Journal of Computational Methods in Sciences and Engineering 10 (2010) 265–289
DOI 10.3233/JCM-2010-0323
IOS Press

Sudden birth and sudden death of entanglement

Zbigniew Ficek[a,*] and Ryszard Tanaś[b]
[a]*The National Centre for Mathematics and Physics, KACST, P.O. Box 6086, Riyadh 11442, Saudi Arabia*
[b]*Nonlinear Optics Division, Department of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland*

Abstract. We discuss a simple model for creation of entanglement in a system composed of two two-level atoms interacting with a common environment. The role of the environment in entangling and disentangling of the atoms is explored. We demonstrate how the spontaneous decay of an initial excitation of the system can create a transient entanglement between the atoms. The opposite situation is also discussed where a spontaneous disentanglement of initially entangled atoms may exhibit some unusual features such as entanglement sudden birth, the phenomenon of entanglement sudden death and revival of entanglement. We provide a discussion of these unusual phenomena in terms of the density matrix elements of the system and show the connection of the phenomena with the threshold behaviour of the concurrence.

1. Introduction

Entanglement is a unique feature of quantum mechanics that can be created between different objects such as individual photons, atoms, nuclear spins and even between biological living cells [1]. Theoretical studies have demonstrated that entanglement can be used as a resource for transmission of quantum information over long distances. However, a transfer of the information requires ingredients for the control and transmission in the form of long-lived entangled states that are immune to environmental noise and decoherence. The transfer of information has been accomplished between light, which appears as a carrier of the information, and atoms whose stable ground states serve as a storage system [2–4]. The ability to store quantum information in long-lived atomic ground states opens possibilities for a long time controlled processing of information and communication of the information on demand.

An important issue in the creation of entanglement between the qubits is the presence of unavoidable decoherence which is a source of an irreversible loss of the coherence. It is often regarded as the main obstacle in practical implementation of coherent effects and entanglement. A typical source of decoherence is spontaneous emission resulting from the interaction of a system with an external environment. For a composed system, each part of the system can interact with own independent environments or all the parts can be coupled to a common environment. In both cases, an initial entanglement encoded into the system degrades during the evolution. Nevertheless, the degradation process can be much slower when the parts of the system are coupled to a common rather than separate environments and, contrary to our intuition, might even entangle initially unentangled qubits. This effect,

*Corresponding author. E-mail: zficek@kacst.edu.sa.

called the environment induced entanglement has been studied for discrete (atomic) [5–11], as well as for continuous variable systems [12,13]. A crucial parameter in the case of the atoms coupled to the same external environment is the collective damping [14–19], which results from an incoherent spontaneous exchange of photons between the qubits.

The destructive effect of spontaneous emission on entanglement can take different time scales. The decoherence time depends on the damping rate of the state in which the entanglement was initially encoded and usually the decay process induced by spontaneous emission occurs exponentially in time. However, some entangled states can have interesting unusual decoherence properties that an initial entanglement can vanish abruptly in a finite time [20–23]. This drastic non-asymptotic feature of entanglement has been termed as the "entanglement sudden death", and is characteristic of the dynamics of a special class of initial entangled states. The qubits may remain separable for the rest of the evolution or entanglement may revive after a finite time [24–28].

Although the sudden death feature is concerned with the disentangled properties of spontaneous emission there is an interesting "sudden" feature in the temporal creation of entanglement from initially independent qubits [29,30]. The phenomenon termed as sudden birth of entanglement, as it is opposite to the sudden death of entanglement arises dynamically during the spontaneous evolution of an initially separate qubits. The sudden birth of entanglement is now intensively studied as it would provide a resource for a controlled creation of entanglement on demand in the presence of a dissipative environment [23].

In this paper we illustrate the effect of spontaneous emission and the interactions between atoms on entanglement creation and its transient evolution. We introduce the concurrence, the necessary and sufficient conditions for entanglement in a two-atom system. Then we turn into the dynamics of atoms interacting with a common environment and examine a possibility for entanglement creation between two atoms. We employ the dynamics of the atoms to illustrate some unusual temporary behaviours of the disentanglement process, the sudden birth and sudden death of entanglement. Next, we demonstrate under what kind of conditions the already destroyed entanglement could revive during the evolution. Finally, we discuss the role of thermal reservoir on the evolution of an initial entanglement.

The collective properties of interacting atoms, which we consider here, is a fascinating research topic in which Professor Stanisław Kielich was interested for many years. In a series of publications, he has explored the role of the interactions in light scattering [31], especially intermolecular interactions play crucial role in the cooperative light scattering discovered by Kielich and his French colleagues [32]. Another important topic investigated by Kielich since the beginning of his scientific career was the problem of optical reorientation of molecules, which started before the first laser became operational [33]. More on the research done in Kielich's group can be found in articles published in [34].

2. Two-atom system

We consider a system composed of two identical atoms located at fixed positions $\vec{r}_i$, and separated by a distance $r_{ij} = |\vec{r}_j - \vec{r}_i|$ large compared to the atomic diameter, so that overlap between the atoms can be ignored. The atoms are modeled as two-level systems with ground states $|g_i\rangle$ and excited states $|e_i\rangle$, separated by transition frequencies ω_0 and connected by a transition dipole moment $\vec{\mu}$. Both atoms are assumed to be damped with the same rates $\gamma-$the spontaneous emission rates arising from the coupling of the atoms to a common reservoir.

The dynamics of the atoms interacting with a reservoir of electromagnetic field modes at some temperature T are determined by the master equation of the density operator of the system, which in the limit of $\omega_0 t \gg 1$ has the form [17,18]

$$\frac{\partial\rho}{\partial t} = -i\omega_0 \sum_{i=1}^{2} [S_i^z, \rho] - i \sum_{i\neq j=1}^{2} \Omega_{ij} \left[S_i^+ S_j^-, \rho\right] \\ - \frac{1}{2}\sum_{i,j=1}^{2} \gamma_{ij}(1+N)\left\{\left[\rho S_i^+, S_j^-\right] + \left[S_i^+, S_j^- \rho\right]\right\}, \\ - \frac{1}{2}\sum_{i,j=1}^{2} \gamma_{ij}\, N \left\{\left[\rho S_i^-, S_j^+\right] + \left[S_i^-, S_j^+ \rho\right]\right\}, \tag{1}$$

where S_i^+, S_i^-, and S_i^z are the dipole raising, lowering, and population difference operators, respectively, of the ith atom, $\gamma_{ii} \equiv \gamma$ are the spontaneous decay rates of the atoms, equal to the Einstein A coefficient for spontaneous emission, and N is the mean number of photons of the reservoir at temperature T. The photon number $N \equiv N(\omega_0)$ is given by the Planck distribution

$$N(\omega) = \frac{1}{\mathrm{e}^{\frac{\hbar\omega}{k_B T}} - 1}, \tag{2}$$

where k_B is the Boltzmann constant.

The terms in the master equation that depend on γ_{ij} and Ω_{ij} $(i \neq j)$ are the so-called *collective* terms and determine collective properties of the system. The parameter γ_{ij} represents the collective damping, while the parameter Ω_{ij} represents the dipole-dipole interaction between the atoms. The effect of Ω_{12} on the atomic system is the shift of the energy of the single excitation collective states from the single-atom energy. The collective parameters can be evaluated explicitly and are given by the expressions [15–18, 35,36]

$$\gamma_{ij} = \frac{3}{2}\gamma \left\{ \left[1-(\hat{\mu}\cdot\hat{r}_{ij})^2\right] \frac{\sin(kr_{ij})}{kr_{ij}} + \left[1-3(\hat{\mu}\cdot\hat{r}_{ij})^2\right]\left[\frac{\cos(kr_{ij})}{(kr_{ij})^2} - \frac{\sin(kr_{ij})}{(kr_{ij})^3}\right]\right\}, \tag{3}$$

and

$$\Omega_{ij} = \frac{3}{4}\gamma \left\{ -\left[1-(\hat{\mu}\cdot\hat{r}_{ij})^2\right] \frac{\cos(kr_{ij})}{kr_{ij}} + \left[1-3(\hat{\mu}\cdot\hat{r}_{ij})^2\right]\left[\frac{\sin(kr_{ij})}{(kr_{ij})^2} + \frac{\cos(kr_{ij})}{(kr_{ij})^3}\right]\right\}, \tag{4}$$

where $\hat{\mu}$ is the unit vector along the dipole moments of the atoms, which we have assumed to be parallel $(\hat{\mu} = \hat{\mu}_i = \hat{\mu}_j)$, $\hat{r}_{ij}$ is the unit vector in the direction of $\vec{r}_{ij}$, $k = \omega_0/c$, and r_{ij} is the distance between the atoms. For small distances between the atoms, $\gamma_{12} \approx \gamma$, whereas for large distances the collective damping vanishes as $(kr_{12})^{-1}$ and becomes negligible for $kr_{12} \gg 1$. In this limit, the atoms decay independently with a rate identical to that of a single atom.

The density operator of the system can be represented in a complete set of basis states spanned by four product (separable) states

$$|1\rangle = |g_1\rangle \otimes |g_2\rangle, |2\rangle = |g_1\rangle \otimes |e_2\rangle, \\ |3\rangle = |e_1\rangle \otimes |g_2\rangle, |4\rangle = |e_1\rangle \otimes |e_2\rangle. \tag{5}$$

Written in the basis (5), the density operator can have a diagonal or non-diagonal form and the presence of any non- diagonal terms simply indicates the existence of coherence effects in the system. For the examples considered in the following sections, the density matrix of a two-atom system will occur in the so-called X-state form [37,38]

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{6}$$

where non-zero matrix elements occur only along the main diagonal and anti-diagonal. Physically, the X-state form corresponds to a situation where all coherences between the ground state $|1\rangle$ and the single excitation states $|2\rangle$ and $|3\rangle$, and between $|2\rangle$, $|3\rangle$ and the double excitation state $|4\rangle$ are zero. The X-state density matrix can be easily created by an appropriate initial preparation of a two-atom system. Also, one can find processes that not only retain the initial X-state form of the density matrix, but even could lead to the X-state form under the evolution.

One can see from Eq. (6) that in the presence of coherence, the density matrix is not diagonal. This means that the product states (5) do not correspond in general to the eigenstates of the system. In this case, a different choice of basis states is found particularly useful to work with, the basis of collective states of the system, or the Dicke states, defined as [14,15,17]

$$\begin{aligned} |g\rangle &= |g_1\rangle \otimes |g_2\rangle, \\ |s\rangle &= \frac{1}{\sqrt{2}}\left(|e_1\rangle \otimes |g_2\rangle + |g_1\rangle \otimes |e_2\rangle\right), \\ |a\rangle &= \frac{1}{\sqrt{2}}\left(|e_1\rangle \otimes |g_2\rangle - |g_1\rangle \otimes |e_2\rangle\right), \\ |e\rangle &= |e_1\rangle \otimes |e_2\rangle. \end{aligned} \tag{7}$$

It is interesting to note that the collective basis contains two states, $|s\rangle$ and $|a\rangle$ that are linear symmetric and antisymmetric superpositions of the product states, respectively. The most important is that the states are in the form of maximally entangled states. The remaining two states $|e\rangle$ and $|g\rangle$ are separable states.

