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# Evolution of quantum correlations in a two-atom system

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**Abstract**
We discuss the evolution of quantum correlations for a system of two two-level atoms interacting with a common reservoir. The Markovian master equation is used to describe the evolution of various measures of quantum correlations. It is shown that different measures of quantum correlations exhibit qualitatively different behaviours in their evolution.

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(Some figures may appear in colour only in the online journal)

## 1. Introduction

It is now well established that quantum entanglement, which is a measure of quantum correlations, is a necessary resource for performing some quantum information algorithms [1]. However, it has been realized that quantum entanglement does not include all quantum correlations, and there are separable states that exhibit quantum correlations different from entanglement. It became important to recognize the nature of correlations and properly discriminate the quantum correlations from the classical ones. A number of measures of quantum correlations have been introduced, and the most popular of these is quantum discord [2, 3] (see also the review [4] and papers cited therein). It is crucial, from the point of view of future applications, to understand how quantum correlations evolve in time when a multipartite system interacts with the dissipative environment.

The time evolution of entanglement for a system of two qubits, or two two-level atoms, can be qualitatively described for various physical situations, and it has been studied extensively in recent years [5–12]. A great deal of discussion has been devoted to the problem of disentanglement of the two-qubit system in finite time, despite the fact that all the matrix elements of the two-atom system decay only asymptotically. Yu and Eberly [9] coined the term 'entanglement sudden death' (ESD) for the process of finite-time disentanglement. ESD has recently been confirmed experimentally [10]. Another problem related to the entanglement evolution that has attracted attention is the evolution of the entangled qubits interacting with non-Markovian reservoirs [13–15]. It has also been shown that a squeezed reservoir leads to steady-state entanglement [16] and revivals of entanglement [17].

Besides entanglement, a considerable amount of attention has been paid in recent years to quantum discord. Generally, quantum discord is difficult to calculate because it requires an optimization procedure, which usually amounts to extensive numerical calculations. In the case of two qubits analytical results can be obtained for some specific families of states, such as Bell-diagonal states or states having maximally mixed marginals [18]. Another interesting family of states is a family of the two-qubit X-states, which are of interest here. Ali *et al* [19] reported a closed form solution for quantum discord of the X states. However, it turned out that their algorithm is not universal. Lu *et al* [20] have proven that a universal set of orthogonal projective measurements cannot be found for the full family of X states. Some counterexamples have been given in [20, 21]. Chen *et al* [21], however, confirmed the applicability of the algorithm for several special cases of X states. Lu *et al* found that the probability distribution of measurements is centralized around a specific von Neumann measurement, which they called the maximal–correlation–direction measurement. This observation justifies the Ali *et al* algorithm in a statistical sense, because it turns out that for 99.4% of the cases of numerically generated X states the algorithm gives correct results. The situation with arbitrary two-qubit states is more complicated and the best of what has been achieved so far

 

are the two transcendental equations obtained by Girolami and Adesso [22] that must be solved numerically to obtain quantum discord.

Dakić *et al* [23] have introduced another measure of quantum correlation based on the Hilbert–Schmidt distance measure, which is called geometric discord. The advantage of this measure is that it allows obtaining analytical formulae for general two-qubit states. Girolami and Adesso [24] introduced the so-called observable measure of bipartite quantum correlation, which is a lower bound to the geometric discord. The behaviour of the geometric discord under decoherence has been studied in [25]. Recently, Bellomo *et al* [26] compared the dynamics of geometric and entropic quantifiers of different kinds of correlations in a non-Markovian open two-qubit system under local dephasing.

In this paper, we study the evolution of concurrence (a measure of entanglement), quantum discord and geometric discord in a system of two two-level atoms interacting with a common reservoir that is in a vacuum state. The evolution of the system is described by the Markovian master equation introduced by Lehmberg [27] and Agarwal [28], taking into account the cooperative behaviour of the atoms. It is shown that different measures of quantum correlations evolve in time quite differently. We compare their evolution for a family of pure initial states. Since the evolution of the system is described by a realistic master equation, which is a good testing ground for studying physical processes involving two atoms, we believe that the results we have obtained shed new light on understanding quantum correlations.

