Comments

Comments are short papers which comment on papers of other authors previously published in the **Physical Review.** *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on "Higher-order squeezing from an anharmonic oscillator"

Ryszard Tanaś
Nonlinear Optics Division, Institute of Physics, Adam Mickiewicz University, 60-780 Poznań, Poland
(Received 7 December 1987)

In this Comment certain questions of higher-order and intrinsic higher-order squeezing from an anharmonic oscillator recently discussed by Gerry and Rodrigues [Phys. Rev. A **35**, 4440 (1987)] are clarified.

In their paper[1] Gerry and Rodrigues have shown that an anharmonic oscillator model, previously shown[2] to give a high degree of second-order squeezing, leads also to a high degree of higher-order squeezing. They calculated the degrees of higher-order squeezing up to the sixth order, and finally state that "at the point of high squeezing, $\langle :(\Delta\hat{E}_1)^4: \rangle$ and $\langle :(\Delta\hat{E}_1)^6: \rangle$ are indeed negative indicating that the higher-order squeezing is not intrinsic." This statement, however, is hardly correct and is a misinterpretation of the otherwise correct numerical results obtained by them. In my Comment I would like to clarify this point.

The notion of higher-order squeezing of quantum electromagnetic fields has been introduced by Hong and Mandel.[3] Introducing the two slowly varying Hermitian quadrature components

$$\begin{aligned} \hat{E}_1 &\equiv \hat{E}^{(+)} e^{i(\omega t-\varphi)} + \hat{E}^{(-)} e^{-i(\omega t-\varphi)} , \\ \hat{E}_2 &\equiv \hat{E}^{(+)} e^{i(\omega t-\varphi-\pi/2)} + \hat{E}^{(-)} e^{-i(\omega t-\varphi-\pi/2)} , \end{aligned} \tag{1}$$

where φ is an arbitrary phase and the operators $\hat{E}^{(+)}$ and $\hat{E}^{(-)}$ satisfy the commutation relation

$$[\hat{E}^{(+)},\hat{E}^{(-)}]=C \tag{2}$$

with C a positive number, one obtains the commutation relation

$$[\hat{E}_1,\hat{E}_2]=2iC , \tag{3}$$

and consequently the uncertainty relation

$$\langle (\Delta\hat{E}_1)^2 \rangle \langle (\Delta\hat{E}_2)^2 \rangle \geq C^2 , \tag{4}$$

where $\Delta\hat{E} \equiv \hat{E} - \langle \hat{E} \rangle$. The state is said to be squeezed if one of the dispersions is less than its value in the coherent state, i.e., if there exists some phase angle φ for which

$$\langle (\Delta\hat{E}_1)^2 \rangle < C . \tag{5}$$

The state is then squeezed to the second order in $\hat{E}_1$. The normally ordered variance $\langle :(\Delta\hat{E}_1)^2: \rangle$ is, according to (2), given by

$$\langle :(\Delta\hat{E}_1)^2: \rangle = \langle (\Delta\hat{E}_1)^2 \rangle - C , \tag{6}$$

and the state is squeezed to the second order in $\hat{E}_1$ if $\langle :(\Delta\hat{E}_1)^2: \rangle$ is negative.

This definition may be generalized for higher-order moments.[3] The state is squeezed to any even order N if

$$\langle (\Delta\hat{E}_1)^N \rangle < (N-1)!!C^{N/2} . \tag{7}$$

This implies that

$$\begin{aligned} &\langle :(\Delta\hat{E}_1)^2: \rangle < 0 , \\ &\langle :(\Delta\hat{E}_1)^4: \rangle + 6C\langle :(\Delta\hat{E}_1)^2: \rangle < 0 , \\ &\langle :(\Delta\hat{E}_1)^6: \rangle + 15C\langle :(\Delta\hat{E}_1)^4: \rangle + 45C^2\langle :(\Delta\hat{E}_1)^2: \rangle < 0 . \\ &\langle :(\Delta\hat{E}_1)^8: \rangle + 28C\langle :(\Delta\hat{E}_1)^6: \rangle + 210C^2\langle :(\Delta\hat{E}_1)^4: \rangle \\ &\qquad + 420C^4\langle :(\Delta\hat{E}_1)^2: \rangle < 0 , \end{aligned} \tag{8}$$

for second-, fourth-, sixth-, and eighth-order squeezing, respectively. This means that to obtain higher-order squeezing it is not necessary to have all normally ordered moments negative. The conditions (8) can be satisfied even if the higher (normally ordered) moments are positive provided that the term $\langle :(\Delta\hat{E}_1)^2: \rangle$ predominates. Squeezing is said to be *intrinsically* of Nth order[3] if

$$\langle :(\Delta\hat{E}_1)^N: \rangle < 0 . \tag{9}$$

That is, Nth-order squeezing does not necessarily imply Nth-order intrinsic squeezing. As Hong and Mandel[3] have shown, the situation differs from one nonlinear optical process to another.