3. Entanglement measure

The usual way to identify entanglement in a system being in a mixed state and composed of two sub-systems (qubits) is to examine the concurrence [39,40]. For a system described by the density matrix ρ, the concurrence $\mathcal{C}$ is defined as

$$\mathcal{C}(t) = \max\left(0, \lambda_1(t) - \lambda_2(t) - \lambda_3(t) - \lambda_4(t)\right), \tag{8}$$

where $\{\lambda_i(t)\}$ are the square roots of the eigenvalues of the non-Hermitian matrix $R = \rho(t)\tilde{\rho}(t)$ with

$$\tilde{\rho}(t) = \sigma_y \otimes \sigma_y \, \rho^*(t) \, \sigma_y \otimes \sigma_y, \tag{9}$$

and σ_y is the Pauli matrix. The range of concurrence is from 0 to 1. For unentangled (separated) atoms $\mathcal{C}(t) = 0$, whereas $\mathcal{C}(t) = 1$ for the maximally entangled atoms.

A particularly interesting behaviour of concurrence is found for a system of qubits determined by the density matrix of the X-state form, Eq. (6). To determine the concurrence, we first find the square roots of the eigenvalues of the matrix R:

$$\sqrt{\lambda_{1,2}} = \sqrt{\rho_{22}(t)\rho_{33}(t)} \pm |\rho_{14}(t)|,$$

$$\sqrt{\lambda_{3,4}} = \sqrt{\rho_{11}(t)\rho_{44}(t)} \pm |\rho_{23}(t)|, \tag{10}$$

from which it is easily verified that for a particular value of the matrix elements there are two possibilities for the largest eigenvalue, either $\sqrt{\lambda_1}$ or $\sqrt{\lambda_3}$. The two possibilities result in the concurrence of the form

$$\mathcal{C}(t) = \max\{0,\, \mathcal{C}_1(t),\, \mathcal{C}_2(t)\}\,, \tag{11}$$

where

$$\begin{aligned}\mathcal{C}_1(t) &= 2\left\{|\rho_{14}(t)| - \sqrt{\rho_{22}(t)\rho_{33}(t)}\right\}\\ &= 2\left\{|\rho_{ge}(t)| - \sqrt{\left[\frac{\rho_{ss}(t)+\rho_{aa}(t)}{2}\right]^2 - [\mathrm{Re}\,\rho_{sa}(t)]^2}\right\},\end{aligned} \tag{12}$$

and

$$\begin{aligned}\mathcal{C}_2(t) &= 2\left\{|\rho_{23}(t)| - \sqrt{\rho_{11}(t)\rho_{44}(t)}\right\}\\ &= 2\left\{\sqrt{\left[\frac{\rho_{ss}(t)-\rho_{aa}(t)}{2}\right]^2 + [\mathrm{Im}\,\rho_{sa}(t)]^2} - \sqrt{\rho_{ee}(t)\rho_{gg}(t)}\right\}.\end{aligned} \tag{13}$$

From this it is clear that the concurrence $\mathcal{C}(t)$ can always be regarded as being made up of either $\mathcal{C}_1(t)$ or $\mathcal{C}_2(t)$ that are associated with two different coherences which can be generated in a two qubit system. From the forms of $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$, it is obvious that $\mathcal{C}_1(t)$ provides a measure of an entanglement produced by the two-photon coherence $\rho_{14}(t)$, whereas $\mathcal{C}_2(t)$ provides a measure of an entanglement produced by the one-photon coherence $\rho_{23}(t)$. The two alternative contributions $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$ are traditionally understood as the *criteria* for one and two-photon entanglement, respectively.

Inspection of Eqs (12) and (13) shows that non-zero coherences $\rho_{14}(t)$ and $\rho_{23}(t)$ are the necessary condition for entanglement, but not in general sufficient one since there are also threshold terms in the concurrence measures involving the diagonal (population) terms. For example, there might be situations where the quantity $\sqrt{\rho_{11}(t)\rho_{44}(t)}$ is different from zero, and $|\rho_{23}(t)|$ may be positive but not large enough to enhance the concurrence $\mathcal{C}_2(t)$ above the threshold for entanglement. Thus, the necessary and sufficient condition for entanglement between two qubits whose the dynamics are determined by the density matrix (6) is

$$|\rho_{14}(t)| \neq 0 \quad \text{and} \quad |\rho_{14}(t)| > \sqrt{\rho_{22}(t)\rho_{33}(t)}, \tag{14}$$

or

$$|\rho_{23}(t)| \neq 0 \quad \text{and} \quad |\rho_{23}(t)| > \sqrt{\rho_{11}(t)\rho_{44}(t)}, \tag{15}$$

The thresholds for the coherences are sort of "boundaries" between classical and quantum behaviour of a two-qubit system. Above the threshold, the behaviour of the system is determined in terms of superposition states, the basic principle of quantum mechanics. Below the threshold, the qubits are separable and no superposition principle is required to determine their properties.

4. Evolution of the density matrix elements

We choose the collective states (7), as a basis for the representation of the density operator and the analysis of entanglement in the system. The equations of motion for the density matrix elements are found from the master equation (1), and are given by

$$
\begin{aligned}
\dot{\rho}_{ee} &= -2\gamma(1+N)\rho_{ee} + N\left[(\gamma+\gamma_{12})\rho_{ss} + (\gamma-\gamma_{12})\rho_{aa}\right], \\
\dot{\rho}_{ss} &= (\gamma+\gamma_{12})\left[\rho_{ee} - (1+3N)\rho_{ss} - N\rho_{aa} + N\right], \\
\dot{\rho}_{aa} &= (\gamma-\gamma_{12})\left[\rho_{ee} - N\rho_{ss} - (1+3N)\rho_{aa} + N\right], \\
\dot{\rho}_{as} &= -\left[\gamma(1+2N) + 2i\Omega_{12}\right]\rho_{as}, \\
\dot{\rho}_{ge} &= -\gamma(1+2N)\rho_{ge}.
\end{aligned}
\tag{16}
$$

Equations (16) describe the evolution of the two-atom system in a quite general case of thermal reservoir with nonzero mean number of photons N. In case of the ordinary vacuum, the mean number of photons of the reservoir is equal to zero, $N = 0$, and Eqs (16) simplify considerably.

We begin with the simple case of reservoir being the vacuum of the electromagnetic modes, so that we put $N = 0$. One can see from Eqs (16) that the transitions rates to and from the symmetric and antisymmetric states are modified by the collective damping γ_{12}. Provided that $\gamma_{12} > 0$, the transitions to and from the symmetric state occur with an enhanced rate $\gamma+\gamma_{12}$, whereas the transitions to and from the antisymmetric state occur with a reduced rate $\gamma-\gamma_{12}$. For small separations between the atoms, $\gamma_{12} \approx \gamma$, and then the state $|s\rangle$ becomes *superradiant* with a decay rate double that of the single atom γ, whereas the state $|a\rangle$ becomes *subradiant*, with a decay rate of order $(kr_{12})\gamma$ which vanishes in the limit of small distances $kr_{12} \ll 1$.

The set of coupled equations for the populations of the collective states can be easily solved, and the solution, valid for arbitrary initial conditions and $N = 0$ is given by [15,17]

$$
\begin{aligned}
\rho_{ee}(t) &= \rho_{ee}(0)\,\mathrm{e}^{-2\gamma t}, \\
\rho_{ss}(t) &= \rho_{ss}(0)\,\mathrm{e}^{-(\gamma+\gamma_{12})t} + \rho_{ee}(0)\frac{\gamma+\gamma_{12}}{\gamma-\gamma_{12}}\left[\mathrm{e}^{(\gamma-\gamma_{12})t} - 1\right]\mathrm{e}^{-2\gamma t}, \\
\rho_{aa}(t) &= \rho_{aa}(0)\,\mathrm{e}^{-(\gamma-\gamma_{12})t} + \rho_{ee}(0)\frac{\gamma-\gamma_{12}}{\gamma+\gamma_{12}}\left[\mathrm{e}^{(\gamma+\gamma_{12})t} - 1\right]\mathrm{e}^{-2\gamma t},
\end{aligned}
\tag{17}
$$

and $\rho_{gg}(t) = 1 - \rho_{ee}(t) - \rho_{ss}(t) - \rho_{aa}(t)$.

We see from Eq. (17) that the decay of the populations depends strongly on the initial state of the system. When the system is initially prepared in the state $|s\rangle$, the population of the initial state decays exponentially with an enhanced rate $\gamma+\gamma_{12}$, while the initial population of the antisymmetric state decays with a reduced rate $\gamma-\gamma_{12}$. This occurs because the photons emitted from the excited atom can be absorbed by the atom in the ground state, so that the photons do not escape immediately from the system. For a general initial state that includes in the state $|e\rangle$, the populations of the symmetric and the antisymmetric states do not decay with a single exponential.

Similarly, equations of motion for the coherences, Eq. (16), are straightforward to solve, giving

$$\begin{aligned}\rho_{sa}(t) &= \rho_{sa}(0)\,\mathrm{e}^{-(\gamma+2i\Omega_{12})t},\\ \rho_{ge}(t) &= \rho_{ge}(0)\,\mathrm{e}^{-\gamma t}.\end{aligned} \tag{18}$$

The time evolution of the density matrix elements depends on the initial state of the system. In the following sections, we will apply the solutions to analyse the time evolution of the concurrence for different initial states. We consider both separable and entangled initial states to see how spontaneous emission can affect separable and entangled properties of the system.

5. Spontaneous creation of entanglement

Let us consider first the problem of creation of entanglement in spontaneous emission from an initial separable state. We take for the initial state of our system the single-excitation state $|I\rangle = |3\rangle = |e_1\rangle \otimes |g_2\rangle$, which corresponds to atom 1 initially prepared in the excited state and atom 2 in the ground state. The initial state is, of course, a separable state. In other words, there is no entanglement in the system at $t = 0$. It is easily verified that in this case, the only non-vanishing matrix elements are

$$\rho_{ss}(0) = \rho_{aa}(0) = \rho_{sa}(0) = \rho_{as}(0) = \frac{1}{2}, \tag{19}$$

and then the initial density matrix has the following form

$$\rho(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{ss}(0) & \rho_{sa}(0) & 0 \\ 0 & \rho_{as}(0) & \rho_{aa}(0) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

According to the solutions (17) and (18), the matrix elements which are zero at the initial time $t = 0$ will remain zero for all time, except the population of the ground state $|g\rangle$ which will buildup during the evolution. Hence, the diagonal form of the density matrix will be preserved during the evolution. Moreover, the dynamics of the systems can be confined to the subspace spanned by three state vectors only, $|g\rangle, |s\rangle$ and $|a\rangle$.

We now examine the concurrence, Eq. (11). Looking at the density matrix (20), two conclusions can be made. First of all, since the coherence $\rho_{ge}(t)$ is equal to zero for all times t, we see from Eq. (12) that the criterion $\mathcal{C}_1(t)$ is always negative. Consequently, $\mathcal{C}_1(t)$ will not contribute to the concurrence. Secondly, the upper state $|e\rangle$ is not involved in the dynamics of the system which, according to Eq. (13) rules out a possibility for the threshold behaviour of the criterion $\mathcal{C}_2(t)$. Under this situation, the time evolution of the concurrence is determined solely by the time evolution of the coherence $|\rho_{23}(t)|$, which in terms of the collective states can be written as [19]

$$\begin{aligned}\mathcal{C}(t) \equiv \mathcal{C}_2(t) &= \sqrt{\left[\rho_{ss}(t) - \rho_{aa}(t)\right]^2 + 4\left[\mathrm{Im}\rho_{sa}\right]^2}\\ &= \mathrm{e}^{-\gamma t}\sqrt{\sinh^2(\gamma_{12}t) + \sin^2(2\Omega_{12}t)}.\end{aligned} \tag{21}$$

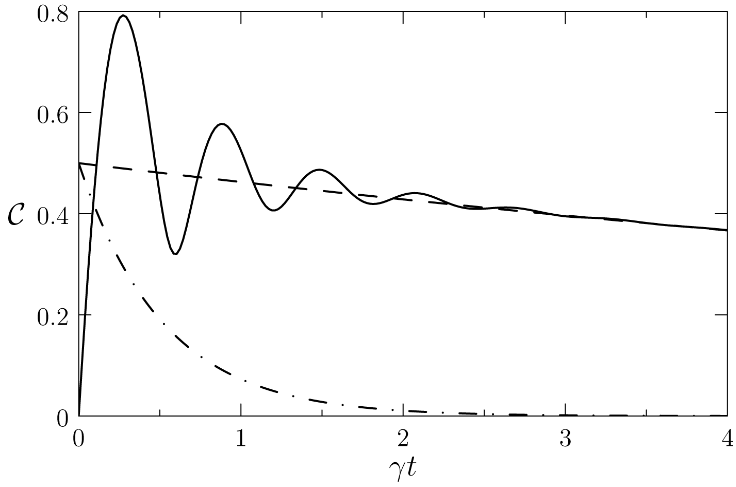

Fig. 1. Time evolution of the concurrence (solid line) and the populations of the antisymmetric (dashed line) and the symmetric (dashed-dotted line) states for the atoms prepared initially in the separable state $|3\rangle = |e_1\rangle \otimes |g_2\rangle$, with $\hat{\mu} \perp \hat{r}_{12}$, and $r_{12} = \lambda/10$.