## 2. The master equation

We consider a system of two two-level atoms, A and B, with ground states $|g_i\rangle$ and excited states $|e_i\rangle$ $(i = \mathrm{A}, \mathrm{B})$ connected by dipole transition moments $\boldsymbol{\mu}_i$. The atoms are located at fixed positions $r_\mathrm{A}$ and $r_\mathrm{B}$ and are coupled to all modes of the electromagnetic field being in the vacuum state.

The reduced two-atom density matrix evolves in time according to the Markovian master equation given by [27–29]

$$\frac{\partial\rho}{\partial t} = -\mathrm{i}\sum_{i=1}^{2}\omega_i\left[S_i^z,\rho\right] - \mathrm{i}\sum_{i\neq j}^{2}\Omega_{ij}\left[S_i^+S_j^-,\rho\right] - \frac{1}{2}\sum_{i,j=1}^{2}\Gamma_{ij}\left(\rho S_i^+S_j^- + S_i^+S_j^-\rho - 2S_j^-\rho S_i^+\right), \quad (1)$$

where $S_i^+$ $(S_i^-)$ are the raising (lowering) operators, and $S_i^z$ is the energy operator of the $i$th atom, $\Gamma_{ii} \equiv \Gamma$ are the spontaneous decay rates. We assume that the two atoms are identical. The parameters $\Gamma_{ij}$ and $\Omega_{ij}(i \neq j)$ depend on the distance between the atoms and describe the collective damping and the dipole–dipole interaction defined, respectively, by

$$\Gamma_{ij} = \frac{3}{2}\Gamma\left(\frac{\sin kr_{ij}}{kr_{ij}} + \frac{\cos kr_{ij}}{(kr_{ij})^2} - \frac{\sin kr_{ij}}{(kr_{ij})^3}\right) \quad (2)$$

and

$$\Omega_{ij} = \frac{3}{4}\Gamma\left(-\frac{\cos kr_{ij}}{kr_{ij}} + \frac{\sin kr_{ij}}{(kr_{ij})^2} + \frac{\cos kr_{ij}}{(kr_{ij})^3}\right), \quad (3)$$

where $k = \omega_0/c$, and $r_{ij}$ is the distance between the atoms. Here, we assume, with no loss of generality, that the atomic dipole moments are parallel to each other and are polarized in the direction perpendicular to the interatomic axis.

To describe the evolution of the two-qubit system the standard basis of atomic product states can be used: $|1\rangle = |e_\mathrm{A}\rangle\otimes|e_\mathrm{B}\rangle$, $|2\rangle = |e_\mathrm{A}\rangle\otimes|g_\mathrm{B}\rangle$, $|3\rangle = |g_\mathrm{A}\rangle\otimes|e_\mathrm{B}\rangle$, $|4\rangle = |g_\mathrm{A}\rangle\otimes|g_\mathrm{B}\rangle$. It is easier, however, to find the solutions of the master equations when using, instead of the standard basis, a basis of the collective states: $|e\rangle = |e_\mathrm{A}\rangle\otimes|e_\mathrm{B}\rangle$, $|s\rangle = \frac{1}{\sqrt{2}}(|e_\mathrm{A}\rangle\otimes|g_\mathrm{B}\rangle + |g_\mathrm{B}\rangle\otimes|e_\mathrm{B}\rangle)$, $|a\rangle = \frac{1}{\sqrt{2}}(|e_\mathrm{A}\rangle\otimes|g_\mathrm{B}\rangle - |g_\mathrm{A}\rangle\otimes|e_\mathrm{B}\rangle)$, $|g\rangle = |g_\mathrm{A}\rangle\otimes|g_\mathrm{B}\rangle$. The states $|s\rangle$ and $|a\rangle$ are the symmetric and antisymmetric states of the two-atom system. They are maximally entangled states, or the Bell states of the two-atom system. Assuming that initially the system density matrix has the so-called X form, which is preserved during the evolution according to the master equation (1), we obtain the following system of equations for the obtain matrix elements [29]:

