SQUEEZING FROM AN ANHARMONIC OSCILLATOR MODEL: $(a^+)^2a^2$ VERSUS $(a^+a)^2$ INTERACTION HAMILTONIANS

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A comparison is made between squeezing obtained from two versions of the anharmonic oscillator model. The periodic revivals of squeezing in the long-time scale are shown to exist in both cases and the differences in this scale are shown explicitly. It is shown that in the short-time scale, for large numbers of photons, both versions of the model lead to the same results. The approximate formula describing the normally ordered variances in this case is derived and illustrated graphically. Some recent misinterpretations are clarified.

1. Introduction

The anharmonic oscillator model is probably the simplest model to study nonlinear interactions of light in a nonlinear medium. This strictly solvable model can give some insight into the nonlinear dynamics. It has been shown [1], for example, that a high degree of squeezing can be obtained in such a model. A possible realization of the model is strong light propagating through a nonlinear Kerr medium. Such light can squeeze itself during the propagation and squeezing obtained in this way was referred to as self-squeezing [2]. In recent years, a number of papers appeared [3–9] in which many aspects of squeezing obtainable from the anharmonic oscillator model have been discussed.

There are basically two versions of the model that are used in discussions of squeezing: (i) with the interaction Hamiltonian $H_I=\frac{1}{2}\hbar\kappa(a^+)^2a^2$ in which the annihilation and creation operators are taken in the normal order, and (ii) with the interaction Hamiltonian $H_I=\frac{1}{2}\hbar\kappa(a^+a)^2$ in which the nonlinear term is proportional to the square of the linear oscillator Hamiltonian. The two interaction Hamiltonians differ by the term $\frac{1}{2}\hbar\kappa a^+a$ which can be incorporated into the free part of the Hamiltonian by changing appropriately the frequency of the oscillator. In squeezing, which is a phase-sensitive phenomenon, however, this extra phase shift may have some significance, especially in the homodyne detection when the local oscillator frequency is just the free oscillation frequency.

Recently, Bužek [10], using version (ii) of the anharmonic oscillator, discussed periodic revivals of squeezing and obtained what he called the "absolute" minimum of the variance of the field. He also suggested some basic discrepancies between his results and my earlier results [1] obtained for version (i) of the anharmonic oscillator. The aim of this Letter is to make an explicit comparison of the results obtained for squeezing in the two versions of the anharmonic oscillator and to clarify certain points.

2. The model

The two versions of the anharmonic oscillator model that are going to be compared in this paper are defined by the Hamiltonians

$$\text{(i)}\quad H=\hbar\omega a^+a+\tfrac{1}{2}\hbar\kappa(a^+)^2a^2\,, \tag{1}$$

$$\text{(ii)}\quad H'=\hbar\omega a^+a+\tfrac{1}{2}\hbar\kappa(a^+a)^2\,, \tag{2}$$

where κ is the nonlinearity parameter, which real and assumed the same in both cases.

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The Heisenberg equations of motion for the annihilation operators are then

$$\text{(i)}\quad \dot{a} = -\frac{\mathrm{i}}{\hbar}[a, H] = -\mathrm{i}(\omega + \kappa a^{+}a)a\,, \tag{3}$$

$$\text{(ii)}\quad \dot{a} = -\frac{\mathrm{i}}{\hbar}[a, H] = -\mathrm{i}(\omega + \tfrac{1}{2}\kappa + \kappa a^{+}a)a\,. \tag{4}$$

Since $a^{+}a$ is a constant of motion in both cases the solutions are the exponentials

$$\text{(i)}\quad a(t) = \exp\{-\mathrm{i}t[\omega + \kappa a^{+}(0)a(0)]\}a(0)\,, \tag{5}$$

$$\text{(ii)}\quad a(t) = \exp\{-\mathrm{i}t[\omega + \tfrac{1}{2}\kappa + \kappa a^{+}(0)a(0)]\}a(0)\,. \tag{6}$$

Eqs. (5) and (6) are the exact operator solutions describing the dynamics of the two versions of the anharmonic oscillator. It is seen that the only difference is the extra phase shift $\frac{1}{2}\kappa t$ which appeared in (6).

