Sudden death and sudden birth of entanglement*

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We compare the evolution of entanglement for a system of two two-level atoms interacting with (i) a common reservoir being in the vacuum state and (ii) a common thermal reservoir at non-zero temperature. The Markovian master equation is used to describe the evolution of entanglement which is quantified by concurrence. Phenomena of sudden death, revival, and sudden birth of entanglement are discussed. It is shown that when the atoms behave collectively and the reservoir is the vacuum, the entanglement evolution exhibits quite a rich structure. For thermal reservoir with non-zero mean number of photons this structure is gradually degraded as the temperature of the reservoir increases. For thermal reservoir the phenomenon of sudden death of entanglement becomes a standard behavior, and no asymptotic evolution can be observed. As the temperature of the reservoir increases the sudden birth of entanglement, which is a signature of collective behavior of the atoms in the vacuum, gradually disappears. The results are illustrated graphically for a number of initial states of the two-atom system.

Keywords: two-atom system, entanglement, sudden death, sudden birth.

1. Introduction

Entanglement is the key property distinguishing quantum and classical worlds. It is of fundamental importance and a necessary resource for various quantum algorithms.[1] Entanglement is a very fragile phenomenon, and it is quickly deteriorated when the quantum system interacts with the environment, so, knowing the evolution of entanglement of a quantum system in a dissipative environment is of vital importance for the quantum information processing. The time evolution of entanglement for a system of two qubits, or two two-level atoms, has been widely studied in recent years.[2–19]

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For more information and extensive literature on the subject see the review article, Ref. 20. A lot of discussion has been devoted to the problem of disentanglement of the two-qubit system in a finite time, despite the fact that all the matrix elements of the two-atom system decay only asymptotically, *i.e.*, when time goes to infinity. Yu and Eberly[12,21,22] coined the name *entanglement sudden death* to the process of finite-time disentanglement. Entanglement sudden death has recently been confirmed experimentally.[23] Another problem related to entanglement evolution that attracted attention is the evolution of the entangled qubits interacting with the non-Markovian reservoirs.[24–26] It has also been shown that squeezed reservoir leads to the steady-state entanglement[27] and revivals of entanglement.[28]

To describe quantitatively entanglement evolution, it is usually assumed that the two atoms are independent, each of them is embedded in its own reservoir, and they are prepared initially in an entangled state, pure or mixed, and the time evolution of entanglement quantified by the values of concurrence[29] or negativity[30,31] is studied.

If the two atoms are separated by a distance comparable to the wavelength of light emitted by the atom, or smaller, and if both atoms are coupled to a common reservoir, the entanglement evolution becomes much more complex and interesting, exhibiting not only entanglement sudden death or asymptotic decay, but entanglement can also be created during the evolution,[3,7,32] or one can observe *revival of the entanglement*[15] as well as *entanglement sudden birth*.[17,33] Entanglement sudden death and sudden birth has recently been discussed for the two atoms interacting with a common structured (non-Markovian) reservoir, within the Dicke model.[34] Experimental conditions for realization of the collective Dicke model have been studied.[35] It has also been shown[36,37] that for separate reservoirs at finite temperatures, entanglement always disappears at finite time, which means that there is always entanglement sudden death when the reservoir has finite temperature.

In this paper we compare the evolution of entanglement, measured by concurrence, for a system of two two-level atoms interacting with a common reservoir at zero temperature as well as at finite temperature. The evolution of the system is described by the Markovian master equation introduced by Lehmberg[38] and Agarwal,[39] taking into account the cooperative behavior of the atoms. We discuss the role of the cooperative behavior of the two atoms in appearance of new effects in entanglement evolution when the reservoir is the vacuum, and their disappearance when the reservoir becomes thermal field with non-zero temperature. The results are illustrated graphically to visualize the differences.

2. Master equation

We consider a system of two two-level atoms with ground states $|g_i\rangle$ and excited states $|e_i\rangle$ $(i = 1, 2)$ connected by dipole transition moments $\boldsymbol{\mu}_i$. The atoms are located at fixed positions r_1 and r_2 and coupled to all modes of the electromagnetic field, which we assume to be in a thermal state.

The reduced two-atom density matrix evolves in time according to the Markovian master equation given by[38–40]

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -i\sum_{i=1}^{2}\omega_i\,[S_i^z,\rho] - i\sum_{i\neq j}^{2}\Omega_{ij}\left[S_i^+S_j^-,\rho\right]\\ &\quad -\frac{1}{2}\sum_{i,j=1}^{2}\Gamma_{ij}(1+N)\left(\rho S_i^+S_j^- + S_i^+S_j^-\rho - 2S_j^-\rho S_i^+\right)\\ &\quad -\frac{1}{2}\sum_{i,j=1}^{2}\Gamma_{ij}N\left(\rho S_i^-S_j^+ + S_i^-S_j^+\rho - 2S_j^+\rho S_i^-\right), \end{aligned} \tag{1}$$

where S_i^+ (S_i^-) are the raising (lowering) operators, and S_i^z is the energy operator of the ith atom, $\Gamma_{ii} \equiv \Gamma$ are the spontaneous decay rates, and N is the mean number of photons of the reservoir. We assume that the two atoms are identical. The parameters Γ_{ij} and Ω_{ij} $(i \neq j)$ depend on the distance between the atoms and describe the collective damping and the dipole-dipole interaction defined, respectively, by

$$\Gamma_{ij} = \frac{3}{2}\Gamma\left(\frac{\sin kr_{ij}}{kr_{ij}} + \frac{\cos kr_{ij}}{(kr_{ij})^2} - \frac{\sin kr_{ij}}{(kr_{ij})^3}\right), \tag{2}$$

and

$$\Omega_{ij} = \frac{3}{4}\Gamma\left(-\frac{\cos kr_{ij}}{kr_{ij}} + \frac{\sin kr_{ij}}{(kr_{ij})^2} + \frac{\cos kr_{ij}}{(kr_{ij})^3}\right), \tag{3}$$

where $k = \omega_0/c$, and r_{ij} is the distance between the atoms. Here, we assume, with no loss of generality, that the atomic dipole moments are parallel to each other and are polarized in the direction perpendicular to the interatomic axis.