$$\begin{aligned}
\rho_{ee}(t) &= \rho_{ee}(0)\mathrm{e}^{-2\Gamma t},\\
\rho_{ss}(t) &= \rho_{ss}(0)\mathrm{e}^{-\Gamma_+ t} + \rho_{ee}(0)\frac{\Gamma_+}{\Gamma_-}\left[\mathrm{e}^{-\Gamma_+ t} - \mathrm{e}^{-2\Gamma t}\right],\\
\rho_{aa}(t) &= \rho_{aa}(0)\mathrm{e}^{-\Gamma_- t} + \rho_{ee}(0)\frac{\Gamma_-}{\Gamma_+}\left[\mathrm{e}^{-\Gamma_- t} - \mathrm{e}^{-2\Gamma t}\right],\\
\rho_{as}(t) &= \rho_{as}(0)\mathrm{e}^{-(\Gamma+\mathrm{i}2\Omega_{12})t},\\
\rho_{eg}(t) &= \rho_{ge}(0)\mathrm{e}^{-(\Gamma+2\mathrm{i}\omega_0)t},
\end{aligned} \quad (4)$$

where $\Gamma_\pm = \Gamma \pm \Gamma_\mathrm{AB}$. For calculating the quantum correlations measures, we need the solutions for the density matrix elements in the standard product basis, which can be expressed in terms of the matrix elements (4) in the following way:

$$\begin{aligned}
\rho_{11}(t) &= \rho_{ee}(t),\\
\rho_{14}(t) &= \rho_{eg}(t),\\
\rho_{22}(t) &= [\rho_{ss}(t) + \rho_{aa}(t) + \rho_{as}(t) + \rho_{sa}(t)]/2,\\
\rho_{33}(t) &= [\rho_{ss}(t) + \rho_{aa}(t) + \rho_{as}(t) - \rho_{sa}(t)]/2,\\
\rho_{23}(t) &= [\rho_{ss}(t) - \rho_{aa}(t) + \rho_{as}(t) - \rho_{sa}(t)]/2.
\end{aligned} \quad (5)$$

The solutions (4) and (5) are used to find the evolution of various measures of quantum correlations.

## 3. Measures of quantum correlations

### *3.1. Entanglement*

Probably the most celebrated and studied manifestation of quantum correlations is quantum entanglement. To quantify the entanglement, various entanglement measures have been introduced. We use here the concurrence introduced by Wootters [30]. In the case of the X states that we consider,

the concurrence can be calculated analytically, and it has the form [7]

$$\begin{aligned} \mathcal{C}(t) &= \max\{0, \mathsf{C}_1(t), \mathsf{C}_2(t)\}, \\ \mathsf{C}_1(t) &= 2\left(|\rho_{14}(t)| - \sqrt{\rho_{22}(t)\rho_{33}(t)}\right), \\ \mathsf{C}_2(t) &= 2\left(|\rho_{23}(t)| - \sqrt{\rho_{11}(t)\rho_{44}(t)}\right). \end{aligned} \tag{6}$$

Inserting into (6) the solutions (4) and (5), we find the values of $\mathsf{C}_1(t)$ and $\mathsf{C}_2(t)$, and whenever one of the two quantities becomes positive, there is a some degree of entanglement in the system.

### 3.2. Quantum discord

To calculate quantum discord we use the algorithm introduced by Ali *et al* [19], which works quite well in our case. We have checked for many numerically generated X states, for which the algorithm fails to find the correct extremum, that the differences between the numerically found extrema and the values given by the algorithm are so small that they cannot be resolved in the scale of the figures. In our calculation, we assume that the measurement is made on the subsystem B. For the X state we have [2, 3, 19]

$$\begin{aligned} \mathcal{D} &= S(\rho^{\mathrm{B}}) - S(\rho) + \min\{\mathcal{K}_1, \mathcal{K}_2\}, \\ \mathcal{K}_1 &= H\left[\frac{1}{2}\left(1 + \sqrt{(s_z^{\mathrm{A}})^2 + 4\eta^2}\right)\right], \\ \mathcal{K}_2 &= -\sum_i \rho_{ii}\log_2\rho_{ii} - S(\rho^{\mathrm{B}}), \end{aligned} \tag{7}$$

where $S(\rho)$, $S(\rho^{\mathrm{B}})$ mean the von Neumann entropy for the two-atom system and the subsystem B, respectively, $s_z^{\mathrm{A}} = \mathrm{Tr}[\rho\sigma_z^{\mathrm{A}}] = \rho_{11} + \rho_{22} - \rho_{33} - \rho_{44}$ is the third component of the Bloch vector of the subsystem A and $\eta = |\rho_{14}| + |\rho_{23}|$. Note that $H(x)$ is the Shannon entropy.

