrac{1}{4}\langle\exp(\mathrm{i}2\hat{\phi}_\theta)+\exp(-\mathrm{i}2\hat{\phi}_\theta)-2\rangle_{\mathrm{p}}\\&=\langle\hat{\sin}^2\,\phi_{\mathrm{SG}}\rangle_{\mathrm{p}}+\tfrac{1}{4}\langle(|0\rangle\langle 0|)\rangle_{\mathrm{p}},\end{aligned}\tag{14}$$

where the subscript p refers to a physical state expectation value. We will use these relations later in the paper to describe phase properties of the states produced in the anharmonic oscillator model.

3. The anharmonic oscillator evolution

We are interested in phase properties of light propagating through a nonlinear Kerr medium. If the medium is isotropic and the light is circularly polarized, the propagation process can be described by the anharmonic oscillator model [11, 12]. The model is defined by the Hamiltonian

$$H=\hbar\omega a^+a+H_{\mathrm{I}},\tag{15}$$

with

$$H_{\mathrm{I}}=\tfrac{1}{2}\hbar\kappa a^{+2}a^2,\tag{16}$$

where a and a^+ are the annihilation and creation operators of the field mode, and κ is the coupling constant which is real and can be related to the nonlinear susceptibility $\chi^{(3)}$ of the medium. We assume the medium as being lossless.

Since the number of photons $\hat{n}=a^+a$ is a constant of motion, the Heisenberg equations of motion for the field operators can be solved exactly, which allows one to get exact analytical solutions for the field variances and predict a high degree of squeezing [11, 12] in the model.

To study phase properties of the field generated in the anharmonic oscillator model, we need to know the state evolution of the field rather than the operator evolution. Since the interaction Hamiltonian (16) commutes with the free Hamiltonian (15), the free evolution of the state can be factored out (we will drop it

altogether later on) and the state evolution of the system is described by the Schrödinger equation

$$i\hbar \frac{d}{dt} U(t) = H_I U(t), \tag{17}$$

where $U(t)$ is the time evolution operator, and H_I is the interaction Hamiltonian (16). In the propagation problem, when the light propagates in a Kerr medium, one can make the replacement $t = -z/v$ to describe the spatial evolution of the field instead of the time evolution. After this replacement the solution to equation (17) is given by [13]

$$U(\tau) = \exp\left[i \frac{\tau}{2} \hat{n}(\hat{n}-1) \right], \tag{18}$$

where

$$\tau = \kappa z/v, \tag{19}$$

is the dimensionless length of the nonlinear medium (or time in the time domain), and $\hat{n} = a^+ a$ is the photon number operator. If the state of the incoming beam is a coherent state $|\alpha_0\rangle$, the resulting state of the outgoing beam is given by

$$\begin{aligned} |\psi(\tau)\rangle &= U(\tau)|\alpha_0\rangle \\ &= \exp(-|\alpha_0|^2/2) \sum_{n=0}^{\infty} \frac{\alpha_0^n}{(n!)^{1/2}} \exp\left[i \frac{\tau}{2} n(n-1) \right] |n\rangle. \end{aligned} \tag{20}$$

The state (20) has an additional phase factor with respect to the coherent state $|\alpha_0\rangle$, and because of the quadratic dependence on n this extra phase cannot be simply added to the phase of the coherent state. So, the state (20) differs essentially from the coherent state $|\alpha_0\rangle$. It is known [11–16] that such states lead to squeezing. The state (20) differs, however, from the ideal squeezed state [5]. It can be referred to as self-squeezed state. On introducing the notation

$$\alpha_0 = N^{1/2} \exp(i\varphi_0), \tag{21}$$

$$b_n = \exp(-N/2) \frac{N^{n/2}}{(n!)^{1/2}}, \tag{22}$$

equation (20) can be rewritten as

$$|\psi(\tau)\rangle = \sum_{n=0}^{\infty} b_n \exp\left\{ i \left[n\varphi_0 + \frac{\tau}{2} n(n-1) \right] \right\} |n\rangle. \tag{23}$$

Since the number of photons is a constant of motion, and is equal to N, the state (23) is a 'physical state', in the meaning of Pegg and Barnett [2, 3], for any finite N. Of course

$$\lim_{n \to \infty} b_n = 0.$$

This means that all formulas concerning physical states can be directly applied to this state, and all phase properties required can be easily calculated. Some of these properties are discussed in detail in the next Section.

