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Sudden birth and death of entanglement of two atoms in a finite temperature reservoir

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Abstract
We discuss the evolution of entanglement for a system of two two-level atoms interacting with a common reservoir at finite temperature. The Markovian master equation is used to describe the evolution of entanglement measured by concurrence. The phenomena of sudden birth and sudden death of entanglement are discussed. It is shown that entanglement sudden death is a standard feature when the reservoir has finite temperature. Entanglement sudden birth, which is the result of the collective behaviour of atoms, appears only for a sufficiently small mean number of photons of the reservoir (sufficiently low temperatures), and it gradually diminishes as the temperature increases. The results are illustrated for the system prepared in the Werner state.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Entanglement is the crucial feature distinguishing quantum and classical worlds. It is a necessary resource for various quantum algorithms. Since entanglement is a very fragile quantity and quickly deteriorates when the quantum system interacts with the environment, to know the evolution of entanglement of a quantum system in a dissipative environment is of vital importance for quantum information processing. The time evolution of entanglement for a system of two qubits or two two-level atoms can be qualitatively described for various physical situations, and it has been studied extensively in recent years [1–8]. A lot of discussion has been devoted to the problem of disentanglement of the two-qubit system in a finite time, despite the fact that all the matrix elements of the two-atom system decay only asymptotically. Yu and Eberly [5] coined the name 'entanglement sudden death' (ESD) for the process of finite-time disentanglement. ESD has recently been confirmed experimentally [6]. Another problem related to entanglement evolution that has attracted attention is the evolution of entangled qubits interacting with non-Markovian reservoirs [9–11]. It has also been shown that a squeezed reservoir leads to steady-state entanglement [12] and revivals of entanglement [13].

To quantitatively describe entanglement evolution, it is usually assumed that the two atoms are independent, each of them is embedded in its own reservoir, and they are prepared initially in an entangled state, pure or mixed, and the time evolution of entanglement quantified by the values of concurrence [14] or negativity [15, 16] is studied.

If the two atoms are separated by a distance of the order of the wavelength of light emitted by the atom, or smaller, and if both atoms are interacting with a common reservoir, the entanglement evolution becomes richer, exhibiting not only ESD or asymptotic decay, but entanglement can also be created during the evolution [2, 3], or one can observe revival of the entanglement [7] as well as 'entanglement sudden birth' (ESB) [8]. ESD and ESB have recently been discussed for the two atoms interacting with a common structured reservoir [17]. Experimental conditions for the realization of the collective Dicke model have been studied [18]. It has also been shown [19, 20] that for separate reservoirs at finite temperatures, entanglement always disappears at finite time, which means that there is always ESD when the reservoir has finite temperature.

In this paper, we discuss the evolution of entanglement, measured by concurrence, for a system of two two-level atoms interacting with a common reservoir at finite temperature. The evolution of the system is described by the Markovian master equation introduced by Lehmberg [21] and Agarwal [22], taking into account the cooperative behaviour of the atoms. It is shown that the temperature of the reservoir has an important influence on the evolution of entanglement in such a system.

2. Master equation

We consider a system of two two-level atoms with ground states $|g_i\rangle$ and excited states $|e_i\rangle$ $(i = 1, 2)$ connected by dipole transition moments $\boldsymbol{\mu}_i$. The atoms are located at fixed positions r_1 and r_2 and coupled to all modes of the electromagnetic field, which we assume to be in a thermal state.

The reduced two-atom density matrix evolves in time according to the Markovian master equation given by [21–23]

$$\begin{aligned}\frac{\partial\rho}{\partial t} = &-\mathrm{i}\sum_{i=1}^{2}\omega_i[S_i^z,\rho] - \mathrm{i}\sum_{i\neq j}^{2}\Omega_{ij}[S_i^+S_j^-,\rho]\\ &-\frac{1}{2}\sum_{i,j=1}^{2}\Gamma_{ij}(1+N)(\rho S_i^+S_j^- + S_i^+S_j^-\rho - 2S_j^-\rho S_i^+)\\ &-\frac{1}{2}\sum_{i,j=1}^{2}\Gamma_{ij}N\left(\rho S_i^-S_j^+ + S_i^-S_j^+\rho - 2S_j^+\rho S_i^-\right),\end{aligned} \qquad (1)$$