There are two contributions to the time evolution of the concurrence. First, there is an imaginary part of the coherence between the states $|s\rangle$ and $|a\rangle$. Since this is off-resonance coupling, it leads to oscillation in the concurrence with frequency 2 Ω_{12}, the frequency difference between the two states. The second contribution is the difference between the populations of the states $|s\rangle$ and $|a\rangle$. This contribution is non-oscillatory. We see that the effect of the non-oscillatory term is apparently to lengthen the lifetime of the concurrence.

Figure 1 shows the time evolution of the concurrence $\mathcal{C}(t)$, calculated from Eq. (21), together with the populations of the symmetric and the antisymmetric states. The time evolution of the concurrence reflects the time evolution of the entanglement between the atoms. It is seen that the concurrence builds up immediately after $t = 0$ and remains positive for all time, which indicates that spontaneous emission can indeed create entanglement between the initially unentangled atoms. The buildup of the concurrence in time generally consists of oscillatory and non-oscillatory components, so that two time scales of completely different behaviour of the concurrence can be distinguished. At early times, the concurrence builds up in an oscillatory manner, and the oscillatory structure is smoothed out on a time scale of $t = (\gamma + \gamma_{12})^{-1}$, the lifetime of the symmetric state. The oscillations vanish at time close to the point where the symmetric state becomes depopulated. At later times, the concurrence evolves in a non-oscillatory manner and overlaps with the population of the antisymmetric state. As a result, the concurrence decays slowly on the time scale of $t = (\gamma - \gamma_{12})^{-1}$. Although the concurrence (21) involves both symmetric and antisymmetric states, it is clear that crucial for the entanglement is the presence of the antisymmetric state.

The decay time of the population of the antisymmetric state, so that the transient entanglement seen in Fig. 1, varies with the distance between the atoms. The time goes to infinity when $kr_{12} \to 0$. In this limit the transition rates to and from the antisymmetric state vanish and the state decouples from the remaining states. Hence, any initial population encoded into the state will remain there for all times. For example, with the initial state $|3\rangle$ and $t \to \infty$, half of the population, that initially encoded into the antisymmetric state, still remains in the atomic system, and half of the population, that initially encoded into the symmetric state, is emitted into the field. As a result, the concurrence evolves to its stationary value of $\mathcal{C}(\infty) = 0.5$ indicating that in the limit of $t \to \infty$, the spontaneous emission could produce steady state entanglement to the degree of 50% with the corresponding pure state of the system.

The creation of the transient entanglement can be understood by considering the properties of the density matrix of the system. For the initial state of only one atom excited the density matrix is not diagonal due to the presence of coherences $\rho_{sa}(0)$ and $\rho_{as}(0)$. Since the form of the matrix is preserved during the evolution, it remains non-diagonal for all times. Consequently, the collective states are no longer the eigenstates of the system. The density matrix can be diagonalized to give new diagonal states. It is easy to verify that the states $|g\rangle$ and $|e\rangle$ remain unchanged, whereas the states $|s\rangle$ and $|a\rangle$ recombine into new diagonal symmetric

$$|+\rangle = \frac{[\rho_{++}(t)-\rho_{ss}(t)]|a\rangle+\rho_{as}(t)|s\rangle}{\left\{[\rho_{++}(t)-\rho_{ss}(t)]^2+|\rho_{as}(t)|^2\right\}^{\frac{1}{2}}}, \tag{22}$$

and antisymmetric

$$|-\rangle = \frac{\rho_{as}(t)|a\rangle+[\rho_{--}(t)-\rho_{aa}(t)]|s\rangle}{\left\{[\rho_{--}(t)-\rho_{aa}(t)]^2+|\rho_{as}(t)|^2\right\}^{\frac{1}{2}}}, \tag{23}$$

states, with the eigenvalues $\rho_{++}(t)$ and $\rho_{--}(t)$, the populations of the diagonal states, given by

$$\begin{aligned}\rho_{++}(t) &= \frac{1}{2}\left[\rho_{aa}(t)+\rho_{ss}(t)\right] + \frac{1}{2}\left\{[\rho_{aa}(t)-\rho_{ss}(t)]^2+4|\rho_{as}(t)|^2\right\}^{\frac{1}{2}}, \\ \rho_{--}(t) &= \frac{1}{2}\left[\rho_{aa}(t)+\rho_{ss}(t)\right] - \frac{1}{2}\left\{[\rho_{aa}(t)-\rho_{ss}(t)]^2+4|\rho_{as}(t)|^2\right\}^{\frac{1}{2}}.\end{aligned} \tag{24}$$

It follows from Eq. (23) that the coherences $\rho_{sa}(t)$ and $\rho_{as}(t)$ cause the system to evolve between two "new" states $|+\rangle$ and $|-\rangle$, which are linear combinations of the collective states $|s\rangle$ and $|a\rangle$. It is easy verified that for the initial condition (19), the population $\rho_{--}(t) = 0$ for all times, whereas the population $\rho_{++}(t)$ is different from zero and equals to the sum of the populations $\rho_{ss}(t)$ and $\rho_{aa}(t)$. The lack of population in the states $|-\rangle$ together with no population in the state $|e\rangle$ reduces the four-level system to an effective two-level system with the excited non-maximally entangled state $|+\rangle$ and the separable ground state $|g\rangle$. In this case, the density matrix of the system has a simple diagonal form

$$\rho(t) = \rho_{++}(t)|+\rangle\langle+| + \rho_{gg}(t)|g\rangle\langle g|. \tag{25}$$

Since $\rho_{--}(t) = 0$ for all times t, we can easily find from Eq. (24) that in this case $|\rho_{as}(t)|^2 = \rho_{aa}(t)\rho_{ss}(t)$, and then the state $|+\rangle$ can be written as

$$|+\rangle = \frac{\sqrt{\rho_{ss}(t)}|s\rangle + \sqrt{\rho_{aa}(t)}|a\rangle}{\sqrt{\rho_{aa}(t)+\rho_{ss}(t)}}. \tag{26}$$

When the collective states $|s\rangle$ and $|a\rangle$ are equally populated, the state $|+\rangle$ reduces to a separable state $|e_1\rangle \otimes |g_2\rangle$. On the other hand, the state $|+\rangle$ reduces to a maximally entangled state, $|s\rangle$ or $|a\rangle$, when either $\rho_{ss}(t)$ or $\rho_{aa}(t)$ is equal to zero. Since the population of the symmetric state decays faster than the antisymmetric state, see Eq. (16), at time when the state $|s\rangle$ becomes depopulated, the state $|+\rangle$ reduces to the maximally entangled antisymmetric state $|a\rangle$. This explains why at later times the evolution of the concurrence follows the evolution of the population of the state $|a\rangle$.

The above analysis gives clear evidence that the creation of transient entanglement from the separable state by spontaneous emission depends crucially on the presence of the antisymmetric state.

6. Sudden birth of entanglement

We have already seen that an entanglement can be created by spontaneous emission from an initially separable state. However, spontaneous emission offers more opportunities for unusual effects. We shall see that the creation of entanglement by spontaneous emission is not a common phenomenon for all separable states, and that for some initial states the spontaneous creation of entanglement is possible but can be delayed in time. To illustrate this phenomenon, which we shall term as *sudden birth of entanglement*, we consider two initial separable states that differ from that considered in the preceding section. In the first, we assume that initially both atoms were inverted, and in the second, we assume that the atoms were initially separable and each prepared in a linear superposition of its energy states.

We start by assuming that initially the system was prepared in the separable two-excitation state $|e\rangle$. In this case, $\rho_{ee}(0) = 1$ and all the remaining density matrix elements are zero. According to Eq. (18), the off-diagonal terms (coherences) will remain zero for all times, but the populations $\rho_{ss}(t)$, $\rho_{aa}(t)$ and $\rho_{gg}(t)$ will buildup during the evolution. This implies that for $t > 0$, the density matrix of the system spanned in the basis of the collective states (7), will have a diagonal form for all t:

$$\rho(t) = \begin{pmatrix} \rho_{gg}(t) & 0 & 0 & 0 \\ 0 & \rho_{ss}(t) & 0 & 0 \\ 0 & 0 & \rho_{aa}(t) & 0 \\ 0 & 0 & 0 & \rho_{ee}(t) \end{pmatrix}, \tag{27}$$

with the time dependent density matrix elements given by

$$\begin{aligned} \rho_{ee}(t) &= \mathrm{e}^{-2\gamma t}, \\ \rho_{ss}(t) &= \frac{\gamma+\gamma_{12}}{\gamma-\gamma_{12}}\left[\mathrm{e}^{(\gamma-\gamma_{12})t}-1\right]\mathrm{e}^{-2\gamma t}, \\ \rho_{aa}(t) &= \frac{\gamma-\gamma_{12}}{\gamma+\gamma_{12}}\left[\mathrm{e}^{(\gamma+\gamma_{12})t}-1\right]\mathrm{e}^{-2\gamma t}, \end{aligned} \tag{28}$$

and $\rho_{gg}(t) = 1 - \rho_{ee}(t) - \rho_{ss}(t) - \rho_{aa}(t)$.

Looking at the density matrix (27) and the criteria for entanglement, Eqs (12) and (13), one can easily find that in this case $\mathcal{C}_1(t)$ is always negative. Thus, positive values of the concurrence, so that the entanglement are determined only by the criterion $\mathcal{C}_2(t)$:

$$\mathcal{C} = \max\left\{0,\, \mathcal{C}_2(t)\right\}, \tag{29}$$

with

$$\mathcal{C}_2(t) = |\rho_{ss}(t) - \rho_{aa}(t)| - 2\sqrt{\rho_{gg}(t)\rho_{ee}(t)}. \tag{30}$$

It is clear from Eq. (30) that entanglement, if any, may only result from an unequal population of the symmetric and antisymmetric states. When the system is prepared in the state $|e\rangle$, the resulting spontaneous transitions are cascades: The system decays first to the intermediate states $|s\rangle$ and $|a\rangle$, from which then decays to the ground state $|g\rangle$. Since the transition rates to and from the states $|s\rangle$ and $|a\rangle$ are different when $\gamma_{12} \neq 0$, there appears unbalanced population distribution between these states. According to Eq. (30), this may result in a transient entanglement between the atoms.