### 3.3. Geometric quantum discord

Geometric quantum discord is defined as [23, 31]

$$\mathcal{G} = 2\min_{\mathcal{X}} \| \rho - \mathcal{X} \|^2, \tag{8}$$

where $\mathcal{X}$ is a set of zero-discord states, and $\| \cdots \|$ denotes the Hilbert–Schmidt norm. The factor 2 in front has been introduced for normalization. The general formula for the geometric quantum discord takes the form

$$\mathcal{G} = \tfrac{1}{2}\left(\| \vec{s}^{\mathrm{B}} \|^2 + \| T \|^2 - k_{\max}\right), \tag{9}$$

where $\vec{s}^{\mathrm{B}}$ is the Bloch vector for the subsystem B (we assume that measurement is made on subsystem B), and $k_{\max}$ is the largest eigenvalue of the matrix

$$K = \vec{s}^{\mathrm{B}}(\vec{s}^{\mathrm{B}})^{\mathrm{t}} + T^{\mathrm{t}}T, \tag{10}$$

where the superscript t means transposition.

For X states we have

$$\begin{aligned} \mathcal{G} &= \min\{\mathcal{G}_1, \mathcal{G}_2\}, \\ \mathcal{G}_1 &= 4(|\rho_{14}|^2 + |\rho_{23}|^2), \\ \mathcal{G}_2 &= 2(|\rho_{14}|^2 - |\rho_2|^2)^2 + \tfrac{1}{2}\left[(s_z^{\mathrm{B}})^2 + T_{zz}^2\right], \end{aligned} \tag{11}$$

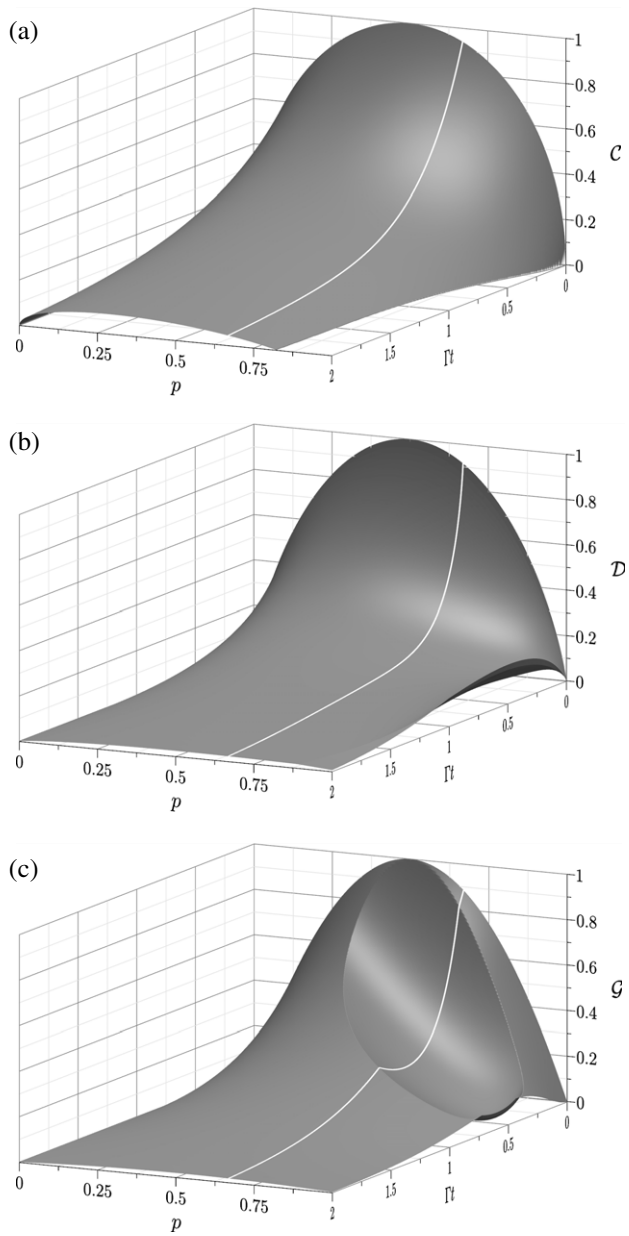


**Figure 1.** Evolution of the concurrence (a), quantum discord (b) and geometric quantum discord (c) for the two-photon Bell-like state (13) and the interatomic distance $r_{\mathrm{AB}} = \lambda/8$. The white lines show the section of the surface for $p = 2/3$.