The density operator of the system can be represented in a complete set of basis states spanned by four product states (standard basis)

$$\begin{aligned} |1\rangle &= |g_1\rangle\otimes|g_2\rangle, \quad |2\rangle = |g_1\rangle\otimes|e_2\rangle,\\ |3\rangle &= |e_1\rangle\otimes|g_2\rangle, \quad |4\rangle = |e_1\rangle\otimes|e_2\rangle. \end{aligned} \tag{4}$$

For further considerations, a special form of the density matrix is especially interesting, which is referred to as X form,[41,42] given by

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix} . \tag{5}$$

Physically, the X form corresponds to a situation where all coherences between the ground state $|1\rangle$ and the single excitation states $|2\rangle$ and $|3\rangle$, and between $|2\rangle$, $|3\rangle$ and the double excitation state $|4\rangle$ are zero. The X form of the density matrix can be easily created by an appropriate initial preparation of a two-atom system. This form can be preserved during the evolution, or it can be created during the evolution.

Since the dipole-dipole interaction Ω_{12} couples the two atoms, the standard basis (4) is usually not the most convenient basis to work with, when describing the evolution of such a system. In this case, it is more convenient to include the dipole-dipole interaction into the Hamiltonian and rediagonalize it. As a result we get a different set of basis states that are particularly useful to describe the two-atom system. These are collective states of the system, or the Dicke states, defined as[38–40,43]

$$\begin{aligned} |g\rangle &= |g_1\rangle \otimes |g_2\rangle, \\ |s\rangle &= \frac{1}{\sqrt{2}} \left(|e_1\rangle \otimes |g_2\rangle + |g_1\rangle \otimes |e_2\rangle \right), \\ |a\rangle &= \frac{1}{\sqrt{2}} \left(|e_1\rangle \otimes |g_2\rangle - |g_1\rangle \otimes |e_2\rangle \right), \\ |e\rangle &= |e_1\rangle \otimes |e_2\rangle. \end{aligned} \tag{6}$$

It is interesting to note that the collective basis contains two states, $|s\rangle$ and $|a\rangle$ that are linear symmetric and antisymmetric superpositions of the product states, respectively. The most important is that the two states, $|s\rangle$ and $|a\rangle$, are in the form of maximally entangled states, or Bell states. The remaining two states, $|e\rangle$ and $|g\rangle$, are separable states.

The density matrix that has X form in the standard basis, has also X form in collective states

$$\rho = \begin{pmatrix} \rho_{gg} & 0 & 0 & \rho_{ge} \\ 0 & \rho_{ss} & \rho_{sa} & 0 \\ 0 & \rho_{as} & \rho_{aa} & 0 \\ \rho_{eg} & 0 & 0 & \rho_{ee} \end{pmatrix} , \tag{7}$$

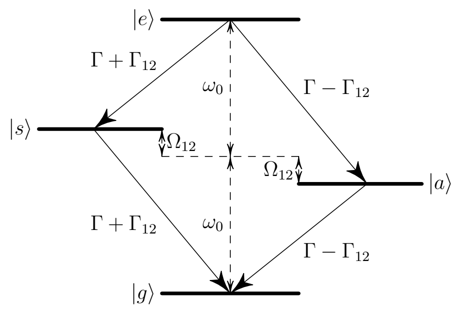

Fig. 1. Collective states and transition rates for the two-atom system

and this form is preserved during the evolution governed by the master Eq. (1). From the master Eq. (1), we get the following system of equations[40] for the matrix elements

$$\begin{aligned}
\dot{\rho}_{ee} &= -2\Gamma(1+N)\rho_{ee} + N\left[(\Gamma+\Gamma_{12})\rho_{ss} + (\Gamma-\Gamma_{12})\rho_{aa}\right] \\
\dot{\rho}_{ss} &= (\Gamma+\Gamma_{12})\left[\rho_{ee} - (1+3N)\rho_{ss} - N\rho_{aa} + N\right] \\
\dot{\rho}_{aa} &= (\Gamma-\Gamma_{12})\left[\rho_{ee} - N\rho_{ss} - (1+3N)\rho_{aa} + N\right] \\
\dot{\rho}_{as} &= -\left[\Gamma(1+2N) + 2i\Omega_{12}\right]\rho_{as} \\
\dot{\rho}_{ge} &= -\Gamma(1+2N)\rho_{ge}
\end{aligned} \tag{8}$$

Solving Eqs. (8), we find all the matrix elements required for calculating the time evolution of the system, in particular, the entanglement evolution.

3. Entanglement measure

To quantify the entanglement, we need some measure of entanglement. In case of two qubits a popular and easy to calculate measure of entanglement is concurrence $\mathcal{C}$ introduced by Wootters.[29] The value of concurrence $\mathcal{C} = 0$ means that there is no entanglement, and the value $\mathcal{C} = 1$ means a maximally entangled state. For the X form of the density matrix of the system, the concurrence can be calculated analytically, and, in collective states, it is given by[7]

$$\begin{aligned}
\mathcal{C}(t) &= \max\{0, \mathsf{C}_1(t), \mathsf{C}_2(t)\} \\
\mathsf{C}_1(t) &= 2|\rho_{ge}(t)| - \sqrt{\left[\rho_{ss}(t)+\rho_{aa}(t)\right]^2 - \left[2\mathrm{Re}\rho_{sa}(t)\right]^2} \\
\mathsf{C}_2(t) &= \sqrt{\left[\rho_{ss}(t)-\rho_{aa}(t)\right]^2 + \left[2\mathrm{Im}\rho_{sa}(t)\right]^2} - 2\sqrt{\rho_{ee}(t)\rho_{gg}(t)}
\end{aligned} \tag{9}$$