4. Phase properties of self-squeezed states

The states that we refer to as self-squeezed states are defined by the superposition (23) of the number states, which for given τ describes a definite state of the outgoing field. The states depend on the value of τ and for some special τ values they become a

discrete superposition of coherent states [23–25]. Initially ($\tau=0$) the state is a coherent state $|\alpha_0\rangle$ with the phase φ_0 defined by equation (21). Since at $\tau=0$ this state belongs to a class of partial phase states, we will choose the initial phase θ_0, appearing in equation (2), in the way convenient for description of partial phase states [3], namely

$$\theta_0=\varphi_0-\frac{\pi s}{s+1}. \tag{24}$$

If we introduce a new phase label

$$\mu=m-\frac{s}{2}, \tag{25}$$

which goes in integer steps from $-(s/2)$ to $(s/2)$ the phase distribution becomes symmetric in μ. According to equations (1), (2) and (23–25), we obtain

$$\langle\theta_\mu|\psi(\tau)\rangle=(s+1)^{-1/2}\sum_{n=0}^{s} b_n \exp\left\{-\mathrm{i}\left[n\theta_\mu-\frac{\tau}{2}\,n(n-1)\right]\right\}, \tag{26}$$

where

$$\theta_\mu=\mu 2\pi/(s+1), \tag{27}$$

and b_n is given by equation (22). From equation (26) we can easily obtain the phase probability distribution in the form

$$|\langle\theta_\mu|\psi(\tau)\rangle|^2=\frac{1}{s+1}+\frac{2}{s+1}\sum_{n>k} b_n b_k \times\cos\left\{(n-k)\theta_\mu-\frac{\tau}{2}\,[n(n-1)-k(k-1)]\right\}. \tag{28}$$

For $\tau=0$ this expression goes over into the corresponding expression for partial phase states given by Pegg and Barnett [3].

In the limit as s tends to infinity, the continuous phase variable can be introduced replacing $\mu 2\pi/(s+1)$ by θ and $2\pi/(s+1)$ by $\mathrm{d}\theta$. This leads to a continuous phase probability distribution given by the formula

$$P(\theta)=\frac{1}{2\pi}\left\{1+2\sum_{n>k} b_n b_k \cos\left[(n-k)\theta-\frac{\tau}{2}\,[n(n-1)-k(k-1)]\right]\right\}, \tag{29}$$

with the normalization

$$\int_{-\pi}^{\pi} P(\theta)\,\mathrm{d}\theta=1. \tag{30}$$

For $\tau=0$, formula (29) describes the phase probability distribution for a coherent state—a member of partial phase states. In this case $P(\theta)=P(-\theta)$, i.e. the phase distribution is symmetric in θ. This symmetry is broken when the nonlinear propagation takes place and $\tau\neq 0$. In this case, the phase probability distribution $P(\theta)$ exhibits some new and very interesting features.

Despite the apparent simplicity of the formula (29), it is not easy to predict the shape of $P(\theta)$ because of the double summations appearing in the formula. Since the amplitudes b_n have Poissonian character, i.e., they are peaked at $n=N$, the series is

rapidly convergent for not too large N, and can be evaluated numerically. On the other hand, for large values of N the summations can be replaced by integrals, and some analytical approximations for $P(\theta)$ are possible.

For small number of photons the direct numerical evaluation of formula (29) can be performed. The results are shown in figures 1–3. In figure 1 the phase probability distribution $P(\theta)$ is plotted against θ in the polar coordinate system for various values of τ, and for $N=0{\cdot}25$ (figure 1 (*a*)) and $N=4$ (figure 1 (*b*)). It is seen from figure 1 (*a*) that for $\tau=0$ the distribution $P(\theta)$ has an elliptic shape which, however, cannot be associated with squeezing, as it is the case for the weakly squeezed vacuum [10], because it describes a coherent state with the mean number of photons $N=0{\cdot}25$. As τ increases, the in-phase quadrature component becomes squeezed [12], but the shape of $P(\theta)$ becomes less elliptic. For $\tau=\pi$ when the maximum of squeezing appears [26], the shape of $P(\theta)$, although symmetric, is far from being elliptic. This means that in the case of self-squeezed states generated by the anharmonic oscillator the simple

