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Stationary two-atom entanglement induced by nonclassical two-photon correlations

R Tanaś[1] and Z Ficek[2]

[1] Nonlinear Optics Division, Institute of Physics, Adam Mickiewicz University, Poznań, Poland
[2] Department of Physics, School of Physical Sciences, The University of Queensland, Brisbane, QLD 4072, Australia

E-mail: tanas@kielich.amu.edu.pl

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Abstract
A system of two two-level atoms interacting with a squeezed vacuum field can exhibit stationary entanglement associated with nonclassical two-photon correlations characteristic of the squeezed vacuum field. The amount of entanglement present in the system is quantified by the well known measure of entanglement called concurrence. We find analytical formulae describing the concurrence for two identical and nonidentical atoms and show that it is possible to obtain a large degree of steady-state entanglement in the system. Necessary conditions for the entanglement are nonclassical two-photon correlations and nonzero collective decay. It is shown that nonidentical atoms are a better source of stationary entanglement than identical atoms. We discuss the optimal physical conditions for creating entanglement in the system; in particular, it is shown that there is an optimal and rather small value of the mean photon number required for creating entanglement.

Keywords: entanglement, squeezing, superpositions

1. Introduction

Entanglement between separate quantum systems is one of the key problems in quantum mechanics. A number of interesting concepts and methods for creating entanglement have been proposed involving trapped and cooled ions or neutral atoms [1–8]. Of particular interest is generation of entangled states in two-atom systems, since they can represent two qubits, the building blocks of the quantum gates that are essential to implement quantum protocols in quantum information processing. It has been shown that entangled states in a two-atom system can be created by a continuous driving of the atoms with a coherent or chaotic thermal field [5, 9–12], or by a pulse excitation followed by a continuous observation of radiative decay [13–15]. Moreover, the effect of spontaneous emission on initially prepared entangled state has also been discussed [16–19]. These studies, however, have been limited to the small sample (Dicke) model [20] or the situation involving noninteracting atoms strongly coupled to a cavity mode. The difficulty of the Dicke model is that it does not include the dipole–dipole interaction among the atoms and does not correspond to realistic experimental situations of atoms located (trapped) at different positions. In fact, the model corresponds to a very specific geometrical configuration of the atoms confined to a volume much smaller compared with the atomic resonant wavelength (the small-sample model). The present atom trapping and cooling techniques can trap two atoms at distances of order of a resonant wavelength [21–23], which makes questionable the applicability of the Dicke model to physical systems.

Recently, we have shown [24] that spontaneous emission from two spatially separated atoms can lead to a transient entanglement of initially unentangled atoms. This result contrasts with the Dicke model where spontaneous emission cannot produce entanglement from initially unentangled atoms [10, 18]. We have also found [25] analytical results for

two measures of entanglement and the relation between them for the two-atom system radiating by spontaneous emission for quite broad range of initial conditions.

In this paper we study the creation of a stationary entanglement in a system of two identical as well as nonidentical two-level atoms separated by an arbitrary distance r_{12} and interacting with a squeezed vacuum. The squeezed vacuum appears here as a source of nonclassical two-photon coherences, essential for the creation of the stationary entanglement. We use the master equation to describe the evolution of the system and find the steady-state solutions for the atomic variables. We present analytical results for *concurrence* which is well known and calculable measure of entanglement. We find a surprising result that nonidentical atoms with significantly different transition frequencies can exhibit a larger entanglement than identical atoms. Under some conditions, the nonidentical atoms can be maximally entangled with the value of the concurrence equal to unity.

2. Master equation

We consider a system of two two-level atoms at fixed positions $\mathbf{r}_1$ and $\mathbf{r}_2$ and coupled to the radiation field, whose the modes are in a squeezed vacuum state. Each atom has energy levels $|g_i\rangle$ and $|e_i\rangle$ $(i = 1, 2)$ such that $E_{e_i} - E_{g_i} = \hbar\omega_i$, transition dipole moment $\boldsymbol{\mu}_i$, which we assume equal for both atoms.