where S_i^+ (S_i^-) are the raising (lowering) operators, and S_i^z is the energy operator of the ith atom, $\Gamma_{ii} \equiv \Gamma$ are the spontaneous decay rates and N is the mean number of photons of the reservoir. We assume that the two atoms are identical. The parameters Γ_{ij} and $\Omega_{ij} (i \neq j)$ depend on the distance between the atoms and describe the collective damping and the dipole–dipole interaction defined, respectively, by

$$\Gamma_{ij} = \frac{3}{2}\Gamma\left(\frac{\sin kr_{ij}}{kr_{ij}} + \frac{\cos kr_{ij}}{(kr_{ij})^2} - \frac{\sin kr_{ij}}{(kr_{ij})^3}\right) \qquad (2)$$

and

$$\Omega_{ij} = \frac{3}{4}\Gamma\left(-\frac{\cos kr_{ij}}{kr_{ij}} + \frac{\sin kr_{ij}}{(kr_{ij})^2} + \frac{\cos kr_{ij}}{(kr_{ij})^3}\right), \qquad (3)$$

where $k = \omega_0/c$ and r_{ij} is the distance between the atoms. Here, we assume, with no loss of generality, that the atomic dipole moments are parallel to each other and are polarized in the direction perpendicular to the interatomic axis.

To describe the evolution of the two-qubit system, the standard basis of atomic product states can be used. We, however, prefer to use, instead of the standard basis, a basis of the collective states: $|g\rangle = |g_1\rangle \otimes |g_2\rangle$, $|e\rangle = |e_1\rangle \otimes |e_2\rangle$, $|s\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle \otimes |g_2\rangle + |g_1\rangle \otimes |e_2\rangle)$ and $|a\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle \otimes |g_2\rangle - |g_1\rangle \otimes |e_2\rangle)$. The states $|s\rangle$ and $|a\rangle$ are the symmetric and antisymmetric states of the two-atom system. They are maximally entangled states or Bell states of the two-atom system. Such a basis is convenient for finding the solutions to the master equation (1). Assuming that initially the system density matrix has the so-called X form, which is preserved during the evolution according to the master equation (1), we obtain the following system of equations [23]:

$$\begin{aligned}\dot{\rho}_{ee} &= -2\Gamma(1+N)\rho_{ee} + N\left[(\Gamma+\Gamma_{12})\rho_{ss} + (\Gamma-\Gamma_{12})\rho_{aa}\right],\\ \dot{\rho}_{ss} &= (\Gamma+\Gamma_{12})\left[\rho_{ee} - (1+3N)\rho_{ss} - N\rho_{aa} + N\right],\\ \dot{\rho}_{aa} &= (\Gamma-\Gamma_{12})\left[\rho_{ee} - N\rho_{ss} - (1+3N)\rho_{aa} + N\right],\\ \dot{\rho}_{as} &= -\left[\Gamma(1+2N) + 2\mathrm{i}\Omega_{12}\right]\rho_{as},\\ \dot{\rho}_{ge} &= -\Gamma(1+2N)\rho_{ge}.\end{aligned} \qquad (4)$$

Solving equations (4) we find all the matrix elements required for calculating the time evolution of the system entanglement. Although analytical solutions are possible, they are rather complicated, and we use numerical solutions in the following.

3. Entanglement evolution

To quantify the entanglement, we use concurrence introduced by Wootters [14]. In the case we consider, concurrence can be calculated analytically, and it has the form [3]

$$\begin{aligned}\mathcal{C}(t) &= \max\{0, \mathsf{C}_1(t), \mathsf{C}_2(t)\},\\ \mathsf{C}_1(t) &= 2|\rho_{ge}(t)| - \sqrt{[\rho_{ss}(t)+\rho_{aa}(t)]^2 - [2\Re\rho_{sa}(t)]^2},\\ \mathsf{C}_2(t) &= \sqrt{[\rho_{ss}(t)-\rho_{aa}(t)]^2 + [2\Im\rho_{sa}(t)]^2} - 2\sqrt{\rho_{ee}(t)\rho_{gg}(t)}.\end{aligned} \qquad (5)$$

Inserting into (5) the corresponding solutions to equations (4), we find the values of $\mathsf{C}_1(t)$ or $\mathsf{C}_2(t)$, and whenever one of the two quantities becomes positive, there is some degree of entanglement in the system.