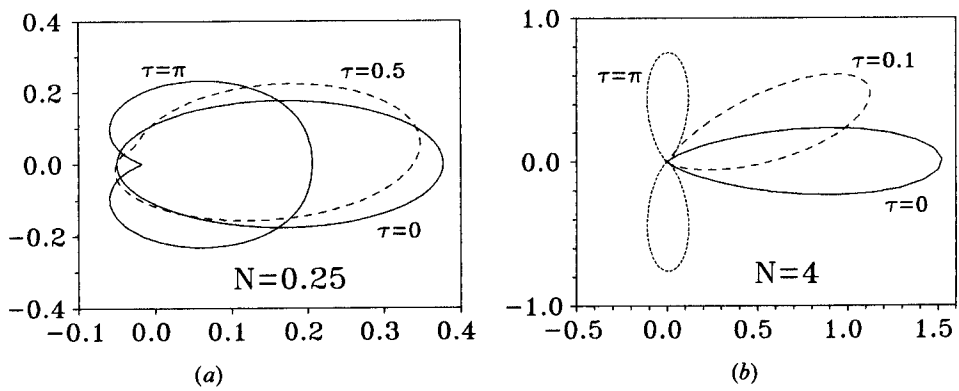

Figure 1. Phase probability distribution $P(\theta)$ plotted against θ in the polar coordinate system for various values of τ, (*a*) $N=0{\cdot}25$, (*b*) $N=4$.

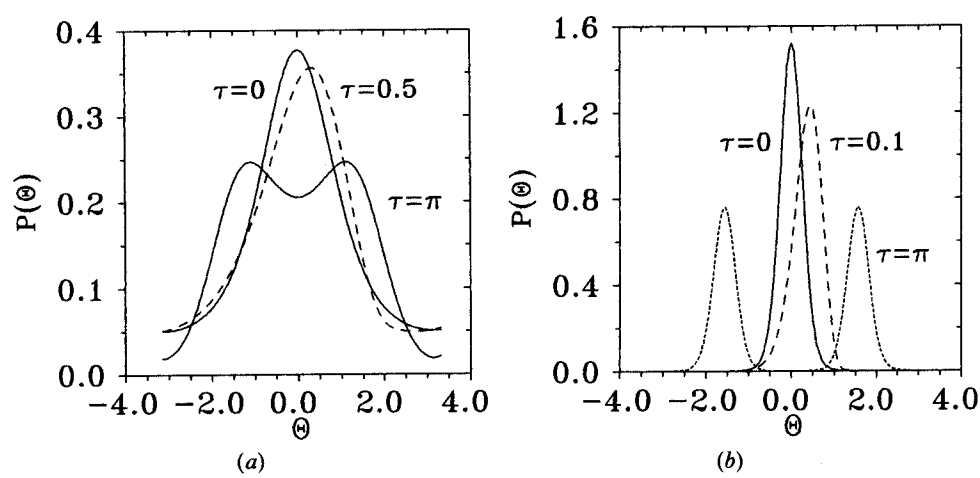

Figure 2. Plot of $P(\theta)$ against θ in the rectangular coordinate system for various values of τ, (*a*) $N=0{\cdot}25$, (*b*) $N=4$.

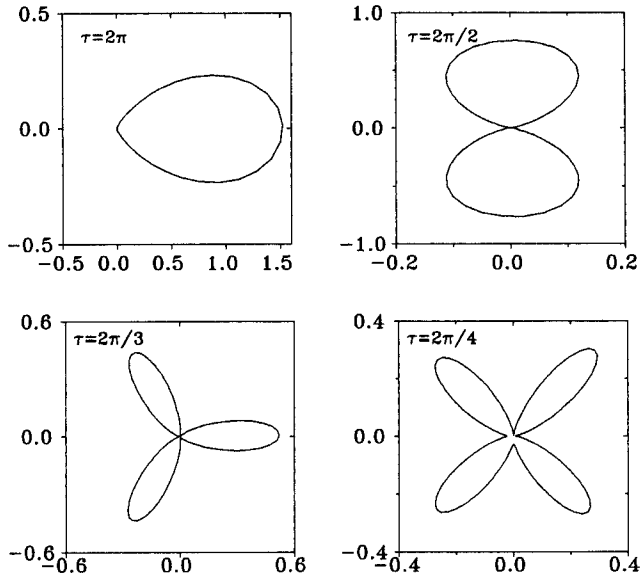

Figure 3. The polar coordinate pictures of $P(\theta)$ for $\tau=2\pi/n$, $n=1,2,3,4$, $N=4$.