where $s_z^{\mathrm{B}} = \rho_{11} - \rho_{22} + \rho_{33} - \rho_{44}$ is the third component of the Bloch vector for the subsystem B and $T_{ij} = \mathrm{Tr}[\rho(\sigma_i^{\mathrm{A}} \otimes \sigma_j^{\mathrm{B}})$ $(i, j = x, y, z)$ are elements of the correlation matrix.

The observable measure of Girolami and Adesso [24] is given by the formula

$$G = \frac{1}{6}\left[2\,\mathrm{Tr}(K) - \sqrt{6\mathrm{Tr}(K^2) - 2\,[\mathrm{Tr}(K)]^2}\right], \tag{12}$$

with the matrix $K$ given by (10).

## 4. Evolution of quantum correlations

To compare the evolution of various measures of quantum correlations in a system of two two-level atoms governed by the master equation (1), we assume that the initial state is the

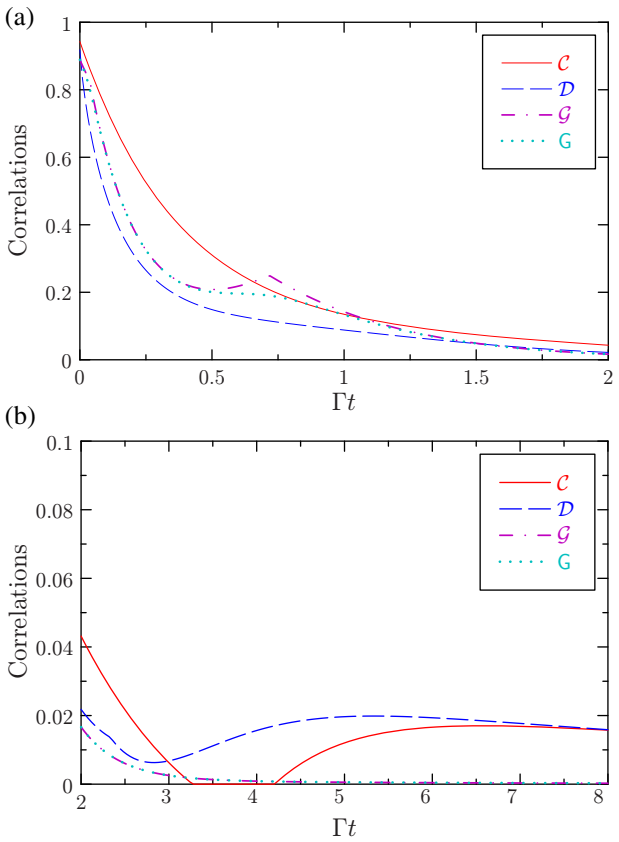


**Figure 2.** Evolution of correlations for the state (13) and interatomic distance $r_{AB} = \lambda/8$ for $p = 2/3$. The evolution is split into two time regions (note the different scales of both parts).