To visualise the behaviour of the concurrence, we plot $\mathcal{C}(t)$ as a function of time and the distance between the atoms in Fig. 2. The system was initially in the separable two-excitation state $|e\rangle$. We

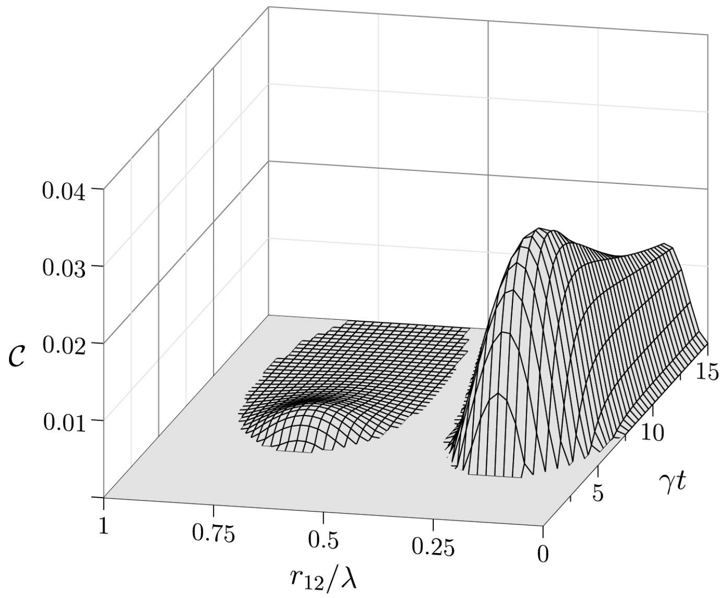

Fig. 2. A three-dimensional plot of the concurrence $\mathcal{C}(t)$ as a function of time γt and the distance r_{12}/λ between two, initially inverted atoms with the polarisation of the atomic dipole moments perpendicular to the inter-atomic axis, $\hat{\mu} \perp \hat{r}_{12}$.

see that there is no entanglement at earlier times, but suddenly at some finite time an entanglement emerges. However, it happens only for a limited range of the distances r_{12}. It is easily verified that the "islands" of entanglement seen in Fig. 2 appear at distances for which γ_{12} is different from zero. Again, it reflects the role of the collective damping which is responsible for creation of entanglement by spontaneous emission from an initial separable state. This result also demonstrates how the collective damping leading to entanglement between the atoms becomes less important as the interatomic distance is increased. Atoms separated by more than just a few wave-lengths become separable.

We have seen that γ_{12} modifies the damping rates of the transitions between the collective states and, on the other hand, introduces two time scales for the decay of the population of the system. One can easily find that the entanglement seen in Fig. 2 decays out on a time scale $(\gamma - \gamma_{12})^{-1}$ that is the time scale of the population decay from the antisymmetric state. Again, this shows that crucial for entanglement creation by spontaneous emission is the presence of the antisymmetric state. This is perhaps not surprising, as the population of the antisymmetric state builds up on a much longer time scale than the population of the symmetric state. At long times, the antisymmetric state will posses a large population with no population left in the symmetric state. What is surprising and, in fact, is very similar to the situation found in the Dicke model that at early times $\gamma t \ll 1$ and $\gamma_{12} \neq 0$, the symmetric state is significantly populated with almost no population in the antisymmetric state, and no entanglement is created.

The above considerations are supported by the analysis of the time evolution of the population of the excited states of the system that is illustrated in Fig. 3. It is quite evident from the figure that at the time $t \approx 4/\gamma$ when the entanglement starts to build up, the antisymmetric state is the only excited state of the system being populated. Clearly, the effect of the delayed creation of entanglement is attributed to the slow decay rate of the antisymmetric state. The state decays on the time scale of $(\gamma - \gamma_{12})^{-1}$ that is much longer than the decay time of the symmetric and the upper states.

Let us consider an another example of a delayed creation of entanglement from an initial separable state. This time, we consider an initial separable state which leads to a very general form of the initial

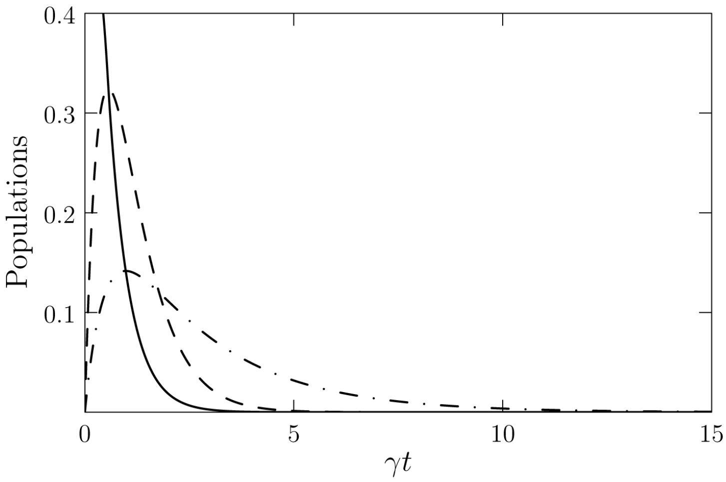

Fig. 3. The time evolution of the populations $\rho_{ee}(t)$ (solid line), $\rho_{ss}(t)$ (dashed line) and $\rho_{aa}(t)$ (dash-dotted line) for $r_{12}/\lambda = 0.25$ and $\vec{\mu} \perp \vec{r}_{12}$.

density matrix, in which all the initial density matrix elements are different from zero. An example of such a state is a product state in which the atoms are prepared in the superposition of their energy states by a short $\pi/2$ pulse

$$|\Phi_0\rangle = \frac{1}{2}\Big(|g_1\rangle + i\mathrm{e}^{i\vec{k}_L\cdot\vec{r}_1}|e_1\rangle\Big)\otimes\Big(|g_2\rangle + i\mathrm{e}^{i\vec{k}_L\cdot\vec{r}_2}|e_2\rangle\Big)\,, \tag{31}$$

where $\vec{k}_L$ is the wave vector of the excitation field.

With the state $|\Phi_0\rangle$, the initial values of the density matrix elements are

$$\begin{aligned}
&\rho_{ea}(0) = \rho_{ag}(0) = -\frac{1}{2\sqrt{2}}\sin\left(\frac{1}{2}\vec{k}_L\cdot\vec{r}_{12}\right),\\
&\rho_{es}(0) = \rho_{sg}(0) = \frac{i}{2\sqrt{2}}\cos\left(\frac{1}{2}\vec{k}_L\cdot\vec{r}_{12}\right),\\
&\rho_{gg}(0) = \frac{1}{4}, \quad \rho_{eg}(0) = -\frac{1}{4},\\
&\rho_{sa}(0) = \frac{1}{4}i\sin\vec{k}_L\cdot\vec{r}_{12}, \quad \rho_{ss}(0) = \frac{1}{4}\left(1+\cos\vec{k}_L\cdot\vec{r}_{12}\right),\\
&\rho_{aa}(0) = \frac{1}{4}\left(1-\cos\vec{k}_L\cdot\vec{r}_{12}\right), \quad \rho_{ee}(0) = \frac{1}{4},
\end{aligned} \tag{32}$$

where $\vec{k}_L\cdot\vec{r}_{12} = k_L r_{12}\cos\theta$, with $\theta-$the angle between the excitation direction $\vec{k}_L$ and the vector $\vec{r}_{12}$ connecting the atoms. In the derivation of the initial values (32) we have assumed, without loss of generality, that the atoms are located at points $\vec{r}_1 = (-\frac{1}{2}r_{12},0,0)$ and $\vec{r}_2 = (\frac{1}{2}r_{12},0,0)$, such that $\vec{r}_1+\vec{r}_2 = 0$. We see that the initial values of the density matrix elements involving the single-excitation states depend on the distance between the atoms and the direction of excitation relative to the interatomic axis.

It is important to note here that the density matrix of the system prepared initially in the state (32) has no longer the X-state form. In this case, all of the density matrix elements are initially different from

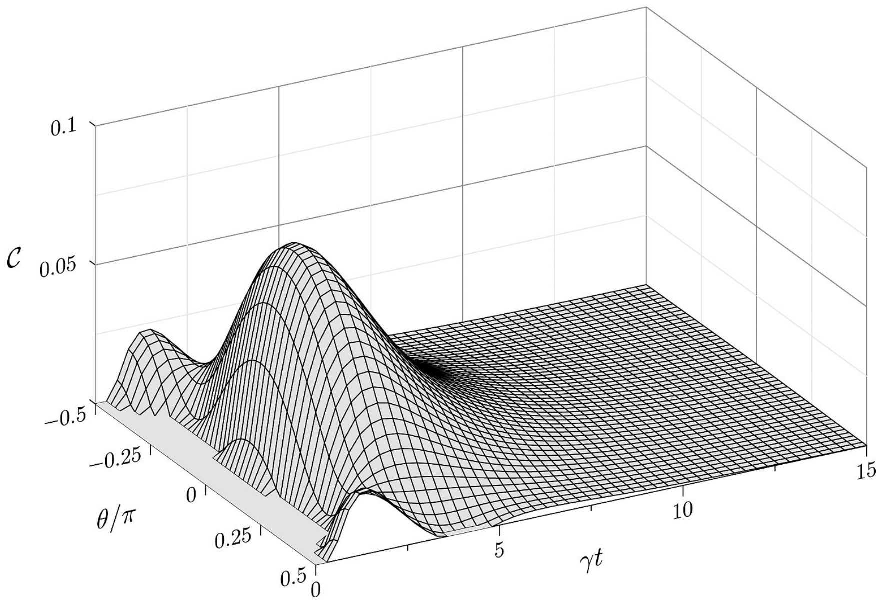

Fig. 4. The time evolution of the concurrence and its dependence on the direction of excitation relative to the inter-atomic axis for $r_{12}/\lambda = 0.25$ and the polarization of the atomic dipole moments $\vec{\mu} \perp \vec{r}_{12}$.

zero. Consequently, the concurrence must be calculated according to the general formula (8) rather than the criteria (12) and (13).

Figure 4 shows the concurrence as a function of time and the angle θ. It is seen that there is no entanglement at earlier times independent of the direction of excitation, and *suddenly* at some finite time an entanglement emerges. The magnitude of the entanglement created depends on the direction of excitation. A large entanglement is created when the system is excited in the direction of the interatomic axis, $\theta = 0$. This means that crucial for entanglement creation by spontaneous emission is to excite the system through the antisymmetric state. This is seen from analysis of the time evolution of the population of the excited states. Figure 5 shows the time evolution of the populations of the excited states of the system. It is clear from the figure that at the time $t \approx 4/\gamma$ when the entanglement starts to build up, the antisymmetric state is the only excited state of the system being populated. This effect is attributed to the slow decay rate of the antisymmetric state. As we have already noticed, the state decays on the time scale of $(\gamma - \gamma_{12})^{-1}$ that is much longer than the decay time of the symmetric and the upper states.

We may summarize that the phenomenon of sudden birth of entanglement is characteristic of initial states which include a population of the two-excitation state. The delayed sudden birth of entanglement is not found in a system with initially only one qubit excited. In experimental practice, the initial conditions of both atoms inverted can be done by using a standard technique of a short π pulse excitation. One could also use a short $\pi/2$ pulse excitation which leaves atoms separable and simultaneously prepared in the superposition of their energy states that populates the state with both atoms inverted.