Inserting into Eqs. (9) the solutions to Eqs. (8), we find the values of $\mathsf{C}_1(t)$ or $\mathsf{C}_2(t)$, the criteria for entanglement. Whenever one of the two quantities becomes positive, there is a some degree of entanglement in the system. From Eqs. (9) it is clear that there is a competition between the inner 2×2 block of states $\{|s\rangle, |a\rangle\}$ and the outer 2×2 block of states $\{|g\rangle, |e\rangle\}$ in contributing to $\mathsf{C}_1(t)$ and $\mathsf{C}_2(t)$. There are two exclusive ways that lead to entanglement in the system. The competition introduces a sort of threshold conditions for obtaining entanglement, which lead to sometimes unexpected behavior of entanglement. One immediate and nice feature emerging from Eqs. (9), is the fact that, if the two-atom system is prepared in either symmetric or antisymmetric state, which both are maximally entangled states, the concurrence evolves in time as the population $\rho_{ss}(t)$ of the symmetric state or as the population $\rho_{aa}(t)$ of the antisymmetric state, respectively. These are exceptionally simple and striking results. In other situations the evolution of concurrence is much more complex, and it is discussed in the following sections.

4. Entanglement evolution: zero temperature reservoir

To discuss the entanglement evolution, we begin with a simple case of the reservoir being the vacuum of the electromagnetic modes, so that we put $N = 0$ in Eqs. (8). One can see from Eqs. (8) that the transition rates to and from the symmetric and antisymmetric states are modified by the collective damping Γ_{12}. Provided that $\Gamma_{12} > 0$, the transitions to and from the symmetric state occur with an enhanced rate $\Gamma + \Gamma_{12}$, whereas the transitions to and from the antisymmetric state occur with a reduced rate $\Gamma - \Gamma_{12}$. The collective damping Γ_{12}, given by Eq. (2), depends on the atomic separation. For small separations between the atoms, $\Gamma_{12} \approx \Gamma$, and then the state $|s\rangle$ becomes *superradiant* with a decay rate double that of the single atom Γ, whereas the state $|a\rangle$ becomes *subradiant*, with a decay rate of order $(kr_{12})\Gamma$ which vanishes in the limit of small distances $kr_{12} \ll 1$.

The set of coupled equations for the populations of the collective states can be easily solved, and the solution, valid for arbitrary initial conditions and $N = 0$, is given by[38,40]

$$\begin{aligned}
\rho_{ee}(t) &= \rho_{ee}(0)\,\mathrm{e}^{-2\Gamma t}, \\
\rho_{ss}(t) &= \rho_{ss}(0)\,\mathrm{e}^{-(\Gamma+\Gamma_{12})t} + \rho_{ee}(0)\frac{\Gamma+\Gamma_{12}}{\Gamma-\Gamma_{12}}\left[\mathrm{e}^{-(\Gamma+\Gamma_{12})t} - \mathrm{e}^{-2\Gamma t}\right], \quad (10)\\
\rho_{aa}(t) &= \rho_{aa}(0)\,\mathrm{e}^{-(\Gamma-\Gamma_{12})t} + \rho_{ee}(0)\frac{\Gamma-\Gamma_{12}}{\Gamma+\Gamma_{12}}\left[\mathrm{e}^{-(\Gamma-\Gamma_{12})t} - \mathrm{e}^{-2\Gamma t}\right],
\end{aligned}$$

and $\rho_{gg}(t) = 1 - \rho_{ee}(t) - \rho_{ss}(t) - \rho_{aa}(t)$.

We see from Eqs. (10) that the decay of the populations depends strongly on the initial state of the system. When the system is initially prepared in the state $|s\rangle$, the population of the initial state decays exponentially with an enhanced rate $\Gamma + \Gamma_{12}$, while the initial population of the antisymmetric state $|a\rangle$ decays with a reduced rate $\Gamma - \Gamma_{12}$. This occurs because the photons emitted from the excited atom can be absorbed by the atom in the ground state, so that the photons do not escape immediately from the system. For a general initial state that includes the state $|e\rangle$, the populations of the symmetric and the antisymmetric states do not decay with a single exponential.

Similarly, equations of motion for the coherences in Eqs. (8), are straightforward to solve, giving

$$\begin{aligned} \rho_{sa}(t) &= \rho_{sa}(0)\,\mathrm{e}^{-(\Gamma+2i\Omega_{12})t}, \\ \rho_{ge}(t) &= \rho_{ge}(0)\,\mathrm{e}^{-(\Gamma-2i\omega_0)t}. \end{aligned} \tag{11}$$

Having at hand solutions (10) and (11), it is just a matter of inserting them into expressions (9) to get analytical formulas describing the criteria for entanglement.

4.1. *Creation of entanglement*

Let us start our discussion of entanglement evolution by assuming that initially one atom is excited and the other one is in its ground state, to be more specific, let $|\Psi(0)\rangle = |e_1\rangle \otimes |g_2\rangle$. Of course, this initial state is separable, so there is no entanglement initially. In terms of collective states, it means that the only non-zero initial matrix elements are: $\rho_{ss} = \rho_{aa} = \rho_{sa} = \rho_{as} = 1/2$. It is easy to verify, that for such initial state the concurrence is equal to[44]

$$\mathcal{C}(t) \equiv \mathcal{C}_2(t) = \mathrm{e}^{-\Gamma t}\sqrt{\sinh^2(\Gamma_{12}t) + \sin^2(2\Omega_{12}t)}. \tag{12}$$

Formula (12) has two parts, an oscillatory part that oscillates with frequency $2\Omega_{12}$, and non-oscillatory part that depends on Γ_{12}. It is evident that for $t > 0$ the concurrence (12) becomes positive, *i.e.*, entanglement is created by spontaneous emission from the system. If $\Omega_{12} \gg \Gamma$ concurrence exhibits oscillations. This is illustrated in Fig. 2, where concurrence is plotted according to (12) for $r_{12} = \lambda/12$. Envelopes of this oscillatory function are given by $\rho_{aa}(t) + \rho_{ss}(t)$ and $\rho_{aa}(t) - \rho_{ss}(t)$, which are plotted for reference. Since $\rho_{ss}(t)$ decays much faster than $\rho_{aa}(t)$, eventually, only the antisymmetric state survives, and for long times, the concurrence is