First of all, before we calculate the time evolution, we find the steady state solutions to equations (4), which are given by

$$\begin{aligned}\rho_{ee}(\infty) &= \frac{N^2}{(2N+1)^2},\\ \rho_{ss}(\infty) = \rho_{aa}(\infty) &= \frac{N(N+1)}{(2N+1)^2},\\ \rho_{gg}(\infty) &= \frac{(N+1)^2}{(2N+1)^2},\end{aligned} \qquad (6)$$

and the steady state values of coherences $\rho_{as}(\infty)$ and $\rho_{eg}(\infty)$ are zero. From solutions (6), it is immediately seen that $\mathcal{C}_1(t)$ must become negative at some finite time t_d. Moreover, since $\sqrt{\rho_{ee}(\infty)\rho_{gg}(\infty)} = \rho_{ss}(\infty) = \rho_{aa}(\infty)$, it means that $\mathcal{C}_2(t)$ must also become negative at some finite time t_d. The steady state solutions (6) do not depend on the collective parameters Γ_{12} and Ω_{12}, which means that independently of the interatomic distance, there is always ESD if the mean number of photons of the reservoir is different from zero. This confirms the results found earlier for atoms in separate reservoirs [19, 20]. For the long-time behaviour of the system in a thermal reservoir, it is not important whether the atoms behave collectively or not.

The collective behaviour of the two atoms, when the interatomic distance is less than the wavelength of the

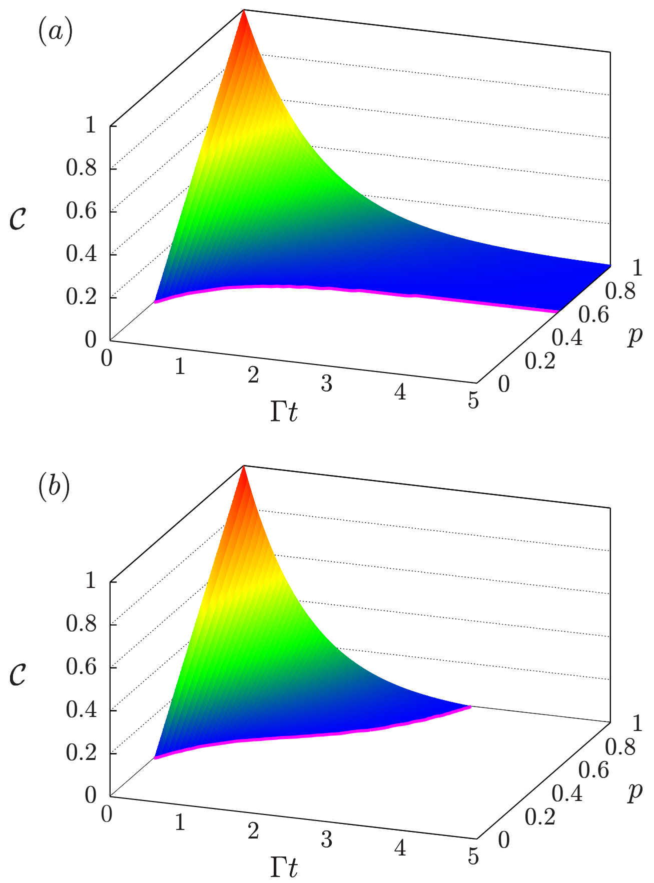

Figure 1. Evolution of concurrence $\mathcal{C}$ for the Werner state (7) and interatomic distance $r_{12} = 10\lambda$ (independent atoms), for (*a*) $N = 0$ and (*b*) $N = 0.01$.