We analyse separately the dynamics of identical and nonidentical atoms. In the case of nonidentical atoms, we assume different transition frequencies ω_1 and ω_2 such that $\Delta = (\omega_2 - \omega_1)/2 \ll \omega_0 = (\omega_1 + \omega_2)/2$, so that the rotating-wave approximation can be applied to calculate the dynamics of the system.

The system can be described by the reduced density operator ρ which, in a Schrödinger picture, satisfies the master equation [26]

$$\begin{aligned}\frac{\partial\rho}{\partial t} = &-\frac{1}{2}\sum_{i,j=1}^{2}\Gamma_{ij}\,(1+N)\left(\rho S_i^+S_j^- + S_i^+S_j^-\rho - 2S_j^-\rho S_i^+\right)\\ &-\tfrac{1}{2}\sum_{i,j=1}^{2}\Gamma_{ij}N\left(\rho S_i^-S_j^+ + S_i^-S_j^+\rho - 2S_j^+\rho S_i^-\right)\\ &+\tfrac{1}{2}\sum_{i,j=1}^{2}\Gamma_{ij}M\left(\rho S_i^+S_j^+ + S_i^+S_j^+\rho - 2S_j^+\rho S_i^+\right)\mathrm{e}^{-2\mathrm{i}\omega_s t}\\ &+\tfrac{1}{2}\sum_{i,j=1}^{2}\Gamma_{ij}M^*\left(\rho S_i^-S_j^- + S_i^-S_j^-\rho - 2S_j^-\rho S_i^-\right)\mathrm{e}^{2\mathrm{i}\omega_s t}\\ &-\mathrm{i}\sum_{i=1}^{2}\omega_i\left[S_i^z,\rho\right] - \mathrm{i}\sum_{i\neq j}^{2}\Omega_{ij}\left[S_i^+S_j^-,\rho\right].\end{aligned} \tag{1}$$

Here, S_i^+ and S_i^- are the raising and lowering operators, respectively, of the ith atom, N and $M = |M|\exp(\mathrm{i}\phi_s)$ characterize squeezing such that $|M|^2 \leqslant N(N+1)$, where the equality holds for a minimum-uncertainty squeezed state, ϕ_s is the squeezing phase and ω_s is the carrier frequency of the squeezed vacuum.

The parameters Γ_{ij}, which appear in equation (1), are spontaneous emission rates, such that

$$\Gamma_{ii} \equiv \Gamma = \frac{\omega_0^3|\boldsymbol{\mu}|^2}{3\pi\varepsilon_0\hbar c^3}, \qquad (i = 1, 2) \tag{2}$$

is the spontaneous emission rate of the ith atom, assumed to be equal for both atoms, and

$$\begin{aligned}\Gamma_{12} = \Gamma_{21} = \frac{3}{2}\sqrt{\Gamma}\Bigg\{&\left[1-(\hat{\boldsymbol{\mu}}\cdot\hat{\mathbf{r}}_{12})^2\right]\frac{\sin(k_0r_{12})}{k_0r_{12}}\\ &+\left[1-3(\hat{\boldsymbol{\mu}}\cdot\hat{\mathbf{r}}_{12})^2\right]\left[\frac{\cos(k_0r_{12})}{(k_0r_{12})^2}-\frac{\sin(k_0r_{12})}{(k_0r_{12})^3}\right]\Bigg\},\end{aligned} \tag{3}$$

are collective spontaneous emission rates arising from the coupling between the atoms through the vacuum field [36–38], and

$$\begin{aligned}\Omega_{12} = \frac{3}{4}\Gamma\Bigg\{&-\left[1-(\hat{\boldsymbol{\mu}}\cdot\hat{\mathbf{r}}_{12})^2\right]\frac{\cos(k_0r_{12})}{k_0r_{12}}\\ &+\left[1-3\left(\hat{\boldsymbol{\mu}}\cdot\hat{\mathbf{r}}_{ij}\right)^2\right]\left[\frac{\sin(k_0r_{12})}{(k_0r_{12})^2}+\frac{\cos(k_0r_{12})}{(k_0r_{12})^3}\right]\Bigg\}\end{aligned} \tag{4}$$

represents the vacuum induced coherent (dipole–dipole) interaction between the atoms.