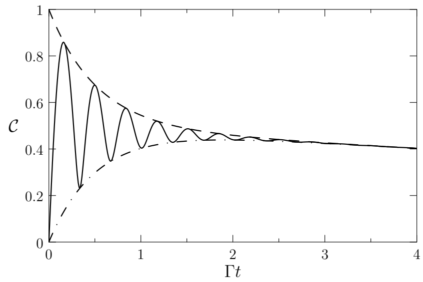

Fig. 2. Concurrence $\mathcal{C}(t)$ for initially one atom excited, according to Eq. (12), (solid). Envelopes are: $\rho_{aa}(t) + \rho_{ss}(t)$ (dashed), $\rho_{aa}(t) - \rho_{ss}(t)$ (dashed-dotted). Parameters are: $r_{12} = \lambda/12$ and $N = 0$.

equal to the population $\rho_{aa}(t)$. The antisymmetric state plays a crucial role in creating entanglement via spontaneous emission. If the antisymmetric state is eliminated from the evolution, as it is the case in the Dicke model, or small sample model, entanglement cannot be created by spontaneous emission.

4.2. *Sudden death of entanglement*

One of the characteristic features of entanglement in a bipartite system is an abrupt disappearance of initial entanglement at finite time, the effect for which Yu and Eberly[12,21,22] introduced the name *sudden death of entanglement*. They considered a two qubit system prepared initially in a state which is diagonal in the collective states basis, and the initial density matrix has the form

$$\rho(0) = \frac{1}{3}\begin{pmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}, \tag{13}$$

where $0 \leq \alpha \leq 1$. With this initial state the concurrence is described by a simple formula

$$\mathcal{C}(t) = \mathsf{C}_2(t) = |\rho_{ss}(t) - \rho_{aa}(t)| - 2\sqrt{\rho_{gg}(t)\rho_{ee}(t)}, \tag{14}$$

where the solutions (10) are to be inserted. The results are illustrated in Fig. 3. Figure (a) presents the evolution of concurrence for the interatomic distance $r_{12} = 10\lambda$, which reproduces exactly the famous picture of Yu

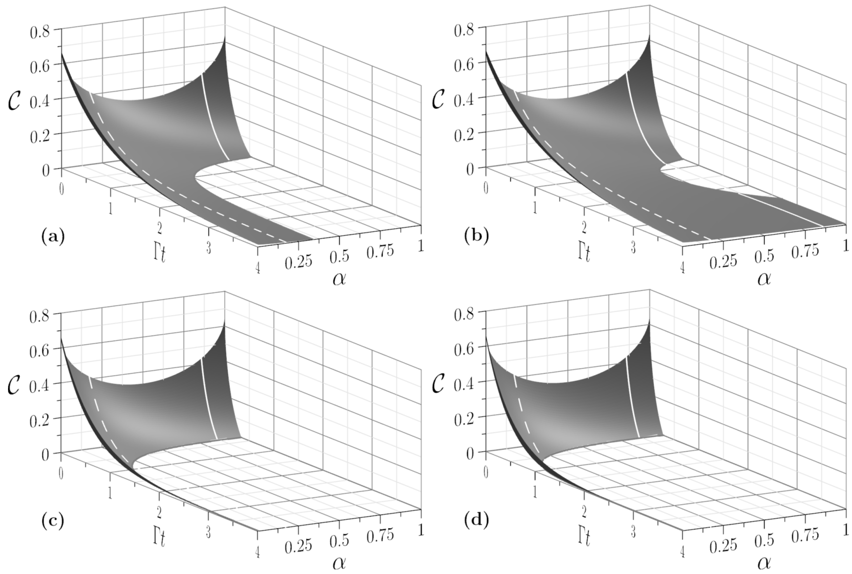

Fig. 3. Concurrence evolution for the vacuum reservoir ($N = 0$) and the initial state (13): (a) $r_{12} = 10\lambda$, (b) $r_{12} = \lambda/2$, (c) $r_{12} = \lambda/4$, (d) $r_{12} = \lambda/12$.

and Eberly[12,21,22] for atoms coupled to independent reservoirs. The interatomic distance $r_{12} = 10\lambda$ appears to be large enough to consider the common reservoir as two independent reservoirs, so both atoms evolve independently. To emphasize the two different types of evolution,*i.e.*, sudden death of entanglement and asymptotic decay, we mark in the figure two trajectories representing the two evolutions: the solid line which represents the sudden death of entanglement and the dashed line which shows the asymptotic decay. It is seen that for $\alpha < 1/3$ evolution is asymptotic and for $\alpha > 1/3$ sudden death of entanglement is observed.

The situation changes radically when the interatomic distance becomes shorter and atoms behave collectively. It is shown in Fig. 3 (c)–(d) that for distances shorter than $\lambda/4$ sudden death appears practically for the whole range of α. Only for $\alpha = 0$ we have asymptotic behavior because in this case the concurrence follows the exponential decay of $\rho_{ss}(t)$. In Fig. 3 (b) an interesting effect of sudden death followed by a revival of entanglement is seen. So, the collective behavior of the atoms makes the entanglement evolution dependent on the interatomic distance and it becomes quite different from that for independent atoms.