identification of the elliptic shape of $P(\theta)$ with squeezing is not possible. In figure 1 (*b*) the polar coordinate shapes of $P(\theta)$ are shown for $N=4$. The initial coherent state phase distribution assumes a lengthened leaf shape, which rotates and changes its shape as the evolution proceeds. For $\tau=\pi$, $P(\theta)$ splits into two separate leaves. The same distributions as in figure 1 are shown in figure 2, but this time $P(\theta)$ is plotted against θ in the Cartesian coordinate system. The splitting of the distribution into two peaks for $\tau=\pi$ is already visible in figure 2 (*a*), and becomes quite evident in figure 2 (*b*). This splitting reflects the fact that the state of the anharmonic oscillator evolves in this case into a superposition of two coherent states [23, 24]. This means that the phase distribution $P(\theta)$ can be related to the shape of the quasi-probability distribution $Q(\alpha, \alpha^*)$ in the complex α plane [14]. If τ is taken as $2\pi/n$ ($n=2,3,4,\ldots$) the shape of $P(\theta)$ in polar coordinates exhibits n-fold symmetry confirming generation of discrete superpositions of coherent states with $2,3,4,\ldots$ components [25]. This is shown convincingly in figure 3.

If the mean number of photons in the field is large, $N \gg 1$, the approximate method used by Barnett and Pegg [2] to describe the phase distribution of coherent states can be applied to finding the phase distribution. In this case the Poisson photon number distribution is well approximated by a continuous Gaussian distribution

$$P(n)=\exp(-N)\frac{N^n}{n!}$$
$$\approx(2\pi N)^{-1/2}\exp\left[-\frac{(N-n)^2}{2N}\right], \tag{31}$$

which is normalized so that

$$\int P(n)\,\mathrm{d}n=1.$$

If the square root of $P(n)$is substituted into equation (26) and the integration over n (instead of the summation) is performed, one eventually arrives at the following result for the phase probability distribution

$$P(\theta)=\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left[-\frac{(\theta-\bar{\theta})^2}{2\sigma^2}\right], \tag{32}$$

where

$$\bar{\theta}=\varphi_0+\tau(N-\tfrac{1}{2}), \tag{33}$$

$$\sigma^2=N\left(\tau^2+\frac{1}{4N^2}\right). \tag{34}$$

The distribution (32) is a Gaussian distribution with the mean given by equation (33) and the dispersion given by equation (34). This means that the mean phase is shifted by τN (we can drop $\frac{1}{2}$ for $N\gg 1$) during the evolution. This result agrees with the result obtained by Gerry [22], and can be understood from the form of the operator solution [12]

$$a(\tau)=\exp[\mathrm{i}\tau a^+(0)a(0)]a(0), \tag{35}$$

If the operators are replaced by the classical amplitudes α, $N=|\alpha|^2$, the shift in phase by τN is immediately seen. Equation (34) says that the dispersion of the resulting Gaussian distribution of phase increases with τ. Since all the time the photon distribution remains Poissonian with the variance $\langle(\Delta N)^2\rangle=N$, the phase-photon number uncertainty relation takes the form

$$\langle(\Delta\hat{\phi}_\theta)^2\rangle\langle(\Delta N)^2\rangle=(\tfrac{1}{4}+N^2\tau^2). \tag{36}$$

which means fast expansion of the uncertainty product during the evolution. One should, however, keep in mind that the approximation (32) works well only when the Gaussian is not too broad. Otherwise, the exact formula (29) should be used to calculate the mean phase and the variance.

The mean value of the phase in the self-squeezed state (23) can be written as

$$\begin{aligned}\langle\psi(\tau)|\hat{\phi}_\theta|\psi(\tau)\rangle&=\varphi_0+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{2\sum_{n>k}b_nb_k\right.\\&\quad\times\cos\left\{(n-k)\theta-\frac{\tau}{2}[n(n-1)-k(k-1)]\right\}\Bigg\}\theta\,\mathrm{d}\theta\\&=\varphi_0-2\sum_{n>k}b_nb_k\frac{(-1)^{n-k}}{n-k}\sin\left\{\frac{\tau}{2}[n(n-1)-k(k-1)]\right\},\end{aligned} \tag{37}$$

and the variance of $\hat{\phi}_\theta$ is given by

$$\begin{aligned}\langle\psi(\tau)|(\Delta\hat{\phi}_\theta)^2|\psi(\tau)\rangle&=\frac{\pi^2}{3}\\&\quad+4\sum_{n>k}b_nb_k\frac{(-1)^{n-k}}{(n-k)^2}\cos\left\{\frac{\tau}{2}[n(n-1)-k(k-1)]\right\}\\&\quad-\left\{2\sum_{n>k}b_nb_k\frac{(-1)^{n-k}}{n-k}\sin\left\{\frac{\tau}{2}[n(n-1)-k(k-1)]\right\}\right\}^2.\end{aligned} \tag{38}$$