In expressions (3) and (4), $\hat{\boldsymbol{\mu}}$ and $\hat{\mathbf{r}}_{12}$ are unit vectors along the atomic transition dipole moments and the vector $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$, respectively, and $k_0 = \omega_0/c$. Later on, we will assume that the atomic dipole moments $\boldsymbol{\mu}$ are perpendicular to the vector $\mathbf{r}_{12}$ joining the two atoms.

The collective parameters Γ_{12} and Ω_{12}, which arise from the mutual interaction between the atoms, significantly modify the master equation of a two-atom system. The parameter Γ_{12} introduces a coupling between the atoms through the vacuum field such that the spontaneous emission from one of the atoms influences the spontaneous emission from the other. The dipole–dipole interaction term Ω_{12} introduces a coherent coupling between the atoms. Owing to the dipole–dipole interaction, the population is coherently transferred back and forth from one atom to the other.

The two-atom system can be described in the basis of product states of the individual atoms

$$\begin{aligned}|1\rangle &= |g_1\rangle\otimes|g_2\rangle = |g\rangle,\\ |2\rangle &= |e_1\rangle\otimes|e_2\rangle = |e\rangle,\\ |3\rangle &= |g_1\rangle\otimes|e_2\rangle,\\ |4\rangle &= |e_1\rangle\otimes|g_2\rangle,\end{aligned} \tag{5}$$

where $|g_i\rangle$ and $|e_i\rangle$ (for $i = 1, 2$) are the ground and excited states of the individual atoms. In this basis, the two-atom system behaves as a single four-level system whose density matrix can be written as a 4×4 matrix. Due to the presence of the dipole–dipole interaction Ω_{12} it is often convenient to introduce collective atomic states that are eigenstates of the system of two identical atoms including the dipole–dipole interaction [20]. They are symmetric and antisymmetric superpositions of the product atomic states $|3\rangle$ and $|4\rangle$, given by

$$\begin{aligned}|s\rangle &= \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle),\\ |a\rangle &= -\frac{1}{\sqrt{2}}(|3\rangle - |4\rangle),\end{aligned} \tag{6}$$

and the states $|g\rangle$ and $|e\rangle$ remain unchanged.

On introducing the collective states (6) and using the master equation (1) we are able to write down equations of motion for the matrix elements of the density matrix for two atoms in a squeezed vacuum. We treat separately two cases of identical and nonidentical atoms.

2.1. Identical atoms

For identical atoms separated by an arbitrary distance r_{12} and interacting with a squeezed vacuum field of the carrier frequency $\omega_s = \omega_0$, we obtain the following set of coupled equations of motion [26]:

$$\begin{aligned}
\dot{\rho}_{ee} &= -2\Gamma\,(N+1)\,\rho_{ee} + N\big[(\Gamma+\Gamma_{12})\,\rho_{ss} \\
&\quad + (\Gamma-\Gamma_{12})\,\rho_{aa}\big] + \Gamma_{12}|M|\rho_u, \\
\dot{\rho}_{ss} &= (\Gamma+\Gamma_{12})\,\big\{N - (3N+1)\,\rho_{ss} \\
&\quad - N\rho_{aa} + \rho_{ee} - |M|\rho_u\big\}, \\
\dot{\rho}_{aa} &= (\Gamma-\Gamma_{12})\,\big\{N - (3N+1)\,\rho_{aa} \\
&\quad - N\rho_{ss} + \rho_{ee} + |M|\rho_u\big\}, \\
\dot{\rho}_u &= 2\Gamma_{12}|M| - (2N+1)\,\Gamma\rho_u - 2|M| \\
&\quad \times \big[(\Gamma+2\Gamma_{12})\,\rho_{ss} - (\Gamma-2\Gamma_{12})\,\rho_{aa}\big],
\end{aligned} \tag{7}$$

