

Collapses, revivals, and phase properties of the field in Jaynes–Cummings type models

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Phase properties of the field in a coherent state interacting with a two-level atom in a lossless cavity (Jaynes–Cummings model) are studied using the new phase formalism of Pegg and Barnett. The phase density distribution, the expectation values and the variances of the hermitian phase operator are calculated. On a polar diagram, the initial phase distribution is shown to split into two counterrotating satellite distributions. When the two satellite distributions overlap, revivals of the atomic inversion occur. Phase properties of the field in a model with an intensity depending coupling, for which the time behavior is exactly periodic, are also discussed.

1. Introduction

The Jaynes–Cummings model [1–3] of quantum optical resonance has been the subject of considerable attention in recent years. The model describes an isolated two-level atom interacting with a single-mode quantized electromagnetic field in a lossless cavity. This model has been studied extensively because of the relatively realistic way that it presents the actual dipole coupling of an atom to an electromagnetic field. Within the rotating wave approximation this model is exactly soluble.

One of the most remarkable effects predicted theoretically and then observed experimentally in the JCM are the collapses and revivals of the atomic inversion. Eberly et al. [4], using a combination of numerical and approximate analytical techniques, have shown that if the atom is initially in its ground state and the field is fully coherent, then the atomic inversion undergoes a cosine oscillation which decays rapidly at short times, but periodically regenerates to large amplitudes on a longer time scale.

The central role in studying the collapse and revival phenomenon is played by the infinite sum

$$\langle R^z(t) \rangle = -\frac{1}{2} \exp(-\bar{n}) \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} \cos[\Omega(n)t], \quad (1)$$

with

$$\Omega(n) = 2g\sqrt{n},$$

where \bar{n} is the mean photon number, g is the atom–field coupling constant. The collapses and revivals of $\langle R^z(t) \rangle$ have a simple interpretation [4]. At $t=0$ the system is prepared in a definite state and therefore all terms in (1) are then correlated. As the time goes on, they start to oscillate with different frequencies and become decorrelated. A revival of the collapsed $\langle R^z(t) \rangle$ occurs when the phases of oscillation of neighboring terms in (1) differ by the factor 2π for $n \propto \bar{n}$. That is, the internal T_R between revivals can be found from the relation

$$[\Omega(\bar{n}+1) - \Omega(\bar{n})] T_R = 2\pi, \quad (2)$$

or

$$T_R = 2\pi(\bar{n})^{1/2}/g \quad (\bar{n} \gg 1). \quad (3)$$

This qualitative result is in agreement with approximate analytical calculations [4]. Here, we discuss only the exact resonance since the features described by Eberly et al. [4] are seen most clearly for this case, and the mathematical expressions simplify considerably.

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The collapse and revival phenomenon can be studied from another point of view. Recently, Eiselt and Risken, using the Q -function, have shown that the collapses and revivals can be understood in terms of interferences in phase space [5]. Phoenix and Knight have mentioned the splitting of the phase probability distribution into two counterrotating satellite distributions in the model consisting of a two degenerate atomic levels coupled through a virtual level by a Raman-type transition [6]. The similarities between $P(\theta)$ and Q -function were marked in connection with the ideal squeezed states [7] and the problem of discrete superpositions of coherent states generated by the anharmonic oscillator [8].

In this paper we examine the phase properties of the field in the Jaynes–Cummings model, using the new phase formalism introduced by Pegg and Barnett [9–11]. We show that the time behavior of the phase density distribution presented on a polar diagram resembles a lot that of the Q -function in phase space. Namely, the interaction forces each phase state to split into two phase states rotating in opposite directions. During the period when the counterrotating distributions are well separated, the atomic inversion shows no oscillations. When the two satellite distributions overlap again, the revival of the atomic inversion occurs. Naturally, the variance of the phase carries some information about the collapses and revivals. However, in this case care must be taken because a particular choice of the reference phase may influence the calculated phase properties of the state [10].

Finally, we shortly discuss the phase properties of the Jaynes–Cummings model with intensity dependent coupling [12]. This model is of interest because it gives rise to commensurable Rabi frequencies. The dynamical behavior of it is exactly periodic and can be compared with the standard JCM.