If we put $\tau = 0$ in formulas (37) and (38), the results for a coherent state with the phase φ_0 are recovered [3]. It is clear from (37) and (38) that the nonlinear evolution of the system leads to essential changes in both the mean value of the phase and its variance.

The results (37) and (38) are illustrated graphically in figures 4–6. In figure 4 the evolution of the mean phase is shown for $N = 0{\cdot}25$. The mean value of phase oscillates within a narrow range of values around zero. In figure 5 the evolution of the phase variance for $N = 0{\cdot}25$ is plotted. The variance goes up initially, reaches a plateau and

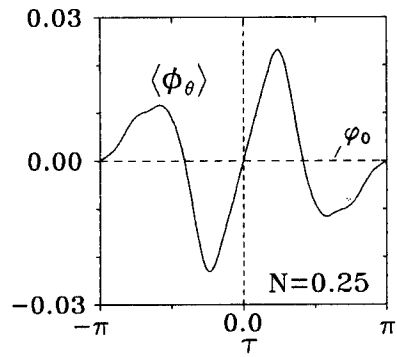

Figure 4. Plot of the mean value of the phase operator as a function of τ, for $N = 0{\cdot}25$ and $\varphi_0 = 0$.

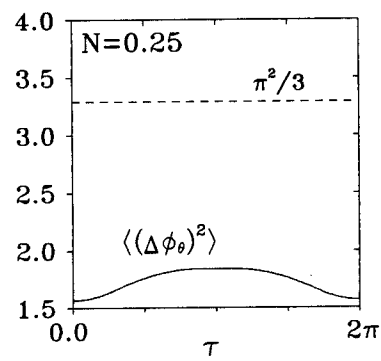

Figure 5. Plot of the variance of the phase operator as a function of τ, for $N = 0{\cdot}25$ and $\varphi_0 = 0$.

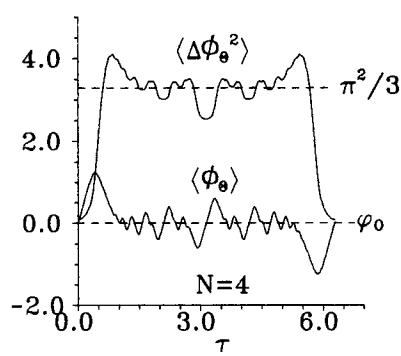

Figure 6. Plot of the mean value and the variance of the phase operator as a function of τ, for $N = 4$ and $\varphi_0 = 0$.

goes down to the initial value when τ approaches 2π. All results are periodic in τ with the period 2π. It is interesting to note that for $\tau=\pi$, when squeezing approaches its maximum, the phase variance has large value, i.e., the phase of the field is badly defined. In fact there is a shallow local minimum at this point, but with the large value of the variance. The results for $N=4$ are presented in figure 6. The amplitude of the phase oscillation becomes larger, and the value of the phase variance goes up even higher and starts to oscillate around the value $\pi^2/3$. This means that the state of the field for most of the period of the evolution has its phase variance close to the state with randomly distributed phase. This tendency is even more pronounced when the mean number of photons increases.

Such phase characteristics of the field as the phase distribution $P(\theta)$, the expectation value of the phase operator and its variance are quantities that can be obtained within the Pegg–Barnett formalism only, and cannot be compared to any other approach so far.