where $\rho_u = \rho_{eg}\exp(-\mathrm{i}\phi_s) + \rho_{ge}\exp(\mathrm{i}\phi_s)$. It is seen from equation (7) that the evolution of the populations depends on the two-photon coherences ρ_{eg} and ρ_{ge}, which can modify population distribution between the collective states. The coherences can also create superposition (entangled) states involving only the ground $|g\rangle$ and the upper $|e\rangle$ states. The evolution of the populations depends on Γ_{12}, but is completely independent of the dipole–dipole interaction Ω_{12}.

The steady-state solutions of equations (7) depend on whether $\Gamma_{12} = \Gamma$ or $\Gamma_{12} \neq \Gamma$. For two atoms separated by an arbitrary distance r_{12}, $\Gamma_{12} \neq \Gamma$, and then the steady-state solutions of equations (7) are

$$\begin{aligned}
\rho_{ee} &= \frac{N^2\left[(2N+1)^2 - 4|M|^2\right] + |M|^2\gamma_{12}^2}{(2N+1)^4 - 4|M|^2\left[(2N+1)^2 - \gamma_{12}^2\right]}, \\
\rho_{ss} &= \frac{N(N+1)\left[(2N+1)^2 - 4|M|^2\right] + |M|^2\gamma_{12}(\gamma_{12}-2)}{(2N+1)^4 - 4|M|^2\left[(2N+1)^2 - \gamma_{12}^2\right]}, \\
\rho_{aa} &= \frac{N(N+1)\left[(2N+1)^2 - 4|M|^2\right] + |M|^2\gamma_{12}(\gamma_{12}+2)}{(2N+1)^4 - 4|M|^2\left[(2N+1)^2 - \gamma_{12}^2\right]}, \\
\rho_u &= \frac{2\,(2N+1)\,|M|\gamma_{12}}{(2N+1)^4 - 4|M|^2\left[(2N+1)^2 - \gamma_{12}^2\right]},
\end{aligned} \tag{8}$$

where $\gamma_{12} = \Gamma_{12}/\Gamma$ is the dimensionless collective damping parameter. This result shows that all the collective states are populated in the steady state even for small interatomic separations ($\gamma_{12} \approx 1$). For large interatomic separations $\gamma_{12} \approx 0$, and then the symmetric and antisymmetric states are equally populated. When the interatomic separation decreases, the population of the state $|a\rangle$ increases, whereas the population of the state $|s\rangle$ decreases and $\rho_{ss} = 0$ for very small interatomic separations. This effect results from the enhanced $(\Gamma + \Gamma_{12})$ damping rate of the symmetric state, as seen from equation (7). The two-photon coherences, represented by ρ_u, affect the population distribution only when both $|M|$ and γ_{12} are nonzero. The coherences are crucial for getting entanglement in the system. Of particular interest are population distributions for maximally squeezed fields with $|M| = \sqrt{N(N+1)}$. In this case, the factor $(2N+1)^2 - 4|M|^2 = 1$, and then the solutions (8) take a very simple form

$$\begin{aligned}
\rho_{ee} &= \frac{N^2 + N(N+1)\gamma_{12}^2}{1 + 4N(N+1)(1+\gamma_{12}^2)}, \\
\rho_{ss} &= \frac{N(N+1)(1-\gamma_{12})^2}{1 + 4N(N+1)(1+\gamma_{12}^2)}, \\
\rho_{aa} &= \frac{N(N+1)(1+\gamma_{12})^2}{1 + 4N(N+1)(1+\gamma_{12}^2)}, \\
\rho_u &= \frac{2\sqrt{N(N+1)}(2N+1)\gamma_{12}}{1 + 4N(N+1)(1+\gamma_{12}^2)}.
\end{aligned} \tag{9}$$

The solutions (8) and (9) will be used for calculation of the degree of entanglement present in the system. For further reference it is important to note that the sum of the populations $\rho_{ss} + \rho_{aa}$ tends to 0.5 as the quantity $N(N+1)(1+\gamma_{12}^2)$ becomes much greater than one, which means that for large values of the mean number of photons N one-half of the population goes eventually to the states $|s\rangle$ and $|a\rangle$.