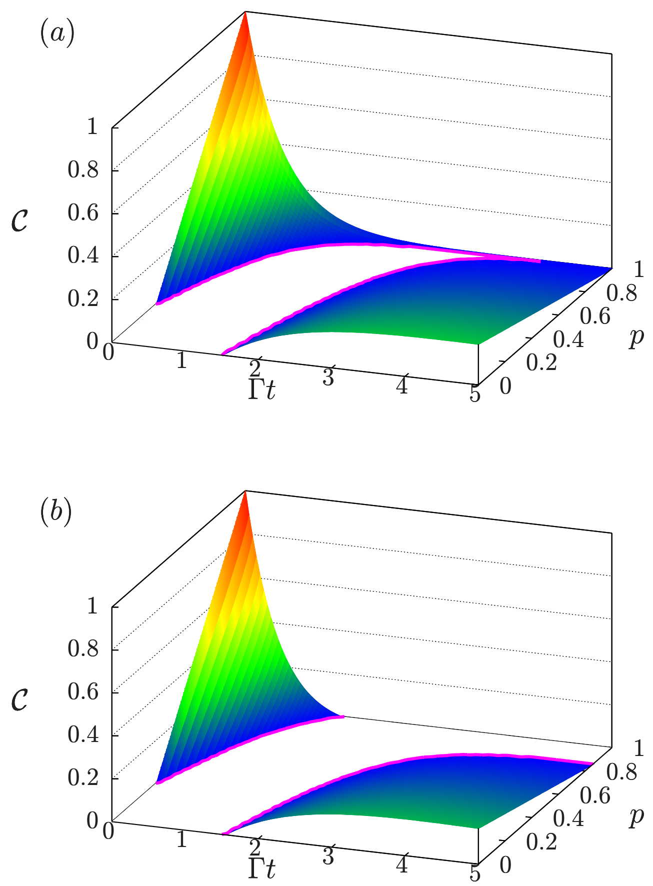

Figure 2. Evolution of concurrence $\mathcal{C}$ for the Werner state (7) and interatomic distance $r_{12} = \lambda/12$ (correlated atoms), for (*a*) $N = 0$ and (*b*) $N = 0.01$.

light emitted by the atom, leads to the sudden birth of entanglement [8]. Here, we illustrate the evolution of entanglement in a system of two two-level atoms interacting with a common reservoir, which is governed by the master equation (1). We assume that the initial state is the Werner state of the form

$$\rho(0) = p|s\rangle\langle s| + (1-p)\frac{\mathbb{I}}{4}, \qquad (7)$$

where p is the population of the symmetric state $|s\rangle$ and $\mathbb{I}$ is the 4×4 unit matrix. In figure 1, we plot concurrence as a function of t for various values of p assuming that the interatomic distance is large ($r_{12} = 10\lambda$), which means that both atoms can be treated as being independent, and this is equivalent to the situation when each atom interacts with its own reservoir. Figure 1(*a*) shows the concurrence evolution for the mean number of photons of the reservoir equal to zero (vacuum). It is seen that for p smaller than $1/3$ there is no entanglement initially, and it never appears during the evolution. For p close to unity there is asymptotic decay (no sudden death), and for some intermediate values of p, ESD is evident. However, when the mean number of photons is non-zero, as in figure 1(*b*), where $N = 0.01$, ESD is observed. The death time is different for different values of p, but there is no asymptotic decay of entanglement.

In figure 2, we show the situation when the interatomic distance is small compared with the resonant wavelength ($r_{12} = \lambda/12$). In this case, ESD appears in almost the whole range of p values, except for the limiting value of $p = 1$, but after the death of entanglement we can observe revival of entanglement. What is even more interesting, is the appearance of entanglement for values of p smaller than $1/3$, which is ESB. Surprisingly, the birth of entanglement is most effective for the initially isotropic state, for which all levels of the system are equally populated ($\rho(0) = \mathbb{I}/4$). Figure 2(*a*) illustrates the situation for $N = 0$, and figure 2(*b*) for $N = 0.01$. From figure 2(*b*) it is seen that the death time becomes shorter as N increases and the region of ESB is gradually shrinking. For longer times, after the birth, there is a subsequent death time after which the entanglement disappears again.

To illustrate the situation more convincingly, we plot in figure 3 the dependence of the entanglement death time t_d on the mean number of photons N, for $p = 1$ ($\rho(0) = |s\rangle\langle s|$) and $r_{12} = \lambda/12$. This initial state is a pure, maximally entangled state (Bell state), and it is seen that for any non-zero value of N there is ESD, and the death time decreases as the mean number of photons increases. Only for $N \to 0$ the death time $t_d \to \infty$, and the asymptotic decay can be observed.