2. The hermitian phase operator

According to Dirac, each quantity that can be measured for a physical system can be represented by a linear hermitean operator [13]. The algebra these operators obey is based on the commutation relations that mark the departure of quantum mechanics from classical mechanics. However, diffi-

culties have been found with proper description of phase variables (see ref. [14] for a recent review). Recently, Pegg and Barnett have suggested a new approach using the states of well-defined phase as a starting point [9–11]. To construct a phase operator that is hermitean they restrict the state space to a $(s+1)$ -dimensional space Ψ spanned by the first $(s+1)$ number states. The value of s can be made arbitrary large. Expectation values are first calculated in Ψ before s is allowed to tend to infinity. The state space Ψ is also spanned by $(s+1)$ orthogonal phase states,

$$|\theta_m\rangle = (s+1)^{-1/2} \sum_{n=0}^{\infty} \exp(in\theta_m) |n\rangle, \quad (4)$$

with

$$\theta_m = \theta_0 + 2\pi m / (s+1), \quad m=0, 1, 2, \dots, S. \quad (5)$$

These states are eigenstates of the hermitian phase operator

$$\hat{\Phi}_\theta \equiv \sum_{m=0}^S \theta_m |\theta_m\rangle \langle \theta_m|. \quad (6)$$

It can be seen from eq. (5) that the eigenvalues of $\hat{\Phi}_\theta$ are restricted to lie within a phase window between θ_0 and $(\theta_0 + 2\pi)$. The value of reference phase θ_0 is arbitrary. Its choice determines the particular range of eigenvalues of $\hat{\Phi}_\theta$ and will influence the results of the calculated phase properties of the states. This emphasizes the need of caution in interpreting the results obtained by employing the phase operator with a particular choice of θ_0 .

Unlike the earlier approach of Susskind and Glogower [14], the Pegg and Barnett formalism allows to discuss a phase distribution function, an expectation value of phase operator and its variance. For a general pure state of field mode

$$|f\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad (7)$$

the phase probability distribution is

$$|\langle \theta_m | f \rangle|^2 = (s+1)^{-1} \left| \sum_n c_n \exp(-in\theta_m) \right|^2, \quad (8)$$

with the expectation value and the variance

$$\langle \hat{\Phi}_\theta \rangle = \sum_m \theta_m |\langle \theta_m | f \rangle|^2, \quad (9)$$

$$\Delta\Phi_\theta^2 = \sum_m (\theta_m - \langle \Phi_\theta \rangle)^2 |\langle \theta_m | f \rangle|^2. \quad (10)$$

These formulae will be used below to describe phase properties of the field in the JCM.

3. Phase properties of the field mode in the JCM

We consider a system of one two-level atom and one mode of the electromagnetic field. These two are coupled by the dipole interaction within the rotating wave approximation, and the system is described by the hamiltonian

$$H = \hbar\omega(a^\dagger a + R^z) + \hbar g(R^+ a + R^- a^\dagger), \quad (11)$$

where a^\dagger , a are the Bose creation and annihilation operators for the photons at frequency ω . The two-level atom is described by the Pauli raising and lowering operators R^+ , R^- and the inversion operator R^z , and g is the coupling constant. For simplicity, we take the exact resonant case only: $\omega_{\text{atom}} = \omega_{\text{field}} = \omega$. This model has been realized in the laboratory with a Rydberg atom contained in a high- Q cavity [16].

To study these properties of the field, we need to know the state evolution of the system. Later on, we will neglect all free evolution terms, as they only change the field phase in a trivial way. For an atom initially in its ground state and a field initially in a coherent state $|\alpha\rangle$, where $\alpha = (\bar{n})^{1/2} \exp(i\beta)$, the wavefunction of the total system is found to be

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} b_n \exp(in\beta) \times [\cos(g\sqrt{n}t)|n; g\rangle - i \sin(g\sqrt{n}t)|n-1; e\rangle]. \quad (12)$$