2.2. Nonidentical atoms

The population distribution is quite different when the atoms are nonidentical with $\Delta = (\omega_2 - \omega_1)/2 \neq 0$. As before for the identical atoms, we use the master equation (1) and find four coupled differential equations for the density matrix elements with time-dependent coefficients oscillating at frequencies $\exp(\pm\mathrm{i}\Delta t)$ and $\exp[\pm 2\mathrm{i}(\omega_s - \omega_0)t + \phi_s]$. If we tune the squeezed vacuum field to the middle of the frequency difference between the atomic frequencies, i.e., $\omega_s = (\omega_1 + \omega_2)/2$, the terms proportional to $\exp[\pm 2\mathrm{i}(\omega_s - \omega_0)t + \phi_s]$ become stationary in time. None of the other time-dependent components is resonant with the frequency of the squeezed vacuum field. Consequently, for $\Delta \gg \Gamma$, the time-dependent components oscillate rapidly in time and average to zero over long times. Therefore, we can make a secular approximation in which we ignore the rapidly oscillating terms and obtain the following equations of motion:

$$\begin{aligned}
\dot{\rho}_{ee} &= -2\Gamma\,(N+1)\,\rho_{ee} + N\Gamma\,(\rho_{ss} + \rho_{aa}) + \Gamma_{12}|M|\rho_u, \\
\dot{\rho}_{ss} &= \Gamma\,[N - (3N+1)\,\rho_{ss} - N\rho_{aa} + \rho_{ee}] - \Gamma_{12}|M|\rho_u, \\
\dot{\rho}_{aa} &= \Gamma\,[N - (3N+1)\,\rho_{aa} - N\rho_{ss} + \rho_{ee}] - \Gamma_{12}|M|\rho_u, \\
\dot{\rho}_u &= 2\Gamma_{12}|M| - (2N+1)\,\Gamma\rho_u - 4\Gamma_{12}|M|\,(\rho_{ss} + \rho_{aa})\,.
\end{aligned} \tag{10}$$

The steady-state solutions of equations (10) are

$$\begin{aligned}
\rho_{ee} &= \frac{1}{4}\left\{\frac{(2N-1)}{2N+1} + \frac{1}{\left[(2N+1)^2 - 4|M|^2\gamma_{12}^2\right]}\right\}, \\
\rho_{ss} = \rho_{aa} &= \frac{1}{4}\left\{1 - \frac{1}{\left[(2N+1)^2 - 4|M|^2\gamma_{12}^2\right]}\right\}, \\
\rho_u &= \frac{2|M|\gamma_{12}}{(2N+1)\left[(2N+1)^2 - 4|M|^2\gamma_{12}^2\right]}.
\end{aligned} \tag{11}$$

Equations (11) are quite different from equations (8) and show that in the case of nonidentical atoms the symmetric and antisymmetric states are equally populated. This fact will be crucial in the entanglement creation in the system

and results from the equal damping rates of the symmetric and antisymmetric states, as seen from equation (10). For maximally squeezed vacuum with $|M| = \sqrt{N(N+1)}$ the solutions (11) simplify to

$$\rho_{ee} = \frac{1}{4}\left\{\frac{2N-1}{2N+1} + \frac{1}{1+4N(N+1)(1-\gamma_{12}^2)}\right\},$$
$$\rho_{ss} = \rho_{aa} = \frac{N(N+1)(1-\gamma_{12}^2)}{1+4N(N+1)(1-\gamma_{12}^2)}, \tag{12}$$
$$\rho_u = \frac{2\sqrt{N(N+1)}\gamma_{12}}{(2N+1)[1+4N(N+1)(1-\gamma_{12}^2)]}.$$