4.3. *Sudden death and revival of entanglement*

Another interesting example of entanglement evolution takes place when the two atoms are prepared initially in the state

$$|\Psi(0)\rangle = \sqrt{p}|e\rangle + \sqrt{1-p}|g\rangle \tag{15}$$

This state is the superposition of the states: one with both atoms excited

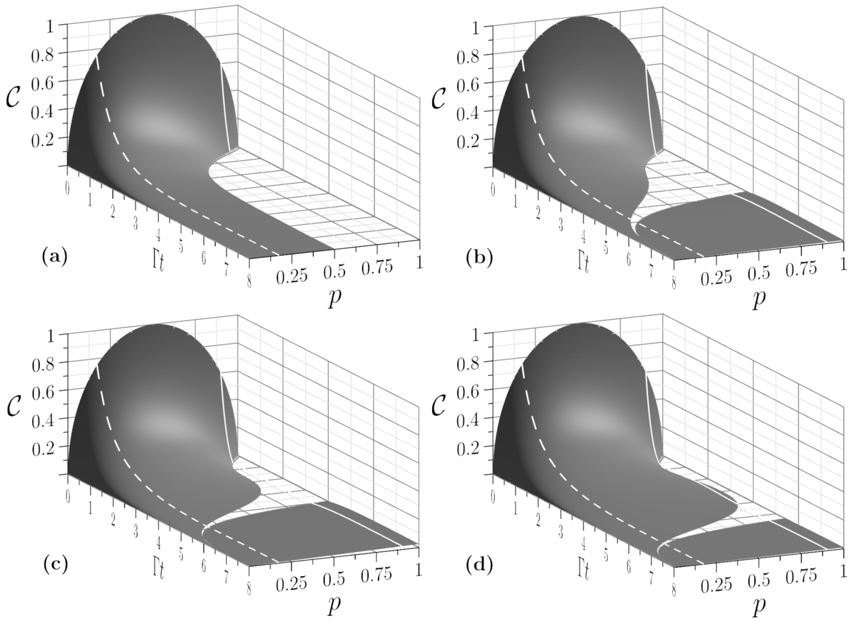

Fig. 4. Concurrence evolution for the vacuum reservoir ($N = 0$) and the initial state (15): (a) $r_{12} = 10\lambda$, (b) $r_{12} = \lambda/4$, (c) $r_{12} = \lambda/8$, (d) $r_{12} = \lambda/20$.

and the other with both atoms in their ground states. Again, for independent atoms clear splitting into two regions is seen, we find the sudden death of entanglement for $p > 0.5$, and the asymptotic evolution for $p < 0.5$, as shown in Fig. 4 (a). However, for interatomic distances shorter than the wavelength of emitted light, the structure of entanglement evolution exhibits very interesting features, as shown in Fig. 4 (b)–(d), which include a broad range of entanglement sudden death accompanied by a subsequent revival. Especially interesting is the evolution of entanglement for values of $p \sim 0.9$, which is shown in Fig. 4 (d), and in more details in Fig. 5. As it is clear, the entanglement experiences sudden death and revival not just once, but twice.[15] This happens for the interatomic distance $r_{12} = \lambda/20$,

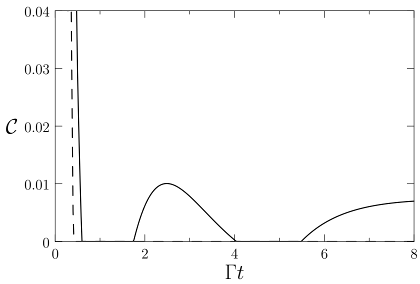

Fig. 5. Concurrence evolution for the initial state (15) and interatomic distance $r_{12} = \lambda/20$; dashed line shows the evolution for independent atoms.

which is quite short, but it illustrates how important for the entanglement evolution can be the collective behavior of the atoms. For reference, the dashed curve in Fig. 5 shows the evolution for independent atoms.

At this point it can be interesting to compare the results obtained above, which were obtained for the model taking into account the collective damping (2) and the dipole-dipole interaction (3), both depending on the interatomic separation, to the Dicke model, or the small sample model, which is based on the assumption that the two atoms are very close to each other, so that $\Gamma_{12} = \Gamma$, and the dipole-dipole interaction can be ignored ($\Omega_{12} = 0$). In this case the antisymmetric state (singlet state) does not evolve in time. The evolution is restricted to the three triplet states ($\{|g\rangle, |s\rangle, |e\rangle\}$, and it is governed by the equations

$$\begin{aligned}\rho_{ee}(t) &= \rho_{ee}(0)\mathrm{e}^{-2\Gamma t}\\ \rho_{ss}(t) &= \rho_{ss}(0)\mathrm{e}^{-2\Gamma t} + 2\Gamma t\, \rho_{ee}(0)\mathrm{e}^{-2\Gamma t}\\ \rho_{ge}(t) &= \rho_{ge}(0)\mathrm{e}^{-(\Gamma - 2i\omega_0)t}.\end{aligned} \tag{16}$$

It is worth to notice the non-exponential evolution of $\rho_{ss}(t)$, if $\rho_{ee}(0) \neq 0$. The concurrence evolution for the Dicke model and the initial state (15) is plotted in Fig. 6. The difference between the Dicke model and the extended model shown in Fig. 4 is clearly visible. Entanglement evolution for the Dicke model and a common structured reservoir has been discussed in Ref. 34. They found revivals of entanglement for non-Markovian reservoir with Lorentzian spectrum. In the limit of very broad spectrum of the reservoir the result should reproduce that for Markovian reservoir, our Fig. 6

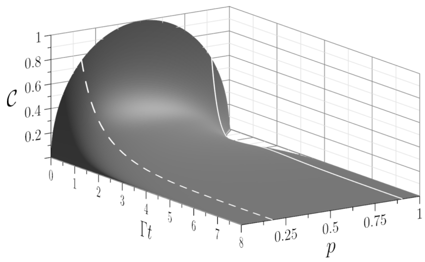

Fig. 6. Concurrence evolution for the vacuum reservoir ($N = 0$) and the initial state (15) for the Dicke model

really agrees with the corresponding figure in Ref. 34. We want to emphasize here that the Dicke model, described by Eqs. (16) and illustrated in Fig. 6, essentially differs from the extended model used in this paper.