Here, we have denoted by $|e\rangle$, $|g\rangle$ the excited and ground states of the atom. The coefficient b_n is given by

$$b_n = \exp(-\bar{n}/2) (\bar{n}^n/n!)^{1/2}. \quad (13)$$

At $t=0$ the cavity field mode is in a coherent state, which is a particular case of the partial physical phase state [11]. Therefore, following Pegg and Barnett we chose the reference phase θ_0 appearing in eq. (5) as

$$\theta_0 = \beta - \pi s/(s+1), \quad (14)$$

and introduce a new phase index

$$\mu = m - s/(s+1), \quad (15)$$

which ranges in integer steps from $-s/2$ to $s/2$. Using eqs. (5), (8), and (12–14) we obtain the following expression for the phase probability distribution,

$$|\langle \theta_m | \psi(t) \rangle|^2 = \frac{1}{s+1} + \frac{2}{s+1} \sum_{n>k} b_n b_k \cos[(n-k)\mu 2\pi/(s+1)] \times \cos[(\sqrt{n}-\sqrt{k})gt]. \quad (16)$$

As s tends to infinity the summation in eq. (16) may be transformed into an integral after replacing $\mu 2\pi/(s+1)$ by θ and $2\pi/(s+1)$ by $d\theta$. This leads to a continuous phase probability distribution

$$P(\theta, t) = \frac{1}{2\pi} \left(1 + 2 \sum_{n>k} b_n b_k \times \cos[(n-k)\theta] \cos[(\sqrt{n}-\sqrt{k})gt] \right), \quad (17)$$

which is normalized so that

$$\int_{-\pi}^{\pi} P(\theta, t) d\theta = 1. \quad (18)$$

From eqs. (9), (17), replacing the summation in (9) by an appropriate integral and taking into account that

$$\theta_m = \theta + \beta \quad (19)$$

in the limit as s tends to infinity, one finds the average value of the phase,

$$\langle \psi(t) | \Phi_\theta | \psi(t) \rangle = \beta. \quad (20)$$

Thus, the interaction of the field and the atom does not give rise to changes of the average phase value. This can be seen immediately from eq. (17). Since at any time t , the phase probability distribution is an even function with respect to θ , the integral from $-\pi$ to π leads to a vanishing average value of θ , and consequently the average value of phase is always equal to the initial quantity β (eqs. (9), (19)). We should note that if a more general situation is taken under consideration, for example, when the resonance is non-exact or when the atom is injected into the field

in a coherent superposition of excited and ground states the time variation of $\langle \hat{\Phi}_\theta \rangle$ will occur.

Now, let us continue to examine the properties of the phase probability distribution. Despite the apparent simplicity of the formula (17), it is difficult to predict the shape of $P(\theta, t)$. But if we rewrite eq. (17) into the form

$$P(\theta, t) = \frac{1}{2} [P_+(\theta, t) + P_-(\theta, t)], \quad (21)$$

where

$$P_{\pm}(\theta, t) = \frac{1}{2\pi} \left(1 + 2 \sum_{n>k} b_n b_k \times \cos[(n-k)\theta \mp (\sqrt{n} - \sqrt{k})gt] \right), \quad (22)$$

from which it can be seen that as the time goes on, the phase probability distribution $P(\theta, t)$ splits into two separate distributions rotating in opposite directions. When we depict them in a polar diagram with θ as the polar angle and P the radius distance they have the same shape and are located quite symmetrically about the line $\theta=0$ (see fig. 1). After a certain interval of time the two counterrotating distributions "collide". They completely overlap when the mean value $\langle \hat{\Phi}_\theta \rangle_+$ ($\langle \hat{\Phi}_\theta \rangle_-$) averaged according to the probability distribution $P_+(\theta, t)$ ($P_-(\theta, t)$) increases by π ($-\pi$). At that time the components of the field oscillate nearly in-phase with one another, and the intuition allows us to think that the atomic inversion will show the revival. The numerical calculations corroborate this convincingly in fig. 1, 2. In fig. 1 the phase probability distribution $P(\theta, t)$ is plotted against θ in a polar coordinate system for various values of time. For comparison, time record of the atomic inversion $\langle R^z(t) \rangle$ is plotted in fig. 2. The time is scaled by the factor $2\pi(\bar{n})^{1/2}/g$: $T = gt/[2\pi(\bar{n})^{1/2}]$ so that the revivals take place when $T=1, 2, 3, \dots$. We have taken everywhere $\bar{n}=20$.