A convenient parameter q_N for measuring the degree of Nth order squeezing is[3]

$$q_N = \frac{\langle (\Delta\hat{E}_1)^N \rangle - (N-1)!!C^{N/2}}{(N-1)!!C^{N/2}} , \tag{10}$$

where q_N is negative whenever there is Nth-order squeezing; $q_N = -1$ means the maximum of Nth-order squeezing.

The anharmonic oscillator has been shown to produce a high degree of second[2]- as well as higher[1]-order squeezing. The model is described by the Hamiltonian

$$\hat{H}=\hbar\omega\hat{a}^{\dagger}\hat{a}+\tfrac{1}{2}K\hat{a}^{\dagger 2}\hat{a}^{2}\ , \tag{11}$$

where K is the anharmonicity parameter related with the third-order susceptibility of the medium.[4] All non-energy-conserving terms have been dropped in Eq. (11), and only a single mode of the electromagnetic field is taken into consideration.

According to (11), the Heisenberg equation of motion for the annihilation operator $\hat{a}$ reads

$$\dot{\hat{a}}=-\frac{i}{\hbar}[\hat{a},\hat{H}]=-i(\omega+K\hat{a}^{\dagger}\hat{a})a\ . \tag{12}$$

Since $\hat{a}^{\dagger}\hat{a}$ is a constant of motion, Eq. (12) has the simple exponential solution

$$\hat{a}(t)=\exp\{-it[\omega+K\hat{a}^{\dagger}(0)\hat{a}(0)]\}\hat{a}(0)\ . \tag{13}$$

Omitting the dimensional constants one can identify the field operators $\hat{E}^{(+)}$ and $\hat{E}^{(-)}$ with $\hat{a}$ and $\hat{a}^{+}$, respectively, and the constant C is then equal to unity. Assuming that the initial state of the field is a coherent state $|\alpha\rangle$, it is easy to calculate the normally ordered moments $\langle:[\Delta\hat{E}_1(\tau)]^N:\rangle$ as

$$\begin{aligned}\langle:[\Delta\hat{E}_1(\tau)]^N:\rangle&=\sum_{r=0}^{N}\begin{bmatrix}N\\ r\end{bmatrix}\langle[\hat{a}^{\dagger}(\tau)-\langle\hat{a}^{\dagger}(\tau)\rangle]^r[\hat{a}(\tau)-\langle\hat{a}(\tau)\rangle]^{N-r}\rangle\\&=\sum_{r=0}^{N}\begin{bmatrix}N\\ r\end{bmatrix}\langle(\hat{a}^{\dagger}e^{i\tau\hat{a}^{\dagger}\hat{a}}-\langle\hat{a}^{\dagger}e^{i\tau\hat{a}^{\dagger}\hat{a}}\rangle)^r(e^{-i\tau\hat{a}^{\dagger}\hat{a}}\hat{a}-\langle e^{-i\tau\hat{a}^{\dagger}\hat{a}}\hat{a}\rangle)^{N-r}\rangle\\&=\sum_{r=0}^{N}\sum_{p=0}^{r}\sum_{q=0}^{N-r}\begin{bmatrix}N\\ r\end{bmatrix}\begin{bmatrix}r\\ p\end{bmatrix}\begin{bmatrix}N-r\\ q\end{bmatrix}(-1)^{N-p-q}\alpha^{*r}\alpha^{N-r}\\&\qquad\times\exp\{\tfrac{1}{2}[p(p-1)-q(q-1)]i\tau+(e^{(p-q)i\tau}-1)|\alpha|^2\\&\qquad\qquad+(r-p)(e^{i\tau}-1)|\alpha|^2+(N-r-q)(e^{-i\tau}-1)|\alpha|^2\}\ ,\end{aligned}\tag{14}$$

where the commutation rules (2) were used and the average values were taken in the coherent state $|\alpha\rangle$, giving, for example,

$$\langle e^{-i\tau\hat{a}^{\dagger}\hat{a}}\hat{a}\rangle=\alpha\exp\{(e^{-i\tau}-1)|\alpha|^2\}\ . \tag{15}$$

The shortened notation $\tau=Kt$ and $\hat{a}(0)=\hat{a}$, $\hat{a}^{\dagger}(0)=\hat{a}^{\dagger}$ was used in (14) and (15). The average number of photons in the mode is $\langle\hat{a}^{\dagger}\hat{a}\rangle=|\alpha|^2$, and the complex amplitude is $\alpha=|\alpha|e^{-i\varphi}$.