7. Sudden death of entanglement

We now turn to an another “sudden” feature of entanglement, the phenomenon of entanglement *sudden death*, i.e., abrupt disappearance of an inital entanglement at a finite time. The subject received its initial

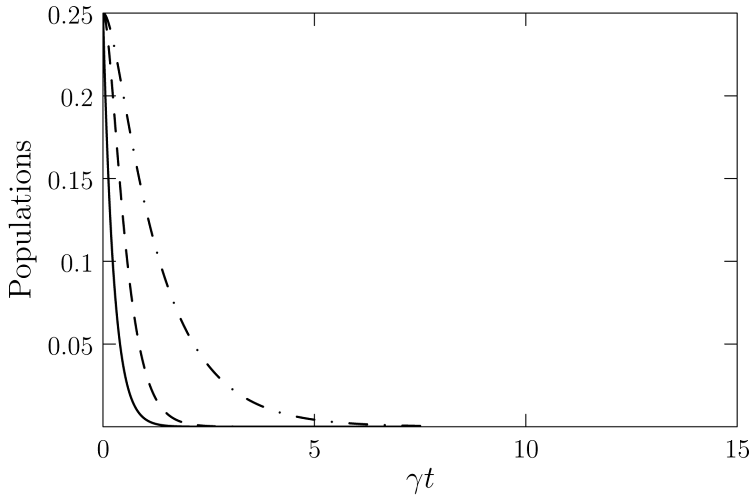

Fig. 5. The time evolution of the populations $\rho_{ee}(t)$ (solid line), $\rho_{ss}(t)$ (dashed line) and $\rho_{aa}(t)$ (dash-dotted line) for $\theta = 0$, $r_{12}/\lambda = 0.25$ and $\vec{\mu} \perp \vec{r}_{12}$.

stimulus in an article by Yu and Eberly [20,21], in which they for the first time introduced the concept of sudden death of entanglement. Many authors since have dealt with the entanglement sudden death in systems composed of two atoms or two harmonic oscillators. The literature on this subject can be divided into two categories: (i) evolution of an initial entanglement of independent atoms interacting with local environments [44], and (ii) entangled evolution of interacting atoms coupled to a common environment [45–50]. We postpone the studies of the latter group to the following section, where we will mostly focus on the phenomenon of entanglement revival. Here, we focus on the original concept of Yu and Eberly, and discuss in details the phenomenon of sudden death of entanglement in a system of two independent atoms interacting with local environments. This model is also applicable to a situation of two distant atoms interacting with a common environment. At large distances, the collective parameters γ_{12} and Ω_{12} are very small, so that the interaction between the atoms can be ignored and the atoms can be treated as independent sub-systems.

Let us suppose that at $t = 0$, a system of two independent atoms is prepared in a non-maximally entangled state of the form

$$|\Psi_0\rangle = \sqrt{q}\,|e_1\rangle \otimes |e_2\rangle + \sqrt{1-q}\,|g_1\rangle \otimes |g_2\rangle, \tag{33}$$

where q is a positive real number such that $0 \leqslant q \leqslant 1$. The state corresponds to an excitation of the system into a coherent superposition of its product states in which both or neither of the atoms is excited. In the special case of $q = 1/2$, the state (33) reduces to the maximally entangled Bell state.

Consider now the time evolution of the concurrence when the system is initially prepared in the state (33). It is not difficult to verify that the initial values for the density matrix elements are

$$\rho_{ee}(0) = q,\ \rho_{ge}(0) = \sqrt{q(1-q)},\ \rho_{gg}(0) = 1 - q, \tag{34}$$

and the other matrix elements, the populations of the symmetric and antisymmetric states, and all one-photon coherences are zero, i.e. $\rho_{ss}(0) = \rho_{aa}(0) = 0$ and $\rho_{es}(0) = \rho_{ea}(0) = \rho_{sg}(0) = \rho_{ag}(0) = \rho_{as}(0) = 0$. According to Eq. (18), the coherences will remain zero for all time, that they cannot be produced by spontaneous decay. However, the populations $\rho_{ss}(t)$ and $\rho_{aa}(t)$ can buildup during the evolution. This implies that for all times, the density matrix of the system spanned in the basis of the collective states (7), is in the X-state form

$$\rho(t) = \begin{pmatrix} \rho_{gg}(t) & 0 & 0 & \rho_{ge}(t) \\ 0 & \rho_{ss}(t) & 0 & 0 \\ 0 & 0 & \rho_{aa}(t) & 0 \\ \rho_{eg}(t) & 0 & 0 & \rho_{ee}(t) \end{pmatrix}, \tag{35}$$

with the density matrix elements evolving as

$$\begin{aligned} &\rho_{ee}(t) = q\,\mathrm{e}^{-2\gamma t}, \\ &\rho_{ge}(t) = \sqrt{q(1-q)}\,\mathrm{e}^{-(\gamma-2i\omega_0)t}, \\ &\rho_{ss}(t) = \rho_{aa}(t) = q\left(1-\mathrm{e}^{-\gamma t}\right)\mathrm{e}^{-\gamma t}, \end{aligned} \tag{36}$$

subject to conservation of the trace of $\rho(t)$: $\rho_{gg}(t) = 1 - \rho_{ss}(t) - \rho_{aa}(t) - \rho_{ee}(t)$. It is to be noticed that the symmetric and antisymmetric states are equally populated for all time. This holds for any initial state and results from the fact that the atoms radiate independently from each other. In this case, the populations of the collective states decay with the same rate, equal to the single-atom damping rate. Therefore, the initial relation between the populations cannot be changed during the spontaneous emission.

The density matrix (35) leads to a particularly simple expression for the concurrence. In general, the concurrence is given in terms of two entanglement criteria $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$, as seen from Eq. (11). However, with the initial state (33), the two-photon coherence $\rho_{ge}(t)$ is different from zero and the symmetric and antisymmetric states are equally populated for all time. As a consequence, the criterion $\mathcal{C}_2(t)$ is always negative, irrespective of q and times t. Therefore, entangled properties of the system are solely determined by the criterion $\mathcal{C}_1(t)$. On substituting from Eq. (36) into Eq. (12), we obtain the following expression for the concurrence

$$\mathcal{C}(t) = \max\left\{0,\ D(t)\,\mathrm{e}^{-\gamma t}\right\}, \tag{37}$$

where

$$D(t) = 2\sqrt{q(1-q)}\left[1 - \sqrt{\frac{q}{1-q}}\left(1-\mathrm{e}^{-\gamma t}\right)\right]. \tag{38}$$

This shows that transient features of the initial entanglement are determined by the properties of the function $D(t)$ which, on the other hand, is dependent on the parameter q. It is evident from Eq. (38) that there is a threshold for values of q; $q = 1/2$, below which $D(t)$ is always positive. However, above the threshold, $D(t)$ can take negative values indicating that the initial entanglement can vanish at a finite time. Consequently, the sudden death of the entanglement is possible for initial states with $q > 1/2$. Since $\rho_{ee}(0) = q$, we can conclude that the entanglement sudden death is ruled out for the initially not inverted system.

Figure 6 shows the concurrence $\mathcal{C}(t)$, calculated from Eq. (37), as a function of time for two different values of the parameter q. It is evident from the figure that for $q < 1/2$ the initial entanglement decays exponentially in time without any discontinuity. The entanglement sudden death appears for $q > 1/2$ that the concurrence decays in a non-exponential way and vanishes at a finite time. In addition, we plot the two-photon coherence $2|\rho_{eg}(t)|$ for $q = 2/3$. It is apparent that the coherence decays exponentially

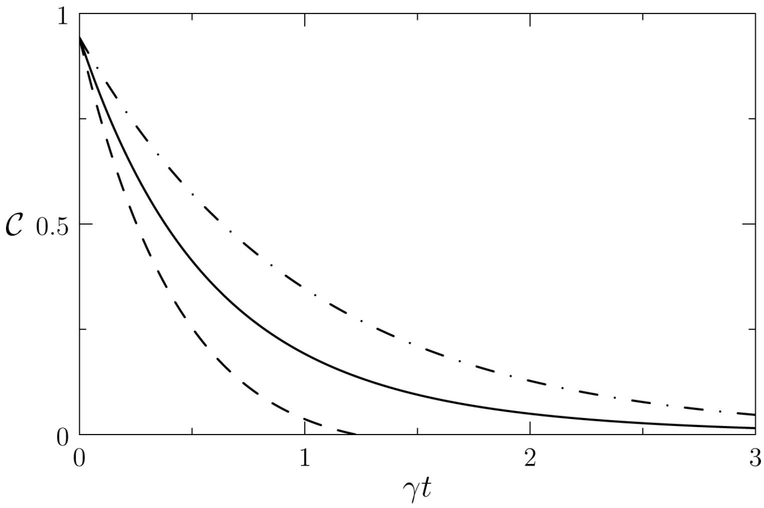

Fig. 6. Time evolution of the concurrence $\mathcal{C}(t)$ for two values of the parameter q: $q = 1/3$ (solid line) and $q = 2/3$ (dashed line). We also plot the time evolution of the two-photon correlation $2|\rho_{eg}(t)|$ (dashed-dotted line), with $q = 2/3$.

in time, which clearly illustrates that the entanglement disappears at finite time despite the fact that the two-photon coherence is different from zero for all time.

As we have already stated, time at which the entanglement disappears is a sensitive function of the initial conditions determined by the parameter q. It is easily verified from Eq. (38) that the time t_d at which the entanglement disappears is given by

$$t_d = \frac{1}{\gamma} \ln \left(\frac{q + \sqrt{q(1-q)}}{2q - 1} \right). \tag{39}$$

The time t_d gives the collapse time of the entanglement beyond which the entanglement disappears. The dead zone of the entanglement continues till infinity that the entanglement never revives. It may continue for a finite rather than infinite time that under some circumstances the already dead entanglement may revive after some finite time. A revival of the entanglement may occur when the atoms directly interact with each other. We leave the discussion of this problem to the following section.

7.1. Experimental demonstration of entanglement sudden death

The phenomenon of entanglement sudden death has been experimentally demonstrated by Almeida et al. [51]. The apparatus used in the experiment involved a tomographic reconstruction of the density matrix and from it the concurrence by measuring polarisation entangled photon pairs produced in the process of spontaneous parametric down-conversion by a system composed of two adjacent nonlinear crystals. One of the crystals produced photon pairs with V-polarisation and the other produced pairs with H-polarisation. Parametric down conversion is a nonlinear process used to produce polarisation entangled photon pairs, which are manifested by the simultaneous or nearly simultaneous production of pairs of photons in momentum-conserving, phase matched modes. Since the pairs of polarised photons are spatially indistinguishable, they are described by a pure state

$$|\Phi\rangle = |\alpha||HH\rangle + |\beta|e^{i\delta}|VV\rangle. \tag{40}$$

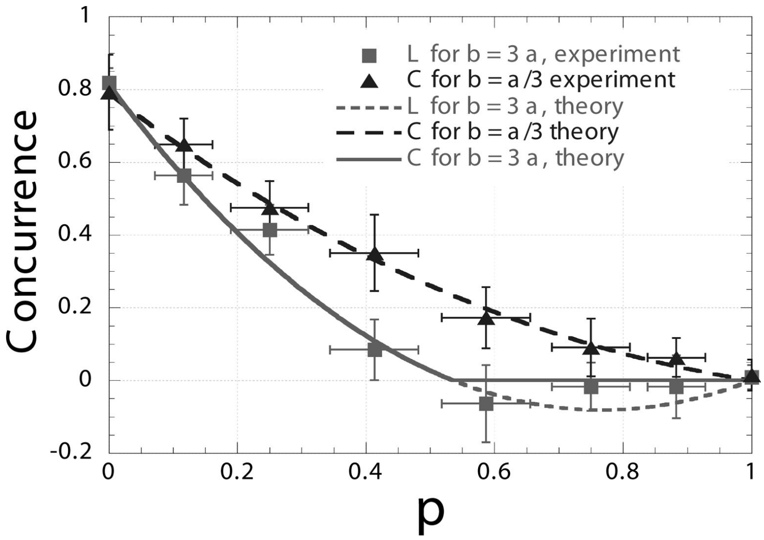

Fig. 7. Experimental results demonstrating the phenomenon of entanglement sudden death. The theoretical prediction for the concurrence is plotted as a function of $p = 1 - \exp(-\gamma t)$ for two values of the ratio $|\beta|/|\alpha|$: $|\beta|/|\alpha| = 3$ (solid line) and $|\beta|/|\alpha| = 0.333$ (dashed line). The squares and triangles are experimentally measured values for the concurrence.

where the coefficients $|\alpha|$ and $|\beta|$, and the phase δ were adjusted by applying half- and quarter-wave plates in the laser beam pumping the crystals to control the creation of pairs of a desired polarisation. In the experiment, they measured the decay of a single polarised beam, serving as a qubit, which was monitored by generating pairs of photons of the same polarisation and registering coincidence counts with one photon propagating through the interferometer and the other serving as a trigger.