The other limiting case is illustrated in figure 4, where the initial state is the isotropic state ($\rho(0) = \mathbb{I}/4$). There is no entanglement initially, of course, but after some finite time t_b (birth time) entanglement is created in the system, it lasts for a finite period of time, and it dies out at the death time t_d. It is interesting to note that the timescale over which the entanglement lasts in this case is much longer

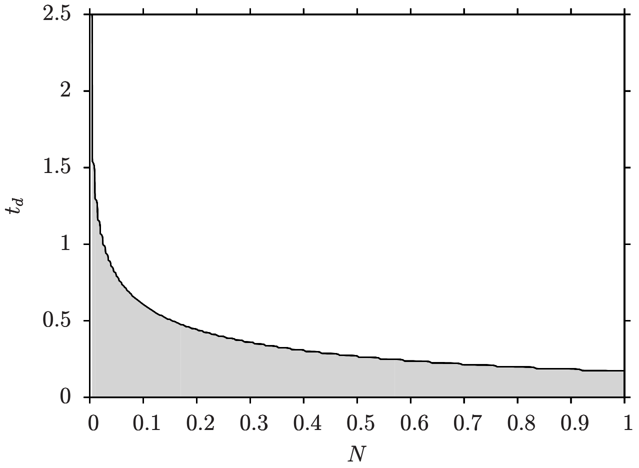

Figure 3. Dependence of the entanglement death time t_d (in units of Γ^{-1}) on the mean number of photons N of the reservoir for the initial state $\rho(0) = |s\rangle\langle s|$ and the interatomic distance $r_{12} = \lambda/12$.

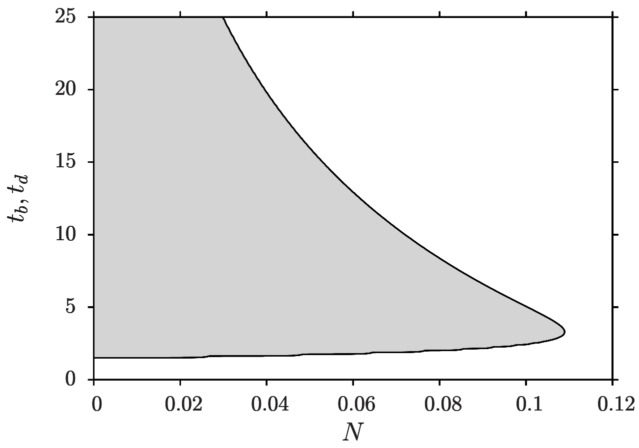

Figure 4. Dependence of the entanglement birth time t_b and death time t_d (in units of Γ^{-1}) on the mean number of photons N of the reservoir, for the initial state $\rho(0) = \mathbb{I}/4$ and the interatomic distance $r_{12} = \lambda/12$.

than it was in the previous case. This is related to the fact that this entanglement comes from the population of the Bell state $|a\rangle$, which decays on a much longer timescale $(\Gamma - \Gamma_{12})^{-1}$ [3]. The grey areas in the figures denote the presence of entanglement. Again, only when $N \to 0$ the asymmetric state decays asymptotically to zero and the asymptotic behaviour of entanglement takes place. Another interesting feature of ESB is the fact that it appears only for rather small values of the mean number of photons N. Above a certain value of N there is no ESB. This value depends on the interatomic distance, it becomes higher as the interatomic distance decreases, and there is no ESB for independent atoms.

4. Conclusions

We have discussed the dynamics of entanglement in a two-atom system interacting with a common reservoir at finite temperature. The evolution of the system is described by the Lehmberg–Agarwal Markovian master equation, which takes into account the collective behaviour of the atoms. The collective spontaneous emission is a source of entanglement even for the isotropic initial state, which is referred to as ESB. We have shown that for non-zero temperature reservoir, the ESD is the standard feature of the evolution, and only in the limit $N \to 0$ is the asymptotic decay of entanglement possible. This means that in real situations of reservoirs at finite temperatures, there is always ESD. On the other hand, ESB created by correlated atoms appears only for reservoirs at sufficiently low temperatures, and it disappears at higher temperatures.

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