4.4. *Sudden birth of entanglement*

As we have seen already, the collective spontaneous emission can create entanglement in the two-atom system in a smooth way, when the atoms initially start from a specific product state. If there is some entanglement in the system initially, it can disappear abruptly in a finite time, or it can decay asymptotically. Here, we want to discuss another interesting feature of entanglement evolution. Let us consider another initial product state of

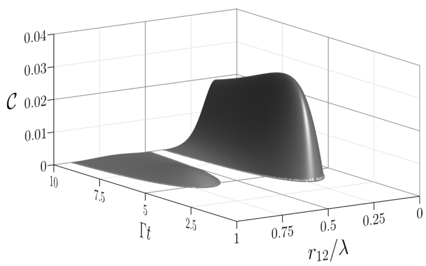

Fig. 7. Concurrence as a function of Γt and r_{12}, for vacuum reservoir ($N = 0$) and initially both atoms excited

the two-atom system, such that initially both atoms are excited, so the only non-zero element of the atomic density matrix is $\rho_{ee}(0) = 1$. In this case, according to solutions (10), we find that the density matrix has a diagonal form, and when we check the criteria for entanglement (9), we easily discover that the only contribution to entanglement, if any, can come from $\mathsf{C}_2(t)$, which is given by Eq. (14), together with the solutions (10) and appropriate initial conditions. It is clear from Eq. (14) that entanglement can result solely from unequal populations of the symmetric and antisymmetric states. When the system starts from the state $|e\rangle$, there are two channels of spontaneous emission: the fast one through the state $|s\rangle$, and the slow one through the state $|a\rangle$, as indicated in Fig. 1. Due to the difference in the emission rates, the difference between the populations builds up, but it is counteracted by the accumulation of the population $\rho_{gg}(t)$ in the ground state $|g\rangle$. In effect, we get another "sudden" phenomenon in the entanglement evolution, which we refer to as *sudden birth of entanglement*.[17] To visualize the behavior of the concurrence for this case, we plot in Fig. 7 the concurrence $\mathcal{C}$ as a function of the dimensionless evolution time Γt and the interatomic distance r_{12}. We see that there is no entanglement at early times of evolution, but suddenly after a finite time an entanglement emerges. However, as it is seen, it happens only for sufficiently small interatomic distances r_{12}, for which atoms behave collectively. Once again, the collective behavior of the atoms appears to be crucial for observing this phenomenon.

Among the states having the X form of the density matrix, which are of great importance in quantum information theory, are Werner states, with the density matrix of the form

$$\rho(0) = p|s\rangle\langle s| + (1-p)\frac{\mathbb{I}}{4}, \tag{17}$$

where $\mathbb{I}$ is the 4×4 identity matrix. Werner states represent a mixture of the maximally entangled state and the isotropic state. In our case the maximally entangled state is the symmetric state $|s\rangle$, which contributes to the mixture with the probability p. Werner states, depending on the value of p, in a sense, interpolate between entangled states and separable states. For $p = 1$, we have a maximally entangled state with concurrence $\mathcal{C} = 1$, while for $p < 1/3$ the state is separable. For $p > 1/3$ the concurrence has the value $\mathcal{C}(0) = (3p-1)/2$. Such states are excellent examples for illustrating all the features of entanglement evolution for the system of two atoms coupled to a common reservoir. It is convincingly shown in Fig. 8, where the concurrence evolution is plotted as a function of Γt and the probability p, for various values of the interatomic distance r_{12}. For independent atoms, Fig. 8 (a), as before, two trajectories are marked, one showing the sudden death of entanglement (solid line) and the other showing the asymptotic evolution (dashed line). The same trajectories are marked in the remaining figures,

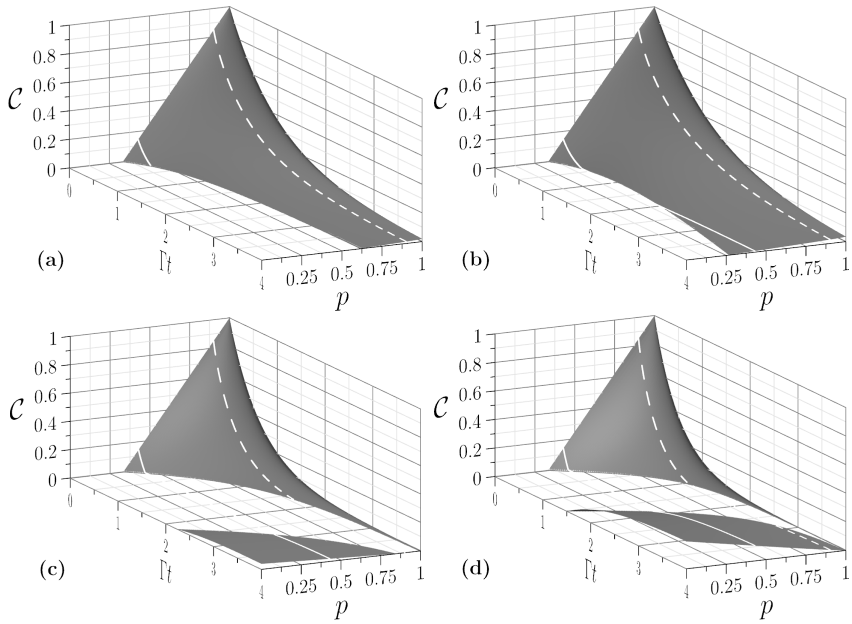

Fig. 8. Concurrence evolution for vacuum reservoir ($N = 0$) and initial state (17): (a) $r_{12} = 10\lambda$, (b) $r_{12} = \lambda/2$, (c) $r_{12} = \lambda/4$, (d) $r_{12} = \lambda/12$.

but, as it is seen, they no longer represent sudden death and asymptotic evolutions. As the atoms behave collectively, the concurrence exhibits much more interesting behavior. In Fig. 8 (b) we see that the trajectory that experienced sudden death, after some time exhibits revival of entanglement. The revival is also seen in figures (c) and (d). Moreover, for independent atoms there is no entanglement for $p < 1/3$, and from figures (c) and (d) we see the presence of entanglement at later times, which means the sudden birth of entanglement. However, for the trajectory representing initially asymptotic evolution, sudden death and subsequent revival of entanglement appears. Generally, we can say that the sudden death is a rather common phenomenon when the two atoms behave collectively, but the sudden death can be followed by a revival, after which asymptotic evolution can take place.