Since we are interested in squeezing, we define the Hermitian quadrature operator

$$Q_{\varphi} = a(t)\,\mathrm{e}^{\mathrm{i}(\omega t - \varphi)} + a^{+}(t)\,\mathrm{e}^{-\mathrm{i}(\omega t - \varphi)}\,, \tag{7}$$

which for $\varphi = 0$ corresponds to the in-phase quadrature component of the field and for $\varphi = \pi/2$ to the out-of-phase component.

The variance of such an operator is given by

$$\begin{aligned}\mathrm{Var}[Q_{\varphi}] &= \langle Q_{\varphi}^{2}\rangle - \langle Q_{\varphi}\rangle^{2}\\ &= 2\,\mathrm{Re}\{\langle a^{2}(t)\rangle\,\mathrm{e}^{2\mathrm{i}(\omega t - \varphi)} - \langle a(t)\rangle^{2}\,\mathrm{e}^{2\mathrm{i}(\omega t - \varphi)}\}\\ &\quad + 2\{\langle a^{+}a\rangle - \langle a^{+}(t)\rangle\langle a(t)\rangle\} + 1\,.\end{aligned} \tag{8}$$

For the vacuum state as well as coherent states this variance is equal to unity. If it becomes smaller than unity the state of the field for which this occurs is referred to as squeezed state, and perfect squeezing is obtained if $\mathrm{Var}[Q_{\varphi}] = 0$. It is convenient to use the normally ordered variance

$$\begin{aligned}V_{\varphi}(t) &= \langle :Q_{\varphi}^{2}(t): \rangle - \langle Q_{\varphi}(t)\rangle^{2}\\ &\quad 2\,\mathrm{Re}\{\langle a^{2}(t)\rangle\,\mathrm{e}^{2\mathrm{i}(\omega t - \varphi)} - \langle a(t)\rangle^{2}\,\mathrm{e}^{2\mathrm{i}(\omega t - \varphi)}\}\\ &\quad + 2\{\langle a^{+}(t)a(t)\rangle - \langle a^{+}(t)\rangle\langle a(t)\rangle\}\,.\end{aligned} \tag{9}$$

Negative values of this variance mean squeezing and its value equal to -1 means perfect squeezing.

Assuming that the initial state of the field is a coherent state $|\alpha\rangle$ with the mean number of photons $N = |\alpha|^{2}$, and using eqs. (5) and (6), one can easily calculate the normally ordered variances (9) for both versions of the nonlinear interactions. The results are as follows:

$$\begin{aligned}\text{(i)}\quad V_{\varphi}(\tau) &= 2N\{\exp[N(\cos 2\tau - 1)]\\ &\quad \times\cos[2(\varphi - \varphi_{0}) + \tau + N\sin 2\tau]\\ &\quad - \exp[2N(\cos\tau - 1)]\\ &\quad \times\cos[2(\varphi - \varphi_{0}) + 2N\sin\tau]\\ &\quad + 1 - \exp[2N(\cos\tau - 1)]\}\,,\end{aligned} \tag{10}$$

$$\begin{aligned}\text{(ii)}\quad V'_{\varphi}(\tau) &= 2N\{\exp[N(\cos 2\tau - 1)]\\ &\quad \times\cos[2(\varphi - \varphi_{0}) + 2\tau + N\sin 2\tau]\\ &\quad - \exp[2N(\cos\tau - 1)]\\ &\quad \times\cos[2(\varphi - \varphi_{0}) + \tau + 2N\sin\tau]\\ &\quad + 1 - \exp[2N(\cos\tau - 1)]\}\,,\end{aligned} \tag{11}$$

where we have introduced the notation $\tau = \kappa t$ and $\alpha = \sqrt{N}\,\mathrm{e}^{\mathrm{i}\varphi_{0}}$ with φ_{0} being the initial phase of the field.

So, when putting $\varphi - \varphi_{0} = 0$ in (10) and (11), one obtains the normally ordered variance for the in-phase quadrature component of the field, or $\varphi - \varphi_{0} = \pi/2$ gives the corresponding variance for the out-of-phase component. Eq. (10) reproduces my earlier results [1], while eq. (11) reproduces the corresponding formulas ((14a) and (14b)) of Bužek [10], although in a slightly different and more transparent form.