From equations (12) it is evident that for the Dicke model, for which $\gamma_{12} = 1$, the populations of the symmetric and asymmetric states are both zero and ρ_u tends to unity for large N. In real situations the separation of the atoms is nonzero, and we always have $\gamma_{12} < 1$, which means that for $N(N+1)(1-\gamma_{12}^2) \gg 1$ the populations ρ_{ss} and ρ_{aa} both approach the value 0.25, i.e., $\rho_{ss} + \rho_{aa} \approx 0.5$. That is, for sufficiently large N one-half of the population is transferred to the block spanned by the symmetric and antisymmetric states, similarly to the identical atoms. There is, however, one essential difference between the identical and nonidentical atoms, which is the scale for the saturation—much slower for nonidentical atoms. For γ_{12} very close to unity, very large intensities N are required to have $N(N+1)(1-\gamma_{12}^2) \gg 1$, and consequently $\rho_{ss} + \rho_{aa} \approx 0.5$. Thus, for small N and $\gamma_{12} \neq 1$, the populations ρ_{ss} and ρ_{aa} are very small, $\rho_{ss} = \rho_{aa} \approx 0$. This fact will have an important effect on the entanglement creation in the system of nonidentical atoms.

3. Steady-state entanglement

To assess how much entanglement is stored in a given quantum system it is essential to have appropriate measures of entanglement. A number of measures have been proposed, which include entanglement of formation [27], entanglement of distillation [28], relative entropy of entanglement [29] and negativity [30–33]. For pure states, the Bell states represent maximally entangled states, but for mixed states represented by a density matrix there are some difficulties with ordering the states according to various entanglement measures; different entanglement measures can give different orderings of pairs of mixed states and there is a problem in the definition of the maximally entangled mixed state [34, 35].

Here we use the concurrence to describe the amount of entanglement created in a two-atom system by the interaction with the squeezed vacuum. The concurrence introduced by Wootters [27] is defined as

$$\mathcal{C} = \max\left(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right), \tag{13}$$

where $\{\lambda_i\}$ are the the eigenvalues of the matrix

$$R = \rho\tilde{\rho} \tag{14}$$

with $\tilde{\rho}$ given by

$$\tilde{\rho} = \sigma_x \otimes \sigma_x \rho^* \sigma_x \otimes \sigma_x; \tag{15}$$

σ_x is the Pauli matrix, and ρ is the density matrix representing the quantum state. The range of concurrence is from zero to unity. For unentangled atoms $\mathcal{C} = 0$ whereas $\mathcal{C} = 1$ for the maximally entangled atoms.

In the basis (5) of the product atomic states the density matrix for two atoms in the squeezed vacuum has in the steady state the following block form:

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & 0 & 0 \\ \rho_{21} & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & \rho_{34} \\ 0 & 0 & \rho_{43} & \rho_{44} \end{pmatrix} \tag{16}$$

with the condition $\mathrm{Tr}\,\rho = 1$. The matrix $\tilde{\rho}$, required for calculation of the concurrence, has the form

$$\tilde{\rho} = \begin{pmatrix} \rho_{22} & \rho_{12} & 0 & 0 \\ \rho_{21} & \rho_{11} & 0 & 0 \\ 0 & 0 & \rho_{44} & \rho_{34} \\ 0 & 0 & \rho_{43} & \rho_{33} \end{pmatrix} \tag{17}$$

and the square roots of the eigenvalues of the matrix R given by (14) are the following:

$$\{\sqrt{\lambda_i}\} = \left\{\sqrt{\rho_{11}\rho_{22}} - |\rho_{12}|, \sqrt{\rho_{11}\rho_{22}} + |\rho_{12}|, \sqrt{\rho_{33}\rho_{44}} - |\rho_{34}|, \sqrt{\rho_{33}\rho_{44}} + |\rho_{34}|\right\}. \tag{18}$$

Depending on the particular values of the matrix elements there are two possibilities for the largest eigenvalue, either the second term or the fourth term in (18). The concurrence is thus given by

$$\mathcal{C} = \max\{0, \mathcal{C}_1, \mathcal{C}_2\}, \tag{19}$$

with

$$\mathcal{C}_1 = 2(|\rho_{12}| - \sqrt{\rho_{33}\rho_{44}}),$$
$$\mathcal{C}_2 = 2(|\rho_{34}| - \sqrt{\rho_{11}\rho_{22}}), \tag{20}$$

and we have two alternative expressions for the concurrence depending on which of them is positive.