There are, however, phase characteristics of the field that have their counterparts in other formalisms. Such are, for example, the cosine and the sine functions of the phase and their variances. To calculate the expectation values of the cosine and the sine functions of the phase we take into account the fact that the self-squeezed states produced by the anharmonic oscillator are physical states, for which the last term on the right-hand side of equation (6) vanishes, and the relation (10) holds. This gives us

$$\begin{aligned}\langle\psi(\tau)|\exp(im\hat{\phi}_\theta)|\psi(\tau)\rangle &= \sum_{n=0}^{\infty}\langle\psi(\tau)|n\rangle\langle n+m|\psi(\tau)\rangle \\ &= N^{m/2}\sum_{n=0}^{\infty} b_n^2\,\frac{\exp\left\{im\left[\varphi_0+\frac{\tau}{2}(m-1+2n)\right]\right\}}{[(n+1)(n+2)\ldots(n+m)]^{1/2}},\end{aligned} \tag{39}$$

where equations (21–23) have been used. From equation (39), in agreement with (11) and (12), we obtain immediately the following expressions for the expectation values of the cosine and sine functions of the phase [22]

$$\langle\psi(\tau)|\cos\hat{\phi}_\theta|\psi(\tau)\rangle = N^{1/2}\sum_{n=0}^{\infty} b_n^2\,\frac{\cos(\varphi_0+n\tau)}{(n+1)^{1/2}}, \tag{40}$$

$$\langle\psi(\tau)|\sin\hat{\phi}_\theta|\psi(\tau)\rangle = N^{1/2}\sum_{n=0}^{\infty} b_n^2\,\frac{\sin(\varphi_0+n\tau)}{(n+1)^{1/2}}, \tag{41}$$

where b_n is given by equation (22).

Similarly, according to equations (13), (14) and (39), we get the results [22]

$$\langle\psi(\tau)|\cos^2\hat{\phi}_\theta|\psi(\tau)\rangle = \frac{1}{2}+\frac{1}{2}N\sum_{n=0}^{\infty} b_n^2\,\frac{\cos[2\varphi_0+(2n+1)\tau]}{[(n+1)(n+2)]^{1/2}}, \tag{42}$$

$$\langle\psi(\tau)|\sin^2\hat{\phi}_\theta|\psi(\tau)\rangle = \frac{1}{2}-\frac{1}{2}N\sum_{n=0}^{\infty} b_n^2\,\frac{\cos[2\varphi_0+(2n+1)\tau]}{[(n+1)(n+2)]^{1/2}}, \tag{43}$$

These are the results for the Pegg–Barnett definition of the phase operator and, of course, we have $\langle\cos^2\hat{\phi}_\theta\rangle+\langle\sin^2\hat{\phi}_\theta\rangle=1$. Again, for $\tau=0$ these results correspond to the coherent state with the phase φ_0 and the mean number of photons N. The Susskind–Glogower results differ from (42) and (43), according to (13) and (14), by the quantity

$$\tfrac{1}{4}|\langle\psi(\tau)|0\rangle|^2=\tfrac{1}{4}\exp(-N), \tag{44}$$

which is negligible for $N \gg 1$, but is essential when N is small. For small values of N all the summations in equations (40–43) can be evaluated numerically for given τ, and the evolution of these quantities can be obtained. This allows us to evaluate the variance of the phase cosine or sine and compare it to the Susskind–Glogower result as well as to the measured phase result [21]. The results for the variance of the phase cosine are shown in figure 7. In figure 7 the evolution of the variance is presented for three different definitions of the phase cosine. The measured phase concept is based on the quadrature phase measurements that are used in squeezing measurements. The phase cosine is defined in this case as appropriately normalized field quadrature [21]

$$\hat{\cos}\,\phi_m = \frac{1}{2(N+\frac{1}{2})^{1/2}}\,(a+a^+), \tag{45}$$

and the cosine variance is simply equal to the appropriately normalized variance of the quadrature field component. The exact explicit expressions for such variances in the anharmonic oscillator model has been given by Tanaś [12], and we will not repeat them here, although we use them to evaluate the variance presented in the figure. The same formulas were used by Lynch [20] who discussed phase uncertainties of the anharmonic oscillator model comparing his results with the results of Gerry [19] obtained within the Susskind–Glogower formalism. All the results are compared in figure 7 (*a*) for $N=0{\cdot}25$. There is no difference in shape between the Pegg–Barnett and the Susskind–Glogower curves, they are only shifted by $(\frac{1}{4})\exp(-N)$. The shape of the curve based on the measured phase concept is slightly different, although it reproduces the main features of the other curves. In fact, this curve is identical, apart from the scale, with the corresponding quadrature phase variance (see [26]). The same curves, but for $N=4$, are drawn in figure 7 (*b*). This time the differences are rather small, but again the resemblance to the corresponding quadrature phase variance (see [14]) is maintained. The differences between various phase definitions in the phase cosine variance will disappear as N will increase.