At $T=0$ the phase distribution assumes a lengthened leaf shape corresponding to the initial coherent state of the field [8]. It gradually splits into two separate leaves as the evolution proceeds (fig. 1a). During this time the collapsed atomic inversion shows no oscillations. At $T=1$ when the two distribution curves are completely mixed up, the amplitude of oscillation reaches its maximum (see fig. 1a and fig.

2). After a while the distribution splits again and the two peaks this time move to the right hand side of the picture, where they collide again (fig. 1b), and so forth. In the course of time the width of the distribution gets broader, and the splitting of the distribution into two leaves becomes difficult to be seen (fig. 1c). This corresponds to the spreading of the revivals in time (fig. 2).

The time behavior of the phase variance together with the density distribution carries some information about the collapses and revivals. To show this, we first obtain the explicit expression for the variance using eqs. (10), (17),

$$\begin{aligned} \langle \psi(t) | \Delta \Phi_\theta^2 | \phi(t) \rangle &= \frac{1}{3} \pi^2 \\ &+ 4 \sum_{n>k} b_n b_k \cos[(\sqrt{n} - \sqrt{k})gt] \\ &\times (-1)^{(n-k)}/(n-k)^2, \end{aligned} \quad (23)$$

where we have transformed the summation in (10) into an integral. $\langle \Delta \Phi_\theta^2 \rangle$ is illustrated graphically in fig. 3 for $\bar{n}=20$. The variance goes up initially and reaches a maximum at the scaled time $T=1$. It looks as it would contradict the fact that the phase density distribution on the polar diagram (fig. 1a) has only one peak then. Nevertheless, as have been pointed out by Pegg and Barnett [10], care must be taken in interpreting the results obtained with a particular choice of θ_0 . Here, θ_0 has been chosen as to minimize the variance of phase in the initial coherent state of the field. This does not hold true at $T=1$. At this time the density distribution $P(\theta, t)$ plotted against θ in a cartesian coordinate system splits into two symmetrical peaks located at $-\pi$ and π (fig. 4). If we move the phase window by π , the variance is minimized again. Thus, with such a particular choice of reference phase θ_0 we may conclude that both the maxima and the minima of the phase variance correspond to the revivals of the atomic inversion. The deeper the extrema are, the more distinctly the corresponding revivals can be seen. For longer times $\langle \Delta \Phi_\theta^2 \rangle$ shows small oscillations around the value $\pi^2/3$, the phase variance of a field state with randomly distributed phase. This reflects the existence of a quasi-irreversibility inherent in the coherent-state JCM [4].

We have shown the relations between the atomic inversion collapses and revivals and the phase prop-