3. Comparison of the results

Even a superficial look at formulas (10) and (11) shows that they are quite similar. The only difference is the extra phase shift by τ, as one would expect according to the solutions (5) and (6). In both cases the variances are periodic in the long-time scale

$$\begin{aligned}V_{\varphi}(\tau) &= V_{\varphi}(\tau + k\times 2\pi)\,,\\ V'_{\varphi}(\tau) &= V'_{\varphi}(\tau + k\times 2\pi)\,,\end{aligned} \tag{12}$$

for $k = 1, 2, \ldots$.

Hence, in both cases there are periodic revivals of squeezing in the long-time scale with exactly the same period. The statement made by Bužek [10] that in case (i) the variance "tends to oscillate irregularly and *never* becomes squeezed again" is completely unjustified. Since the extra phase shift τ is the time-dependent shift, there is an essential difference in the shape of the variances for both cases within the period. This is shown in figs. 1 and 2, where the normally ordered variances for the in-phase and the out-of-phase quadrature components are plotted against τ for the two versions of the anharmonic oscillator. It is seen from fig. 1 that in the long-time scale and for a small mean photon number only the in-phase component can be squeezed in case (i). Fig. 2 reproduces the corresponding figure obtained by Bužek [10] for case (ii) and shows that squeezing can go from the in-phase to the out-of-phase component of the field during the evolution.

If the anharmonic oscillator model is to be used to describe the propagation of light in a nonlinear Kerr medium, the estimates based on the realistic values of the nonlinear susceptibilities give, assuming the

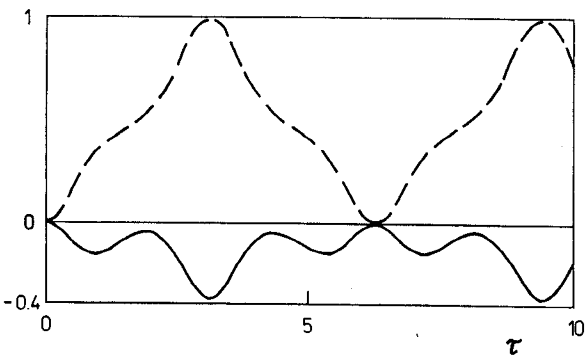

Fig. 1. The normally ordered variances $V_\varphi(\tau)$ (eq. (10)) plotted against τ, for $N=0.25$: solid line: the in-phase component; dashed line: the out-of-phase component.

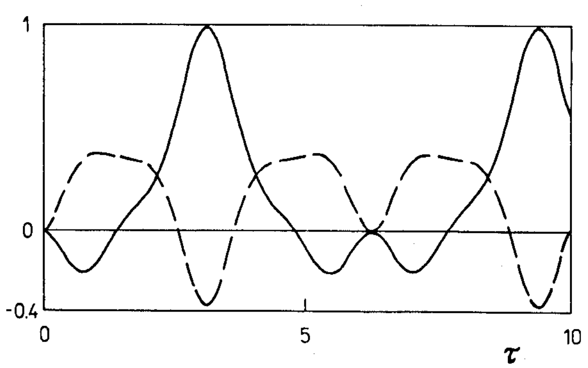

Fig. 2. The same as in fig. 1, but for variances $V'_\varphi(\tau)$ (eq. (11)).

length of the medium to be of the order of meters, for τ values of the order of 1×10^{-6} [2]. So, the long-time scale is far beyond the limits that can be reached in any real system and has a rather academic meaning. In real systems, however, one can still expect a high degree of squeezing when the mean number of photons N is sufficiently large [1,2]. This brings us to the short-time scale. In this case, when $\tau\ll1$ and $N\gg1$, the variable that properly describes the scale on which essential changes in the variances take place is [1]

$$x=\tau N\,. \tag{13}$$

Taking advantage of the inequalities $\tau\ll1$ and $N\gg1$, one can expand the variances (10) and (11) into power series and, after leaving only the leading terms in $x=\tau N$, obtain the following simple expression for the two variances:

$$V_\varphi(x)=V'_\varphi(x)$$
$$=-2[x\sin2\theta-x^2(1-\cos2\theta)]\,, \tag{14}$$

where $\theta=\varphi-\varphi_0+x$.