In terms of the collective atomic states $|g\rangle$, $|e\rangle$, $|s\rangle$ and $|a\rangle$, the expressions for the concurrence (20) take the form

$$\mathcal{C}_1 = 2|\rho_{ge}| - \sqrt{(\rho_{ss}+\rho_{ss})^2 - (\rho_{sa}+\rho_{as})^2},$$
$$\mathcal{C}_2 = |\rho_{ss} - \rho_{aa} + \rho_{sa} - \rho_{as}| - 2\sqrt{\rho_{gg}\rho_{ee}}. \tag{21}$$

Having the concurrence expressed in terms of the density matrix elements, we can apply the steady-state solutions (8) or (11) and obtain analytical results for the stationary concurrence. We will discuss the results separately for identical and nonidentical atoms.

3.1. Identical atoms

In the case of identical atoms the steady-state solutions (8) are valid, and we find

$$\mathcal{C}_1 = |\rho_u| - (\rho_{ss} + \rho_{aa}) = 2\frac{(2N+1)|M|\gamma_{12} - |M|^2\gamma_{12}^2 - N(N+1)\left[(2N+1)^2 - 4|M|^2\right]}{(2N+1)^4 - 4|M|^2\left[(2N+1)^2 - \gamma_{12}^2\right]}. \tag{22}$$

It turns out that $\mathcal{C}_2$ is always negative, so the only contribution to the concurrence comes from $\mathcal{C}_1$, and then $\mathcal{C} = \max(0, \mathcal{C}_1)$.

It is clear from (22) that the perfect entanglement in the system, i.e., the value of concurrence equal to unity, can be achieved when $|\rho_u| = 1$ and $\rho_{ss} + \rho_{aa} = 0$. This is generally impossible for identical atoms because there is always some population stored in the states $|s\rangle$ and $|a\rangle$. However, it is possible to obtain some degree of entanglement in the system for appropriately chosen values of r_{12}, N and $|M|$. The concurrence, measuring the degree of entanglement, depends on the interatomic distance through the collective damping parameter γ_{12}, and the degree of the two-photon coherences $|M|$. It is easy to show from equation (22) that there is no entanglement possible for $|M| \leqslant N$, i.e., for classically correlated fields. For a quantum squeezed field with maximum correlations $|M| = \sqrt{N(N+1)}$, the concurrence can be written as

$$\mathcal{C}_1 = 2\sqrt{N(N+1)}\frac{(2N+1)\gamma_{12} - \sqrt{N(N+1)}(1+\gamma_{12}^2)}{1+4N(N+1)(1+\gamma_{12}^2)}. \tag{23}$$

In this case, $\mathcal{C}_1$ can be positive. To show this, we plot in figure 1 the concurrence for two identical atoms in the maximally squeezed vacuum as a function of the interatomic distance r_{12} and the mean number of photons N. It is evident from figure 1 that there is a range of values of r_{12}/λ for which $\mathcal{C}$ is positive. The maximum of concurrence is obtained for $r_{12} \approx 0$ when the atoms are very close to each other. The values of concurrence decrease as the interatomic distance increases and reduces to zero at $r_{12} \approx \lambda/2$, but we can observe revival of concurrence for longer interatomic distances, although the next maximum is much weaker. It is interesting that the maximum of concurrence appears for not very high values of the mean number of photons $N < 0.1$. It is easy to check, from (23), that for $\gamma_{12} = 1$ the number of photons $N_{\max}$ for which C_1 reaches its maximum is given by $N_{\max} = \left(\sqrt{(1+\sqrt{2})/2} - 1\right)/2 \approx 0.049$. The fact that the maximum of concurrence appears for moderate values of the mean photon numbers can be important from the experimental point of view as the present sources of squeezed fields can produce quantum squeezed fields of intensities $N < 1$.