5. Entanglement evolution: thermal reservoir

Now, we want to discuss the situation when the reservoir is not the vacuum, or zero-temperature reservoir, but it is a thermal reservoir with the mean number of photons N different from zero. To find the evolution of concurrence it is now necessary to solve the full version, with $N \neq 0$, of Eqs. (8). Although analytical solutions are possible, they are rather complicated and

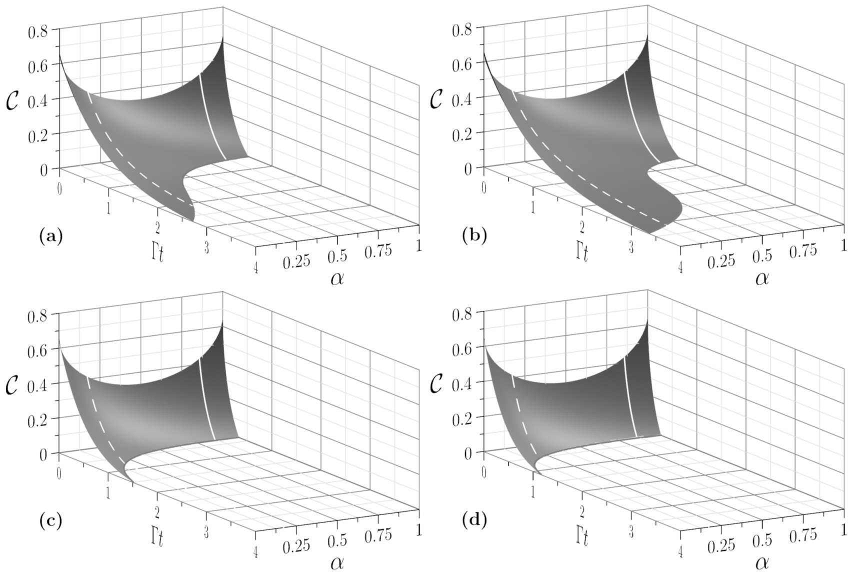

Fig. 9. Same as Fig. 3 but for thermal reservoir with $N = 0.01$.

we do not give them here. In further calculations of the concurrence we will rely on numerical solutions.

However, before we calculate the time evolution of concurrence, we find the steady state solutions to Eqs. (8), which are given by

$$\begin{aligned} \rho_{ee}(\infty) &= \frac{N^2}{(2N+1)^2} \\ \rho_{ss}(\infty) = \rho_{aa}(\infty) &= \frac{N(N+1)}{(2N+1)^2} \\ \rho_{gg}(\infty) &= \frac{(N+1)^2}{(2N+1)^2}, \end{aligned} \tag{18}$$

and the steady state values of coherences $\rho_{as}(\infty)$ and $\rho_{eg}(\infty)$ are zero. From the solutions (18), it is immediately seen that both $\mathsf{C}_1(t)$ and $\mathsf{C}_2(t)$, given by Eqs. (9), take negative values as $t \to \infty$. This means that there is no entanglement in the system for $t \to \infty$. So, no matter how large the degree of entanglement could be in the system at earlier times, there must be a finite time t_d at which entanglement disappears. Of course, the death time t_d depends on the value of the mean number of photons N of the reservoir, but for any $N > 0$, no matter how small it could be, there is entanglement sudden death for thermal reservoir.

The steady state solutions (18) do not depend on the collective parameters Γ_{12} and Ω_{12}, which means that independently of the interatomic distance, there is always entanglement sudden death if the mean number of photons of the reservoir is different from zero. This confirms the results found earlier for atoms coupled to separate reservoirs.[36,37] For the long-time behavior of the system in a thermal reservoir, it is not important whether the atoms behave collectively or not.

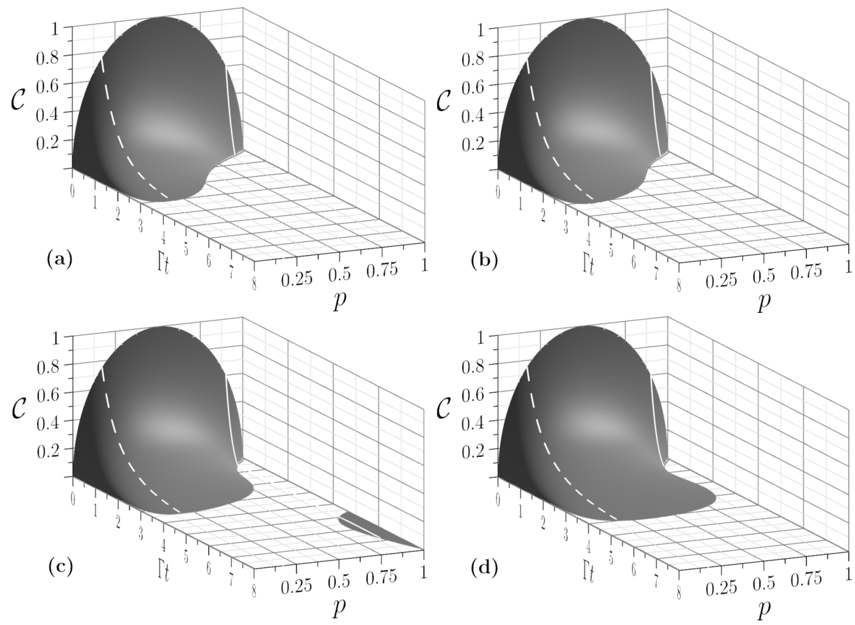

Fig. 10. Same as Fig. 4 but for thermal reservoir with $N = 0.01$.