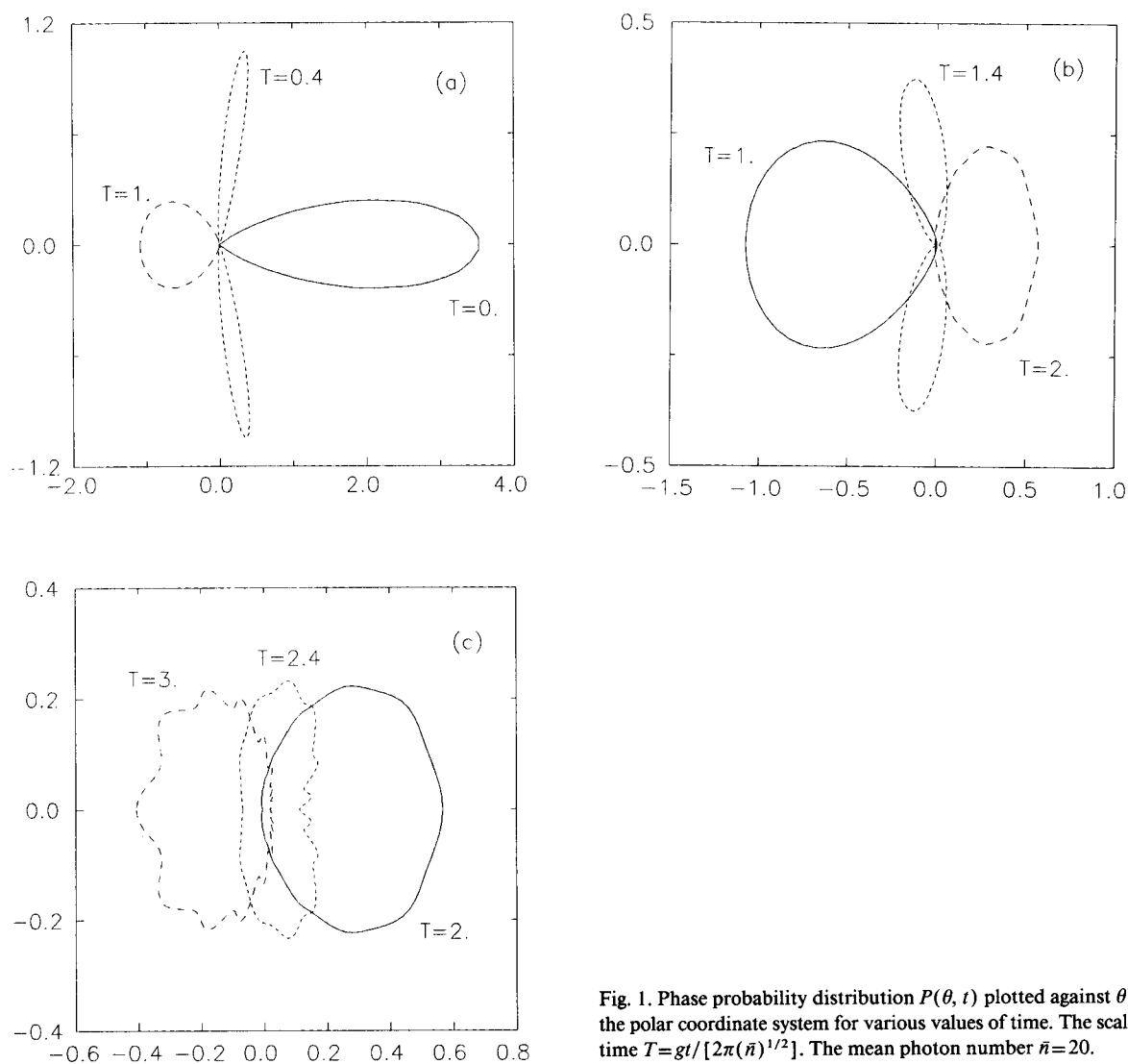


Fig. 1. Phase probability distribution $P(\theta, t)$ plotted against θ in the polar coordinate system for various values of time. The scaled time $T = gt / [2\pi(\bar{n})^{1/2}]$. The mean photon number $\bar{n} = 20$.

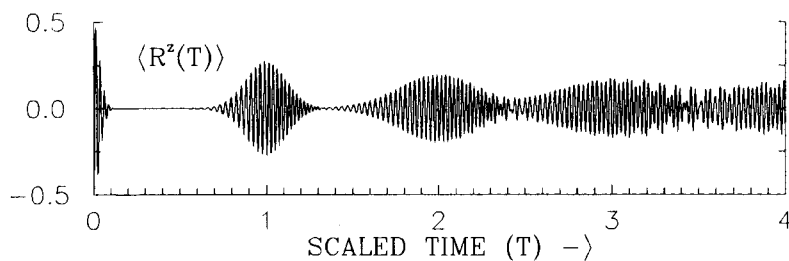


Fig. 2. Time record of the atomic inversion of $\langle R^z(t) \rangle$ with $\bar{n} = 20$. The scaled time $T = gt / [2\pi(\bar{n})^{1/2}]$.

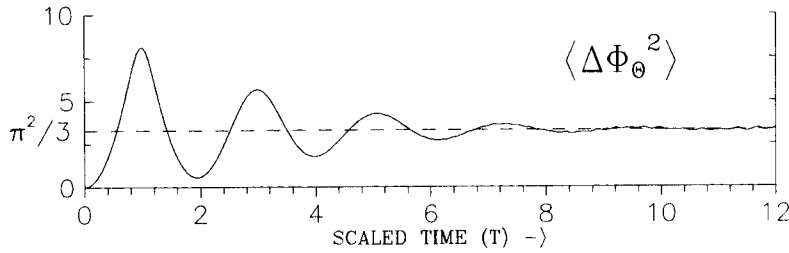


Fig. 3. The variance of the phase operator as a function of the scaled time $T = \tau / [2\pi(\bar{n})^{1/2}]$. The mean photon number $\bar{n} = 20$.