Figure 7 shows the results of the measured concurrence for two different values of the ratio $|\beta|/|\alpha|$. The solid and dashed lines represent the theoretically predicted concurrence. The measured values for the concurrence are found to be in good agreement with the theoretical predictions. Thus, it was confirmed that entanglement may display the sudden death feature and that the spontaneous decay of the initial entanglement depends on the relation between the coefficients $|\alpha|$ and $|\beta|$. For $|\beta| < |\alpha|$ the entanglement decays exponentially in time, while for $|\beta| > |\alpha|$, the entanglement vanishes at finite time.

8. Revival of entanglement

We have already shown that under appropriate conditions, two initially entangled and afterwards not interacting qubits can become completely disentangled in a finite time. We have discussed this phenomenon for the case of independent atoms. In this section, we extend these analysis to include the direct interactions between the atoms and study in details the time evolution of an initial entanglement. As we shall see, the interactions impose strong qualitative changes to the time evolution of entanglement that the irreversible spontaneous decay can lead to a *revival* of the entanglement that has already been destroyed [24–26].

Consider again the initial state of the system given by Eq. (33), but now assume that the atoms can interact with each other. In this case, the initial values of the density matrix of the system are the same as for independent atoms, but the time evolution of the density matrix elements is different. It is now given by

$$\begin{aligned}
\rho_{ee}(t) &= q\,\mathrm{e}^{-2\gamma t}, \\
\rho_{eg}(t) &= \sqrt{q(1-q)}\,\mathrm{e}^{-(\gamma-2i\omega_0)t}, \\
\rho_{ss}(t) &= q\,\frac{\gamma+\gamma_{12}}{\gamma-\gamma_{12}}\left[\mathrm{e}^{(\gamma-\gamma_{12})t}-1\right]\mathrm{e}^{-2\gamma t}, \\
\rho_{aa}(t) &= q\,\frac{\gamma-\gamma_{12}}{\gamma+\gamma_{12}}\left[\mathrm{e}^{(\gamma+\gamma_{12})t}-1\right]\mathrm{e}^{-2\gamma t},
\end{aligned} \tag{41}$$

and $\rho_{gg}(t) = 1 - \rho_{ee}(t) - \rho_{ss}(t) - \rho_{aa}(t)$. As before for independent atoms, the remaining density matrix elements are equal to zero. By comparing Eq. (41) with Eq. (36) for independent atoms, we see that a major difference between the time evolution of the density matrix elements for independent and interacting atoms is that in the later the symmetric and antisymmetric states are populated with different rates. As a result, the transient buildup of the populations from $\rho_{ss}(0) = \rho_{aa}(0) = 0$ at $t = 0$ will lead to unequal populations of the states for all later times $t > 0$.

A direct consequence of unequal populations of the symmetric and antisymmetric states is seen in the transient evolution of an initial entanglement which may now depend on both criteria $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$, that are needed to construct the concurrence $\mathcal{C}(t)$, rather than on one of them. According to Eqs (12) and (34), a two-atom system initially prepared in the state (33) can be entangled according to the criterion $\mathcal{C}_1(t)$, and the degree to which the system is initially entangled is $\mathcal{C}_1(0) = 2\sqrt{q(1-q)}$. The other criterion, $\mathcal{C}_2(t)$, is negative at $t = 0$. Nevertheless, during the spontaneous evolution, a population builds up in the symmetric and antisymmetric states and according to Eq. (41), $\rho_{ss}(t) \neq \rho_{aa}(t)$ for all $t > 0$. As discussed in Sec. 3, an unequal population of the states gives rise to positive values of the criterion $\mathcal{C}_2(t)$ which then may result in an entanglement of the atoms. Of course, in order to create the entanglement, the population difference has to overweight the threshold term in the criterion $\mathcal{C}_2(t)$. If this is the case, we could see an entanglement having its origin in the correlation determined by the criterion $\mathcal{C}_2(t)$.

According to the above discussion, the concurrence, in general, is determined by both, $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$ criteria

$$\mathcal{C}(t) = \max\left\{0,\, \mathcal{C}_1(t),\, \mathcal{C}_2(t)\right\}, \tag{42}$$

where $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$, written in the basis of the collective states are of the form

$$\mathcal{C}_1(t) = 2|\rho_{eg}(t)| - \left[\rho_{ss}(t) + \rho_{aa}(t)\right], \tag{43}$$

and

$$\mathcal{C}_2(t) = |\rho_{ss}(t) - \rho_{aa}(t)| - 2\sqrt{\rho_{gg}(t)\rho_{ee}(t)}. \tag{44}$$

We point out again, that initially at $t = 0$, the criterion $\mathcal{C}_2(0)$ is negative, but it may rise to positive values during the evolution of the system. Thus, an entanglement can be generated during the spontaneous evolution of the system. This kind of entanglement is an example of spontaneously generated entanglement.

Figure 8 shows the deviation of the time evolution of the concurrence for two interacting atoms $(\gamma_{12} \neq 0)$ from that of independent atoms $(\gamma_{12} = 0)$. In both cases, the initial entanglement falls sharply in time. For independent atoms we observe the collapse of the entanglement without any revivals.

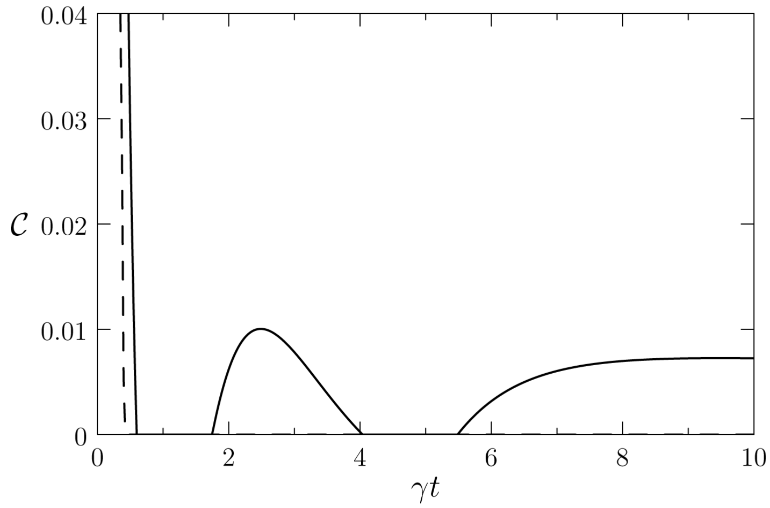

Fig. 8. Time evolution of the concurrence $\mathcal{C}(t)$ for the initial state (33) with $q = 0.9$. The solid line represents $\mathcal{C}(t)$ for the collective system with the interatomic separation $r_{12} = \lambda/20$. The dashed line shows $\mathcal{C}(t)$ for independent atoms, $\gamma_{12} = 0$.

However, for interacting atoms, the system collapses over a short time and remains disentangled until a time $t_r \approx 1.5/\gamma$ at which, somewhat counterintuitively, the entanglement revives. This revival then decays asymptotically to zero, but after some period of time it revives again. Thus, important features of the transient evolution of the entanglement is that there are two time intervals at which the entanglement vanishes and two time intervals at which the entanglement revives.

The origin of the sudden death and revivals of the entanglement can be explained in terms of the populations of the collective states and the rates with which the populations and the two-photon coherence decay. It is easily verified from Eq. (41) that at early times of $\gamma t \ll 1$, the population $\rho_{aa}(t) \approx 0$, but $\rho_{ss}(t)$ is large. Moreover, the two-photon coherence $|\rho_{eg}(t)|$ is also large at that short time. Thus, the entanglement behaviour can be understood entirely in terms of the population $\rho_{ss}(t)$ and the coherence $|\rho_{eg}(t)|$. In this connection, we compare in Fig. 9 the transient behaviour of the concurrence $\mathcal{C}(t)$, with the transient properties of the population $\rho_{ss}(t)$, and the coherence $\rho_{eg}(t)$ for the same choice of parameters as in Fig. 8. As can be seen from the graphs, the entanglement vanishes at the time where $\rho_{ss}(t)$ is maximal, and remains zero until the time t_r at which $\rho_{ss}(t)$ becomes smaller than $|\rho_{eg}(t)|$. Clearly, the first dead zone of the entanglement arises due to the significant accumulation of the population in the symmetric state.

The reason for the first revival of the entanglement is in the unequal decay rates of the population $\rho_{ss}(t)$ and the coherence $|\rho_{ge}(t)|$. According to the criterion $\mathcal{C}_1(t)$, entanglement would be generated if the sum of the populations of the single-photon collective states is small while at the same time the two-photon coherence is large. Since the coherence $\rho_{ge}(t)$ decays more slowly than the population of the symmetric state, and keeping in mind that the population of the antisymmetric state is negligibly small at that time, the two-photon coherence can become significant and entanglement generated over some period of time during the decay. Note, that this is the same coherence that produced the initial entanglement. Therefore, we may call the first revival as an "echo" of the initial entanglement that has been unmasked by destroying the population of the symmetric state.

The revival of the entanglement depends on the initial state of the system, which is determined by the parameter q. There is a range of values of q for which the concurrence is positive. This range defines the values of q for which the first revival occurs. The range depends on the values of γ_{12} and

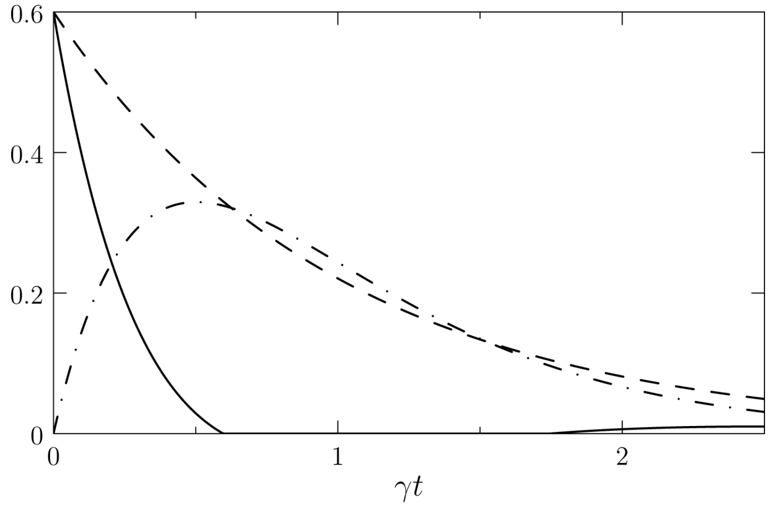

Fig. 9. Time evolution of the concurrence $\mathcal{C}(t)$ (solid line), the two-photon coherence $2|\rho_{eg}(t)|$ (dashed line) and the population $\rho_{ss}(t)$ (dashed-dotted line) plotted for the same parameters as in Fig. 8.

the revival is most pronounced for q over the range $0.85 \leqslant q \leqslant 0.95$. For values outside this range, no revival is observed. Note that the values of q at which a pronounced revival exists are close to one. This is not surprised because for $q > 1/2$ the system is initially inverted that increases the rate of spontaneous emission.