Here, the relation between the phase properties of the field and squeezing becomes more transparent. Apart from the direct relation of the measured phase concept to squeezing, because of the similarity of the curves, one can expect squeezing for the in-phase component of the field if the variance of the phase cosine falls below its value for a coherent state. Since squeezing of the in-phase component

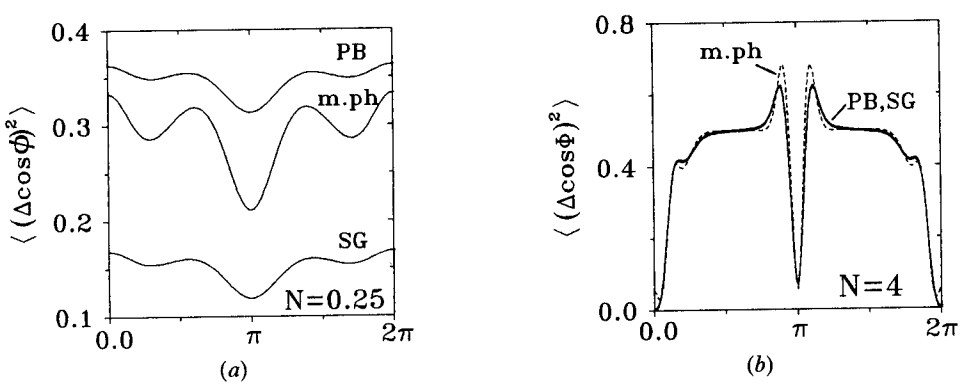

Figure 7. The evolution of the variance of the phase cosine function. Results for different approaches are compared: PB-Pegg–Barnett, SG-Susskind–Glogower, m.ph-measured phase, (*a*) $N=0{\cdot}25$, (*b*) $N=4$.

of the field means squeezing of the uncertainty of the field amplitude, the uncertainty of the phase becomes large. It is clearly seen when comparing figures 5 and 7 (*a*), for $\tau=\pi$, for which the maximum of squeezing in the in-phase component appears (for $N=0{\cdot}25$). The variance of the phase sine has its maximum at this point.

If the anharmonic oscillator model is used to describe propagation of laser light in a nonlinear Kerr medium the realistic values of τ are very small [11, 12] because of the smallness of the nonlinear susceptibility of the medium. Even in this case large degrees of squeezing are possible [11, 12] if the mean number of photons N becomes large.

5. Conclusions

We have discussed the phase properties of the self-squeezed states generated by the nonlinear evolution of the anharmonic oscillator. The new Pegg and Barnett formalism has been used to describe the phase properties of such states. The phase distribution $P(\theta)$ has been obtained and illustrated graphically for various evolution times (lengths of the medium) τ, and different values of the mean number of photons N. This phase distribution exhibits a number of interesting features. It has been shown that in this case an elliptic shape of $P(\theta)$ in the polar coordinate system cannot be associated with squeezing. Another interesting feature that has been predicted is the n-fold symmetry of the phase distribution if τ is taken as $2\pi/n$ ($n=1,2,\ldots$). This confirms earlier results [23–25] that the states of the anharmonic oscillator evolve in this case into a discrete superposition of coherent states with n components. We have calculated the mean value of the phase and its variance in the self-squeezed states. At the initial stage of the evolution the mean value of the phase increases, and later starts to oscillate around the initial coherent state phase φ_0. The variance of the phase increases initially at a high rate, and later oscillates around the value $\pi^2/3$, i.e. the value for the states with random distribution of phase. This means that for most of the period of the evolution, the anharmonic oscillator states (for $N>1$) are close to the states with random distribution of phase. These unique phase properties of the anharmonic oscillator states were possible to obtain due to the new Pegg–Barnett phase formalism. In the limits of very high and very low light intensities, our exact results go over into the results obtained by Gerry [22].

The phase cosine and sine functions as well as their variances have also been calculated using the new formalism and the results compared to the results of the Susskind–Glogower formalism and the measured phase concept. In view of these results, the relation between phase properties of field and squeezing is established. It is shown that, at least qualitatively, squeezing can be predicted from the knowledge of the phase properties of the field.

Finally, it should be noted that the self-squeezed states considered in this paper belong to a class of generalized coherent states discussed by Titulaer and Glauber [27], Stoler [28], and recently by Vourdas and Bishop [29].

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