To illustrate entanglement evolution in a thermal reservoir, we calculate the concurrence for the initial states that we used earlier to discuss the evolution of the two-atom system in the vacuum. In Fig. 9 we plot the concurrence for initial state (13), discussed by Yu and Eberly[12,21,22] for atoms embedded in a common thermal reservoir with the mean number of

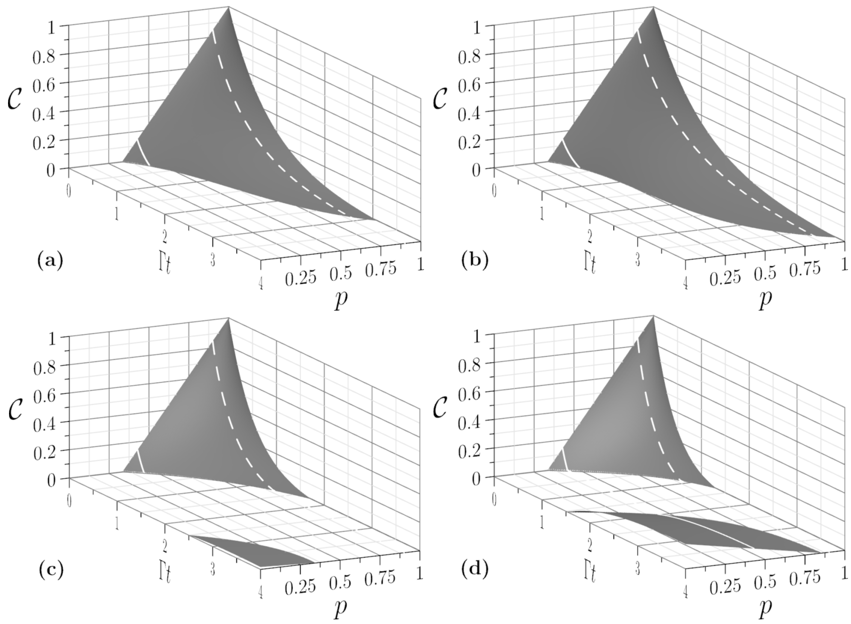

Fig. 11. Same as Fig. 8 but for thermal reservoir with $N = 0.01$.

photons $N = 0.01$. The remaining parameters are the same as in Fig. 3. It is evident from Fig. 9 that there is sudden death of entanglement for the whole range of α values. The death time t_d depends on the interatomic distance and the value of α in a rather complicated way, but sudden death of entanglement is clearly evident, and there is no asymptotic evolution. For small interatomic distances, as a rule, the death time becomes shorter for shorter distances.

In Fig. 10 the concurrence evolution is shown for the initial state (15), for the same values of interatomic distances as in Fig. 4, but for the thermal reservoir with the mean number of photons $N = 0.01$. We see that the evolution has drastically changed with respect to the vacuum reservoir. Again, as in the previous case, the sudden death of entanglement is clearly visible. Moreover, the revivals of entanglement that are so evident in Fig. 4, disappeared almost completely, the remnants are still visible in figure (c), for $r_{12} = \lambda/8$, but they will also disappear if the number of photons will increase. It is also seen that, for the initial state (15), in contrast to the initial state (13) illustrated in Fig. 9, the shorter interatomic distance does not mean the shorter death time of entanglement. The dependence on the interatomic distance is in this case reversed. However, in any case, the death time becomes shorter as the mean number of photons N increases, but this is not illustrated in the figures presented in the paper.

Similarly, in Fig. 11, we plot the concurrence behavior for the initial Werner state (17) and thermal reservoir with the mean number of photons $N = 0.01$. Direct comparison of Fig. 8 and Fig. 11 shows that there is sudden death of entanglement for the cases that previously exhibited asymptotic evolution. In cases, for which sudden death appeared due to the collective behavior of the atoms, in thermal reservoir the death time is shortened. Moreover, for the interatomic distances for which collective evolution of the atoms leads to the revival or sudden birth of entanglement in the vacuum reservoir, for the thermal reservoir both effects are gradually diminished.[45] Eventually, for a reservoir with a higher mean number of photons, they will be completely erased, and the only effect that remains in such a reservoir is the sudden death of entanglement, with the death time being shorter and shorter as the mean number of photons increases. All the examples of entanglement evolution discussed here convincingly show that the collective behavior of the atoms, discussed in the paper, leads to a quite interesting evolution of entanglement, with new features, when the two atoms are embedded in a common reservoir being in the vacuum state. When the reservoir has non-zero temperature, the structure of entanglement evolution simplifies, and eventually becomes quite simple, showing always the sudden death of entanglement. No asymptotic evolution is possible when the temperature is non-zero, no matter whether the atoms are coupled to separate reservoirs or to a common reservoir. It shows that the zero-temperature reservoir plays a special role in entanglement evolution.

6. Conclusion

We have discussed the dynamics of entanglement in a two-atom system interacting with a common reservoir at zero temperature (vacuum) as well as at finite temperature (thermal reservoir). The evolution of the system is described by the Lehmberg-Agarwal Markovian master equation, which takes into account collective behavior of the atoms. The collective spontaneous emission is a source of entanglement for initially separable states. The entanglement can be created continuously from the beginning of the evolution, or it can be created abruptly after a finite time of the evolution that took place without entanglement. The latter is referred to as entanglement sudden birth. The entanglement which is already present in the system can disappear in a finite time, which is the effect of entanglement sudden death, but it can also revive after some time. For some initial states and vacuum reservoir, entanglement can decay asymptotically (for $t \to \infty$). Generally, for atoms behaving collectively, the structure of the entanglement evolution appears to be quite rich. However, this rich structure of the evolution is degraded for the non-zero temperature reservoir, for which the entanglement sudden death becomes the standard feature of the evolution, and only in the limit $N \to 0$ the asymptotic decay of entanglement is possible. This means that, in real physical situations of finite temperature reservoirs, there is always entanglement sudden death. On the other hand, entanglement sudden birth created by correlated atoms appears only for reservoirs at sufficiently low temperatures, and it disappears at higher temperatures. We have made a comparison of the concurrence evolution for a number of initial states of the two-atom system to show the variety of effects that can be observed in the evolution.

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