We may estimate approximate values for the first death and revival times of the entanglement. Using Eqs (12) and (41) we find that in the case of small inter-atomic separations, at which $\gamma_{12} \approx \gamma$, the entanglement criterion $\mathcal{C}_1(t)$ attains zeroth at times satisfying the relation

$$\gamma t \, \exp(-\gamma t) = \sqrt{\frac{1-q}{q}}, \tag{45}$$

which for $q > 0.85$ has two non-degenerate solutions, t_d and $t_r > t_d$. The solution t_d gives the death time of the entanglement beyond which it disappears. At this time, the atoms become disentangled and remain separable until the time t_r, at which the entanglement revives. The analytic expression (45) makes clear that for the parameters of Fig. 8, the system disentangles at $t_d = 0.6/\gamma$ and entangles again at the time $t_r = 1.7/\gamma$.

As we have seen in Fig. 8, the entanglement revives not once but twice, and the second revival occurs at long times. The long time entanglement has completely different origin than that at short times. At long times, all the density matrix elements are almost zero except $\rho_{aa}(t)$, which remains large even for long times due to the small decay rate of the antisymmetric state. These considerations imply that the long time entanglement comes from the population $\rho_{aa}(t)$ which determines positive values of the criterion $\mathcal{C}_2(t)$. The entanglement decays asymptotically with the reduced rate $\gamma - \gamma_{12}$, and vanishes at

$$t_{d2} \approx \frac{1}{\gamma} \text{arcsinh} \left(\sqrt{\frac{1-q}{q}} \frac{2\gamma}{\gamma - \gamma_{12}} \right). \tag{46}$$

This explicitly shows that lifetime of the entanglement depends on the collective damping parameter γ_{12} and approaches infinity when $\gamma_{12} \to \gamma$. The rich structure of the death and revival of entanglement is illustrated in Fig. 10, where the dependence of both, death time t_d and revival time t_r is plotted versus

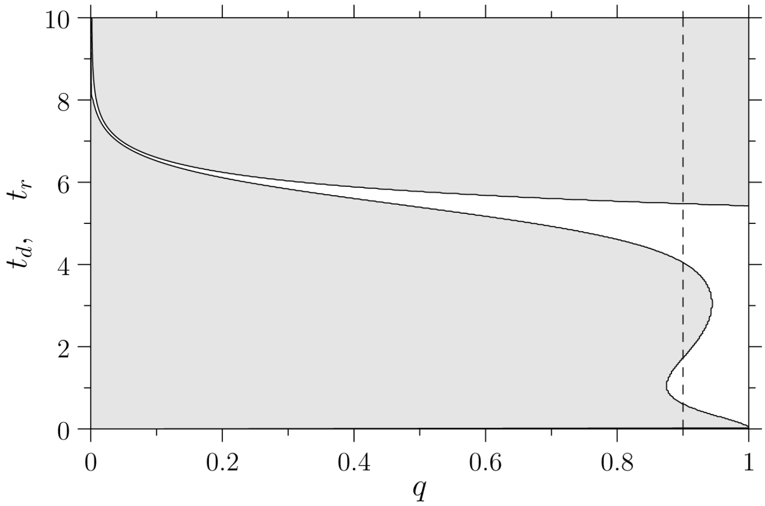

Fig. 10. Dependence of the entanglement death time t_d and revival time t_r on the parameter q, plotted in units of γ^{-1}, for the initial state given by (33) and $r_{12} = \lambda/20$. Shadowed area indicates presence of entanglement. The dashed line shows the trajectory for $q = 0.9$ illustrated in Fig. 8.

q for the whole range of values of the parameter q. The dashed line in Fig. 10 indicates the trajectory illustrated in Fig. 8. Gray area marks the area of nonzero concurrence, i.e., presence of entanglement.

We may conclude that the most interesting consequence of the collective damping is the possibility of entanglement revival. What is even more remarkable, the revival of entanglement occurs under the spontaneous evolution of the system without the presence of any external coherent fields. We may say that entanglement is reversible even if it evolves under the irreversible process.

Finally, we would like to point out that entanglement revival is associated not only with the collective damping, but may occur in a number of other situations. For example, it has been found that entanglement revival may occur under non-Markovian dynamics of two independent atoms coupled to local reservoirs [27]. Here, the memory effects of the non-Markovian dynamics may result in the "return" of the correlation from the reservoirs to the atoms. Entanglement revival has also been found in reversible systems whose dynamics are determined by the Jaynes-Cummings Hamiltonian [28,25,26]. In these models, an initial entanglement encoded into atoms undergoes a coherent transfer forth and back to the field modes, so it returns periodically to the atoms.

9. Thermal reservoir

So far, we have discussed entanglement evolution of the two-atom system interacting with the ordinary vacuum, that can be considered as the thermal reservoir at zero temperature. In this case the mean number of photons of the reservoir is zero, i.e., we have assumed $N = 0$. Now, we extend our discussion to the situation when the temperature of the reservoir is not zero. In this case we have to come back to equations (16). Before we calculate the time evolution, let us note that equations (16) for $N \neq 0$ have nonzero steady state solutions, which are given by [52]

$$\rho_{ee}(\infty) = \frac{N^2}{(2N+1)^2},$$

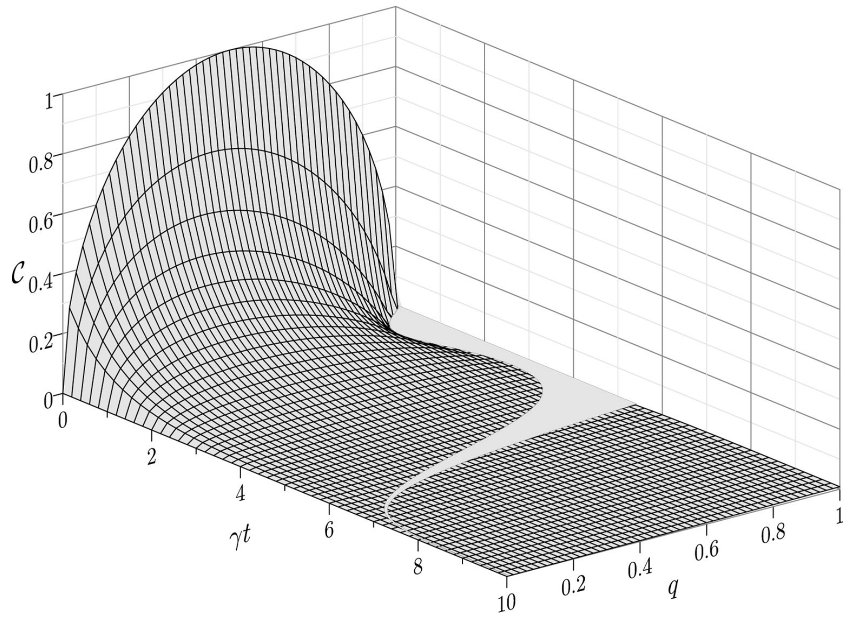

Fig. 11. Concurrence evolution for various values of the parameter q and the initial state (33) with $r_{12}/\lambda = 1/20$ and $N = 0$.

$$\rho_{ss}(\infty) = \rho_{aa}(\infty) = \frac{N(N+1)}{(2N+1)^2}, \qquad \rho_{gg}(\infty) = \frac{(N+1)^2}{(2N+1)^2}, \tag{47}$$

and the steady state values of the coherences $\rho_{as}(\infty)$ and $\rho_{eg}(\infty)$ are zeros. From the solutions (47) and the entanglement criteria (12) and (13), it is immediately seen that $\mathcal{C}_1(t)$ must become negative at some finite time t_d. Moreover, since $\sqrt{\rho_{ee}(\infty)\rho_{gg}(\infty)} = \rho_{ss}(\infty) = \rho_{aa}(\infty)$, it means that $\mathcal{C}_2(t)$ must also become negative at some finite time t_d. Note, that the steady-state solutions (47) do not depend on the collective parameters γ_{12} and Ω_{12}, which means that independently of the interatomic distance, there is always entanglement sudden death if the mean number of photons of the reservoir is different from zero. This confirms the results found earlier for atoms coupled to local, separate reservoirs [53,54]. Thus, one can conclude that for the long time behaviour of the system in a thermal reservoir, it is not important whether the atoms behave collectively or not. The evolution of the concurrence for the initial state given by (33), for the interatomic distance $r_{12} = \lambda/20$ and the reservoir at zero temperature (vacuum reservoir) is illustrated in Fig. 11. It shows the sudden death and revival of entanglement for the whole range of q values. This is in contrast to the situation for two independent atoms for which there is an asymptotic behaviour of entanglement for $q < 0.5$. The death and revival times depend on the values of q and the interatomic distance r_{12}, which has been discussed in details in the previous section and illustrated in Fig. 10.

When the temperature of the reservoir is not zero, however, the evolution of entanglement changes dramatically even if the mean number of photons of the reservoir is small [55]. In Fig. 12 we illustrate the evolution of entanglement for the same initial state and the same values of the other parameters but for the mean number of photons of the reservoir being $N = 0.01$. It is clearly seen that the reservoir with finite temperature degrades entaglement very fastly, and we observe sudden death of entanglement

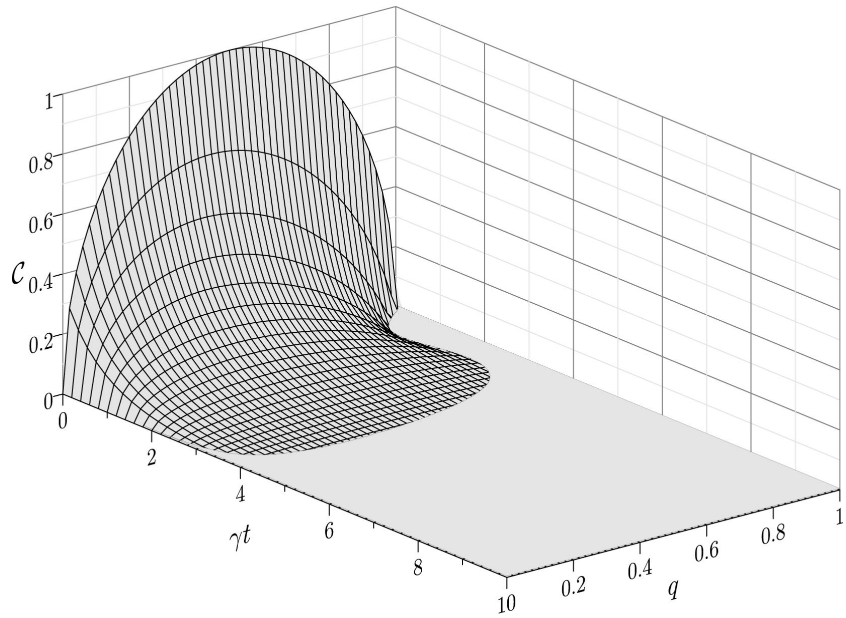

Fig. 12. Same as Fig. 10, but for $N = 0.01$.

for any value of q. Moreover, there is no revival of entanglement even for not very high mean number of photons of the reservoir.