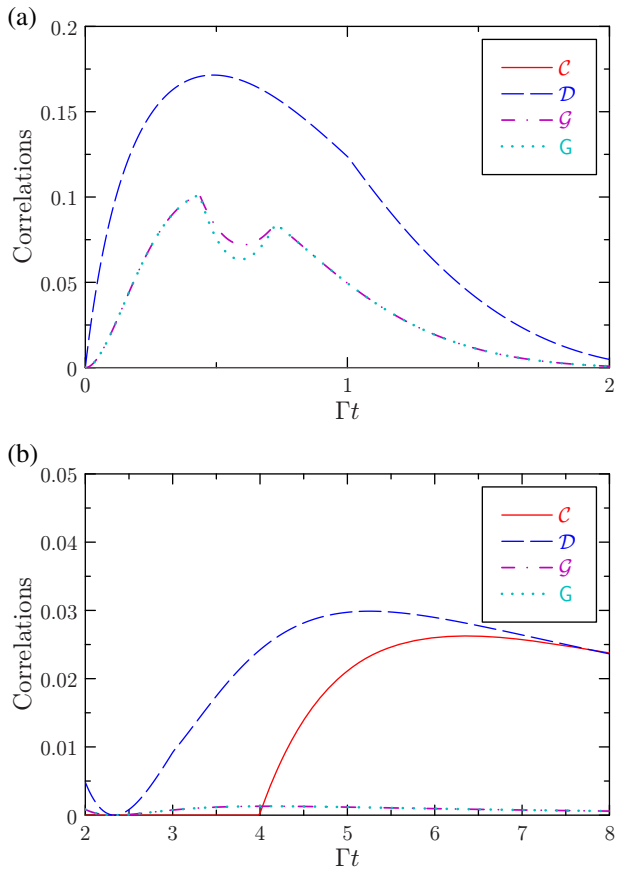


**Figure 3.** The same as figure 2, but for $p = 1$, i.e. initially both the atoms are excited.

Bell-like state

$$|\Psi\rangle = \sqrt{p}\,|1\rangle + \sqrt{1-p}\,|4\rangle, \tag{13}$$

where $p$ is the population of the upper state $|1\rangle$. In figure 1, we illustrate the evolution of concurrence, quantum discord and geometric quantum discord for the whole range of values of $p$. For $p < 1$, the state (13) is a superposition of the states $|1\rangle$ and $|4\rangle$ which has nonzero two-photon coherence $\rho_{14}$, while for $p = 1$ it is a product state of both atoms excited. It is seen from figure 1 that the evolution of different measures of quantum correlations depends on the value of $p$ and shows essential differences, not only quantitative but also qualitative. The white line seen on the figures indicates the section of the surface at $p = 2/3$. The evolution for this value of $p$ is illustrated in more detail in figure 2. It has been shown previously [11] that there is sudden death and revival of entanglement in the system for $p > 0.5$, which is clearly seen in figure 2(b). The other correlations do not exhibit sudden death, but there is an increase in quantum discord for $\Gamma t > 2$. The most striking feature is the behaviour of geometric quantum discord with a deep crater for $\Gamma t < 1$ as seen in figure 1(c), and the cusp seen in figure 2(a). The observable measure of geometric quantum discord given by (12) and shown in figure 2(a) smoothes out the cusp of geometric discord.

For $p = 1$, the state (13) becomes the product state of both the atoms being excited for which all correlations are zero. However, as is evident from figure 1, concurrence remains zero for some time, while the other correlations increase immediately after the start of evolution. This is clearly seen from figure 3. For short times the quantum discord increases, reaches a maximum and goes down to a minimum only increase again to a subsequent maximum, and eventually it goes down to zero asymptotically. This behaviour has been reported in [32]. The concurrence remains zero up to $\Gamma t \sim 4$ and then abruptly becomes nonzero, an effect that has been referred to as sudden birth of entanglement [12]. Both revival of entanglement and the sudden birth of entanglement are due to the collective behaviour of the two atoms when the interatomic distance is smaller than the wavelength of light emitted by individual atoms [11, 12, 33]. Here we take the interatomic distance as equal to $\lambda/8$.

From figure 3(a), it is seen that the geometric quantum discord has a minimum with sharp cusps, and the observable measure of geometric quantum discord, which is a lower bound for geometric quantum discord, is a really tight bound, except for the interval around the minimum, where the bound is not so tight. For this time interval, geometric quantum discord behaves quite differently than quantum discord, and both differ from concurrence.

## 5. Conclusions

We have discussed the dynamics of various measures of quantum correlations in a two-atom system interacting with a common reservoir in a vacuum state. The evolution of the system is described by the Lehmberg–Agarwal Markovian

master equation, which takes into account the collective behaviour of the atoms. The collective spontaneous emission is a source of quantum correlations in the system. We have shown that the evolution of different measures of quantum correlations is qualitatively different, with a rather strange behaviour of the geometric discord. Some aspects of the evolution of quantum correlation in such a system have been studied in [34]. Recently, Piani [35] argued that the geometric discord is not a good measure of the quantumness of correlations.

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