Expression (14) is the main result of this paper. It is essential that an expression such as Eq. (14) in exact, closed form can be obtained for the normally ordered Nth-order moments. For even N, this expression is real. Only even-order moments will be considered now in further discussion. The triple summation in (14) is not easy to perform, especially for small values of τ and high values of $|\alpha|^2$, which is the case in real physical situations. For $\tau\ll 1$, however, the summations can be performed by collecting separately the terms of the same order in τ. The results, for $\tau=1\times 10^{-6}$, are illustrated in Fig. 1, where the normally ordered moments $\langle:(\Delta\hat{E}_1(\tau))^N:\rangle$ ($N=2,4,6,8$ and $\varphi=0$) are plotted against $|\alpha|^2\tau$. One notes that for $N=2$ and 6 the normally ordered moments are negative, whereas for $N=4$ and 8 the moments are positive. This means that according to the definitions of Hong and Mandel,[3] the anharmonic oscillator model leads to intrinsic squeezing of the second and sixth order but not of the forth or eighth order. So, intrinsic squeezing appears only for $N/2$ odd and does not appear for $N/2$ even. Although our calculations have been performed up to $N=8$ only, one can expect similar behavior for $N>8$. This is a situation similar to that obtained for the two-photon coherent states.[3]

The preceding results, however, are in contrast to the statement of Gerry and Rodrigues,[1] who imply that $\langle:(\Delta\hat{E}_1)^4:\rangle$ and $\langle:(\Delta\hat{E}_1)^6:\rangle$ are negative. In fact, $\langle:(\Delta\hat{E}_1)^4:\rangle$ has to be positive. This can be proved as well from the numerical results of Gerry and Rodrigues.[1] For $|\alpha|^2\tau=0.59$ they obtain, for the q_N parameters defined by (10), the following values: $q_2=-0.6600$, $q_4=-0.884$, and $q_6=-0.9667$. Albeit, according to their equation [Eq. (21), Ref. 1], one obtains

$$\langle(\Delta\hat{E}_1)^4\rangle=\langle:(\Delta\hat{E}_1)^4:\rangle+6\langle:(\Delta\hat{E}_1)^2:\rangle+3\ . \tag{16}$$

Since $\langle:(\Delta\hat{E}_1)^2:\rangle=q_2$, one gets $6q_2+3=-0.96$ negative,

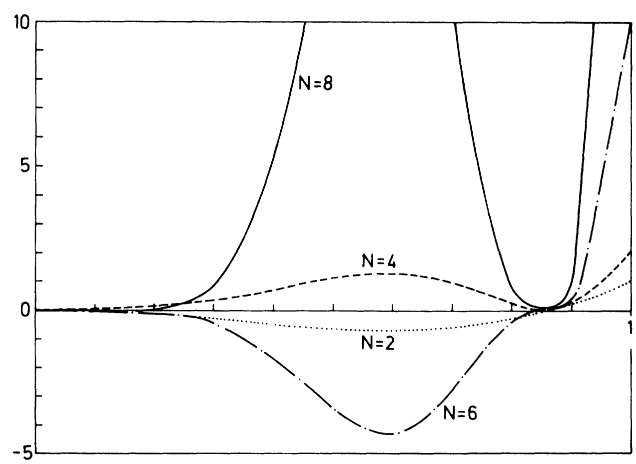

FIG. 1. Normally ordered moments $\langle:[\Delta\hat{E}_1(\tau)]^N:\rangle$ vs $|\alpha|^2\tau$.

meaning that $\langle :(\Delta\hat{E}_1)^4: \rangle$ must be positive in order to have $\langle (\Delta\hat{E}_1)^4 \rangle$ positive. Hence their statement that $\langle :(\Delta\hat{E}_1)^4: \rangle$ is negative appears to us as an obvious misinterpretation of the otherwise correct numerical results obtained by them. They, in fact, have calculated directly ordinary (nor normally ordered) moments. Obviously, the normally ordered moments given by Eq. (14) are easier to calculate. For $|\alpha|^2\tau=0.59$, I obtain

$$\langle :(\Delta\hat{E}_1)^2: \rangle = -0.66 ,$$

$$\langle :(\Delta\hat{E}_1)^4: \rangle = 1.31 ,$$

$$\langle :(\Delta\hat{E}_1)^6: \rangle = -4.31 ,$$

and

$$\langle :(\Delta\hat{E}_1)^8: \rangle = 19.93 ,$$

giving $q_2=-0.66$, $q_4=-0.88$, $q_6=-0.96$, and $q_8=-0.98$. These results are in agreement with those of Gerry and Rodrigues[1] and confirm their statement that higher-order squeezing from an anharmonic oscillator increases as N increases, i.e., q_N becomes closer to -1 with increasing N.

I thank Professor S. Kielich for discussions. This work was supported by Polish Research project No. CPBP 01.07.

[1] C. Gerry and S. Rodrigues, Phys. Rev. A **35**, 4440 (1987).
[2] R. Tanaś, in *Coherence and Quantum Optics V*, edited by L. Mandel and E. Wolf (Plenum, New York, 1984), p. 645.
[3] C. K. Hong and L. Mandel, Phys. Rev. Lett. **54**, 323 (1985); Phys. Rev. A **32**, 974 (1985).
[4] P. D. Drummond and D. F. Walls, J. Phys. A **13**, 725 (1980).