We may conclude that the phenomenon of sudden death of entanglement appears to be a general feature of thermal reservoirs with nonzero temperature. It is rather the asymptotic decay of concurrence that is unusual and takes place only when the reservoir is the ordinary vacuum state.

10. Summary

We have reviewed the recent research on entanglement creation and disentanglement of two-atom systems coupled to a noisy environment. Particular attention has been paid to unexpected behaviours of entanglement, such as entanglement sudden death, spontaneous revival and sudden birth of entanglement. We have explored the role of the irreversible process of spontaneous emission in creation of entanglement and in disentanglement of two atoms. Simple analysis have shown that spontaneous emission does not necessary lead to disentanglement of an initial entangled atoms. Under some circumstances, this irreversible process can entangle already disentangled atoms. We have also discussed the effect of thermal fluctuations on the evolution of entanglement.

References

[1] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.
[2] C. Schori, B. Julsgaard, J.L. Sorensen and E.S. Polzik, *Phys Rev Lett* **89** (2002), 057903.
[3] A. Dantan, G. Reinaudi, A. Sinatra, F. Laloe, E. Giacobino and M. Pinard, *Phys Rev Lett* **95** (2005), 123002.
[4] J.F. Sherson, H. Krauter, R.K. Olsson, B. Julsgaard, K. Hammerer, I. Cirac and E.S. Polzik, *Nature* **443** (2006), 557.
[5] D. Braun, *Phys Rev Lett* **89** (2002), 277901.

[6] M.S. Kim, J. Lee, D. Ahn and P.L. Knight, *Phys Rev A* **65** (2002), 040101(R).
[7] X.X. Yi, C.S. Yu, L. Zhou and H.S. Song, *Phys Rev A* **68** (2003), 052304.
[8] A.M. Basharov, *JETP Lett* **75** (2002), 123.
[9] L. Jakóbczyk, *J Phys A: Math Theor* **35** (2002), 6383.
[10] Z. Ficek and R. Tanaś, *J Mod Opt* **50** (2003), 2765.
[11] F. Verstraete, M.M. Wolf and J.I. Cirac, *Nature Physics* **10** (2009), 1.
[12] K.-L. Liu and H.-S. Goan, *Phys Rev A* **76** (2007), 022312.
[13] C. Hörhammer and H. Büttner, *Phys Rev A* **77** (2008), 042305.
[14] R.H. Dicke, *Phys Rev* **93** (1954), 99.
[15] Z. Ficek and R. Tanaś, *Phys Rep* **372** (2002), 369.
[16] Z. Ficek and S. Swain, *Quantum Interference and Coherence: Theory and Experiments*, Springer, New York, 2005.
[17] R.H. Lehmberg, *Phys Rev A* **2** (1970), 883, 889.
[18] G.S. Agarwal, Quantum Statistical Theories of Spontaneous Emission and their Relation to other Approaches, edited by G. Höhler, *Springer Tracts in Modern Physics* **70** Berlin, Springer–Verlag, 1974.
[19] R. Tanaś and Z. Ficek, *J Opt B* **6** (2004), S90.
[20] T. Yu and J.H. Eberly, *Phys Rev Lett* **93** (2004), 140404.
[21] J.H. Eberly and T. Yu, *Science* **316** (2007), 555; T. Yu and J.H. Eberly, *Science* **323** (2009), 598.
[22] K. Życzkowski, P. Horodecki, M. Horodecki and R. Horodecki, *Phys Rev A* **65** (2001), 012101.
[23] Z. Ficek, *App Math Inf Sci* **3** (2009), 375.
[24] Z. Ficek and R. Tanaś, *Phys Rev A* **74** (2006), 024304.
[25] M. Yonac, T. Yu, and J.H. Eberly, *J Phys B: At Mol Opt Phys* **39** (2006), S621.
[26] M. Yonac, T. Yu and J.H. Eberly, *J Phys B: At Mol Opt Phys* **40** (2007), S45.
[27] J.-H. An and W.-M. Zhang, *Phys Rev A* **76** (2007), 042127; X. Cao and H. Zheng, *Phys Rev A* **77** (2008), 022320; F.-Q. Wang, Z.-M. Zhang and R.-S. Liang, *Phys Rev A* **78** (2008), 062318; Z.Y. Xu and M. Feng, *Phys Lett A* **373** (2009), 1906; X.F. Cao and H. Zheng, *Eur Phys J B* **68** (2009), 209; X.L. Huang, L.C. Wang and X.X. Yi, *Int J Quant Inf* **7** (2009), 385; B. Bellomo, R. Lo Franco and G. Compagno, *Phys Rev A* **77** (2008), 032342; C.H. Chou, T. Yu and B.L. Hu, *Phys Rev E* **77** (2008), 011112; Y.J. Zhang, Z.X. Man and Y.J. Xia, *Eur Phys Jour D* **55** (2009), 173; Z.-Z. Li, X.-T. Liang and X.-Y. Pan, *Phys Lett A* **373** (2009), 4028; J. Zhou, C. Wu, M. Zhu and H. Guo, *J Phys B: At Mol Opt Phys* **42** (2009), 215505.
[28] S. Chan, M.D. Reid and Z. Ficek, *J Phys B: At Mol Opt Phys* **42** (2009), 065507.
[29] Z. Ficek and R. Tanaś, *Phys Rev A* **77** (2008), 054301.
[30] C.E. Lopez, G. Romero, F. Lastra, E. Solano and J.C. Retamal, *Phys Rev Lett* **101** (2008), 080503.
[31] S. Kielich, *Acta Phys Polonica* **19** (1960), 149; *ibidem* **23** (1963), 321; *ibidem* **25** (1964), 85.
[32] S. Kielich, *Acta Phys Polonica* **32** (1967), 297; S. Kielich, J.R. Lalanne and F.B. Martin, *Phys Rev Lett* **26** (1971), 1295.
[33] A. Piekara and S. Kielich, *J Chem Phys* **29** (1958), 1297; S. Kielich and A. Piekara, *Acta Phys Polonica* **18** (1959), 439; S. Kielich, *Acta Phys Polonica* **37** (1970), 719.
[34] *Modern Nonlinear Optics*, Vol. 85 of *Advances in Chemical Physics*, eds. M. Evans and S. Kielich, (Wiley, New York, 1993), Part 1; *Modern Nonlinear Optics*, Vol. 119 of *Advances in Chemical Physics*, ed. M. Evans, (Wiley, New York, 2001), Part 1.
[35] S.I. Schmid and J. Evers, *Rev A* **77** (2008), 013822.
[36] M.J. Stephen, *J Chem Phys* **40** (1964), 669.
[37] T. Yu and J.H. Eberly, *Quant Inf and Comp* **7** (2007), 459.
[38] B. Corn and T. Yu, *Quant Inf Process* **8** (2009), 565.
[39] S. Hill and W.K. Wootters, *Phys Rev Lett* **78** (1997), 5022.
[40] W.K. Wootters, *Phys Rev Lett* **80** (1998), 2245.
[41] Z. Ficek, R. Tanaś and S. Kielich, *Optics Commun* **36** (1981), 121.
[42] Z. Ficek, R. Tanaś and S. Kielich, *Optica Acta* **30** (1983), 713.
[43] H.S. Freedhoff, *Phys Rev A* **26** (1982), 684.
[44] L. Jakóbczyk and A. Jamróz, *Phys Lett A* **333** (2004), 35; F. Lastra, G. Romero, C.E. Lopez, M. Frana Santos and J.C. Retamal, *Phys Rev A* **75** (2007), 062324; K. Ann and G. Jaeger, *Phys Rev A* **76** (2007), 044101; I. Sainz and G. Björk, *Phys Rev A* **76** (2007), 042313; J.-H. Huang and S.-Y. Zhu, *Phys Rev A* **76** (2007), 062322; A. Vaglica and G. Vetri, *Phys Rev A* **75** (2007), 062120; M. Macovei and G.-X. Li, *Phys Rev A* **76** (2007), 023818; L. Aolita, R. Chaves, D. Cavalcanti, A. Acn and L. Davidovich, *Phys Rev Lett* **100** (2008), 080501; I. Sainz and G. Björk, *Phys Rev A* **77** (2008), 052307; A. Salles, F. de Melo, M.P. Almeida, M. Hor-Meyll, S.P. Walborn, P.H. Souto Ribeiro and L. Davidovich, *Phys Rev A* **78** (2008), 022322; N. Yamamoto, H.I. Nurdin, M.R. James and I.R. Petersen, *Phys Rev A* **78** (2008), 042339; B. Bellomo, R.L. Franco, S. Maniscalco and G. Compagno, *Phys Rev A* **78** (2008), 060302; S.-Y. Lin, C.-H. Chou and B.L. Hu, *Phys Rev D* **78** (2008), 125025; J. Leon and C. Sabin, *Phys Rev A* **79** (2009), 012301; L. Mazzola, S. Maniscalco, J. Piilo, K.-A. Suominen and B.M. Garraway, *Phys Rev A* **79** (2009), 042302; W. Song, L. Chen and S.-L. Zhu, *Phys Rev A* **80** (2009), 012331; M. Al-Amri, G.-X. Li, R. Tan and M.S. Zubairy, *Phys Rev A* **80** (2009), 022314; L. Derkacz and L.

Jakóbczyk, *Phys Rev A* **76** (2007), 042304; X.Q. Yan, *Mod Phys Lett B* **22** (2008), 2503; M. Scala, R. Migliore and A. Messina, *J Phys A: Math Theor* **41** (2008), 435304; M. Ban, *Opt Commun* **281** (2008), 3943; Z.X. Man, Y.J. Xia and N.B. An, *J Phys B: At Mol Opt Phys* **41** (2008), 085503; M. Abdel-Aty, *Eur Phys Jour D* **46** (2008), 537; H.T. Cui, K. Li and X.X. Yi, *Phys Lett A* **365** (2007), 44; Y.J. Zhang and Y.J. Xia, *Int J Quant Inf* **7** (2009), 949; J.S. Zhang and J.B. Xu, *Opt Commun* **282** (2009), 3652; A. Kowalewska-Kudlaszyk and W. Leoński, *J Opt Soc Am B* **26** (2009), 1289; K. Xia, M. Macovei, J. Evers and C.H. Keitel, *Phys Rev B* **79** (2009), 024519.

[45] C.-Y. Lai, J.-T. Hung, C.-Y. Mou and P. Chen, *Phys Rev B* **77** (2008), 205419.

[46] J. Dajka and J. Łuczka, *Phys Rev A* **77** (2008), 062303.

[47] J.P. Paz and A.J. Roncaglia, *Phys Rev Lett* **100** (2008), 220401.

[48] M. Hernandez and M. Orszag, *Phys Rev A* **78** (2008), 042114.

[49] J.P. Paz and A.J. Roncaglia, *Phys Rev A* **79** (2009), 032102.

[50] G.-X. Li, L.H. Sun and Z. Ficek, *J Phys B: At Mol Opt Phys* **43** (2010), 135501.

[51] M.P. Almeida, F. de Melo, M. Hor-Meyll, A. Salles, S.P. Walborn, P.H.S. Ribeiro and L. Davidovich, *Science* **316** (2007), 579.

[52] R. Tanaś and Z. Ficek, *J Opt B* **6** (2004), S610.

[53] M. Ikram, F.-L. Li and M.S. Zubairy, *Phys Rev A* **75** (2007), 062336; R. Tahira, M. Ikram, T. Azim and M.S. Zubairy, *J Phys B: At Mol Opt Phys* **41** (2008), 205501.

[54] A. Al-Qasimi and D.F.V. James, *Phys Rev A* **77** (2008), 012117.

[55] L. Jakóbczyk, *J Phys B: At Mol Opt Phys* **43** (2010), 015502.