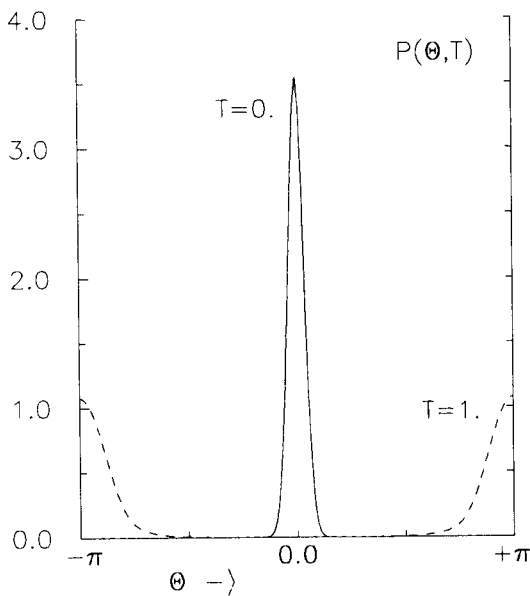


Fig. 4. Phase probability distribution $P(\theta, t)$ plotted against θ in the rectangular coordinate system for the values of the scaled time $T=0$ and $T=1$. The mean photon number $\bar{n} = 20$.

erties of the field based mainly on the numerical calculations. A rigorous analysis is difficult because of the square root appearing in the summation (17). However, there exist some other models close in spirit to the original JCM, for which the dynamical behavior is exactly periodic. One among them is the JCM with an intensity dependent coupling. This model was introduced by Buck and Sukumar [12] with the hamiltonian

$$H^{\text{BS}} = \hbar\omega(a^\dagger a + R^z) + \hbar g(R^\dagger s + R^- s), \quad (24)$$

where $s = a(a^\dagger a)^{1/2}$, $s^\dagger = (a^\dagger a)^{1/2} a^\dagger$. It is interesting to

compare the two models from the point of view of phase properties. Repeating the procedure used above we arrived at the following results for the phase density distribution,

$$P^{\text{BS}}(\theta, t) = \frac{1}{2} [P_+^{\text{BS}}(\theta, t) + P_-^{\text{BS}}(\theta, t)], \quad (25)$$

with

$$P_\pm^{\text{BS}}(\theta, t) = \frac{1}{2\pi} \left(1 + 2 \sum_{n>k} b_n b_k \cos[(n-k)(\theta \mp gt)] \right). \quad (26)$$

From eqs. (25), (26) it can be seen that, different from the standard JCM, the phase density distribution $P^{\text{BS}}(\theta, t)$ is exactly periodic with period $2\pi/g$. The two counterrotating distributions $P_\pm^{\text{BS}}(\theta, t)$ are nothing but the phase density distributions for coherent field with phase θ replaced by $(\theta \mp gt)$ respectively. They completely overlap after every subsequent time interval of π/g . This is just the time when the revivals of the atomic inversion in this model occur [12]. The variance of the phase,

$$\begin{aligned} \langle \Delta \Phi_\theta^2 \rangle^{\text{BS}} &= \frac{1}{3} \pi^2 + 4 \sum_{n>k} b_n b_k \cos[(n-k)gt] \\ &\quad \times (-1)^{(n-k)/(n-k)^2}, \end{aligned} \quad (27)$$

naturally also oscillates with the same period as the phase density distribution. The exact periodicity of the phase density distribution and the phase variance in the JCM with the intensity dependent coupling is not surprising because, in this model, the revivals of the atomic inversion restore exactly its initial value [12].

4. Conclusion

Using the new phase formalism of Pegg and Barnett, we have shown how the collapse and revival phenomenon is reflected in the phase properties of the field. When the components of the field oscillate in-phase with one another the revivals occur. When the components of the field oscillate with strongly different phases the atomic inversion goes over into the collapse regime. The phase distribution is a one more example revealing the quantum nature of the field-atom interaction. We have also examined the modified JCM with an intensity dependent coupling. For this model the period between the overlappings of the counterrotating satellite distributions has been obtained analytically. The average value of phase shows no changes in time in both the models. But, we have found that this property disappears in more general cases, for example, when we take into account the field-atom frequency detuning. This will be reported elsewhere.

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