Again, for $\varphi-\varphi_0=0$ we have the variance for the in-phase component of the field, while for $\varphi-\varphi_0=\pi/2$ we have it for the out-of-phase component. The generalization of formula (14) to include higher order nonlinearities is given elsewhere [11].

Strikingly, in the short-time scale ($\tau\ll1$) and for a large number of photons ($N\gg1$), there is no difference between the two versions of the anharmonic oscillator model. One could expect this because, for $\tau\ll1$, the extra phase shift which is just τ is small. However, one should be warned of dropping the free τ's in the cosine functions altogether. This would be wrong. The reason why both versions give the same result is the same difference in phase, equal to τ, between the two phase-sensitive cosine terms which are subtracted in each variance. The variances (14) are illustrated graphically in figs. 3 and 4. In fig. 3 the in-phase and the out-of-phase component variances are plotted against x. This figure is just the reproduction of my earlier results [1]. Both variances show oscillatory behaviour with minima that exhibit a considerable amount of squeezing. In fig. 4 the variance for the in-phase component of the field is plotted against the extended range of x to visualize how the subsequent minima become deeper and nar-

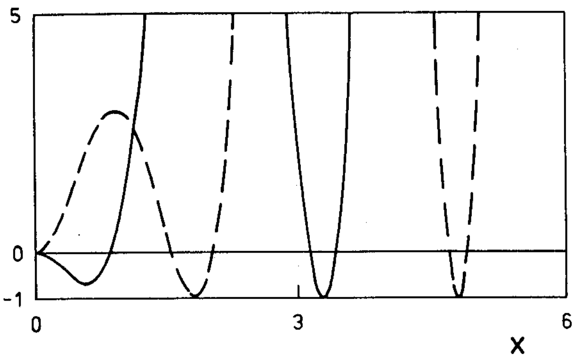

Fig. 3. The normally ordered variances (eq. (14)) for short times and large numbers of photons plotted against x; solid line: the in-phase component; dashed line: the out-of-phase component.

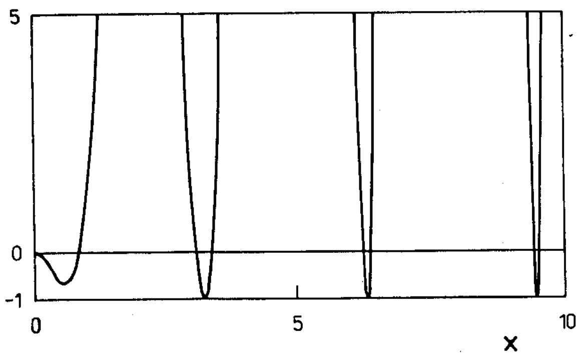

Fig. 4. Appearance of the subsequent minima in the variance (eq. (14)) for the in-phase component.

rower. This means that squeezing in the subsequent minima becomes closer to perfect squeezing, but this large squeezing lasts over a very narrow range of x.

Bužek [10] has calculated what he called the "absolute minimum" of the variance obtaining the value -0.63. To find this minimum he calculated the first derivative of the variance and equated it to zero to derive the point of the minimum. However, he somehow overlooked that there are more than one zeros of the derivative and, consequently, more than one minima of the variance.

So, the "absolute minimum" is not at all absolute. It is simply the first minimum and there are other minima which are deeper than the first one. In fact, the limit of the variance for $N\to\infty$ calculated by Bužek (his formula (23)) after appropriate change of variables agrees with my formula (14), which has been obtained in a direct way from the exact formulas (10) and (11).

I would like to emphasize that there is no difference in squeezing between the two versions of the anharmonic oscillator for short times ($\tau \ll 1$) and large numbers of photons ($N \gg 1$).

4. Conclusions

In this Letter the explicit formulas for the normally ordered variances for two versions of the anharmonic oscillator have been obtained. It has been shown that in both versions the variances are periodic with the same period, which means periodic revivals of squeezing in the long-time scale. The difference in the long-time scale behaviour is illustrated in figs. 1 and 2. Moreover, it has been shown that in the short-time scale, for large numbers of photons, both versions of the anharmonic oscillator give the same formula for squeezing, which exhibits oscillatory behaviour with the subsequent minima being deeper and narrower. A comparison has been made to the recent results by Bužek [10] and, as I believe, certain misinterpretations have been clarified.

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