The steady-state entanglement and its presence for only quantum squeezed fields of small intensities N is associated with nonclassical two-photon correlations characteristic of the squeezed vacuum field. To show this, we introduce a parameter

$$\mathrm{TC} = \frac{|M|}{N}, \tag{24}$$

which characterizes two-photon correlations normalized to the intensity of the squeezed field. For classical fields, $|M| \leqslant N$, and then TC < 1 for all N. For a quantum squeezed field with $|M| = \sqrt{N(N+1)}$, the parameter becomes

$$\mathrm{TC} = \sqrt{1+\frac{1}{N}}, \tag{25}$$

which is always greater than one. The result is a strong two-photon correlation, which is greatest for $N < 1$. Thus, the nonclassical two-photon correlations are significant for $N < 1$ and lead to a large entanglement in the system.

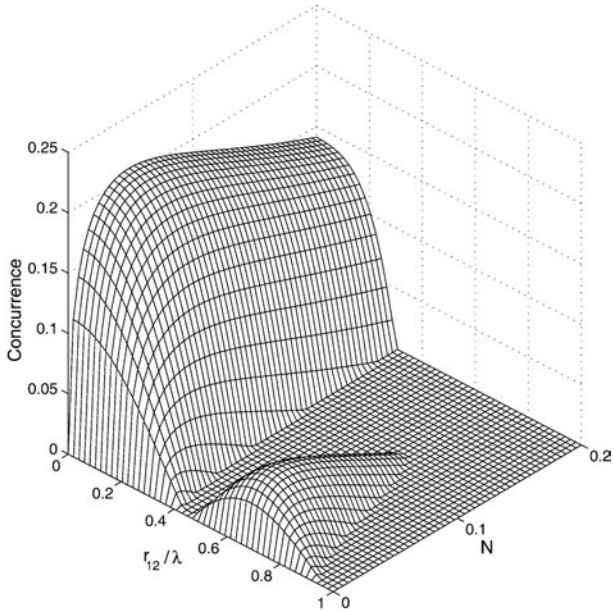

Figure 1. Concurrence $\mathcal{C}$ for two identical atoms as a function of the interatomic distance r_{12}/λ and the mean number of photons N for $|M| = \sqrt{N(N+1)}$.

3.2. Nonidentical atoms

In the case of nonidentical atoms with $\Delta \neq 0$, the steady-state values for the density matrix elements are given in equation (11). As above for identical atoms, the concurrence (19) can be expressed by formula (20) which, with solutions (11), leads to

$$\mathcal{C}_1 = |\rho_u| - (\rho_{ss} + \rho_{aa}) = \frac{2|M||\gamma_{12}|}{(2N+1)\left[(2N+1)^2 - 4|M|^2\gamma_{12}^2\right]} - \frac{1}{2}\left\{1 - \frac{1}{(2N+1)^2 - 4|M|^2\gamma_{12}^2}\right\}. \tag{26}$$

Similarly to the case of identical atoms, $\mathcal{C}_2$ is always negative. Moreover, $\mathcal{C}_1$ is always negative for $|M| \leqslant N$ independent of γ_{12}. Thus, entanglement is possible only for quantum squeezed fields, which for the maximum correlations $|M| = \sqrt{N(N+1)}$ gives

$$\mathcal{C}_1 = 2\sqrt{N(N+1)}\frac{\gamma_{12} - (2N+1)\sqrt{N(N+1)}(1-\gamma_{12}^2)}{(2N+1)[1+4N(N+1)(1-\gamma_{12}^2)]}. \tag{27}$$

Equation (27) is significantly different from that for identical atoms, equation (23). For example, if the atoms are close together, $\gamma_{12} \approx 1$, and then equation (27) reduces to

$$\mathcal{C}_1 = \frac{2\sqrt{N(N+1)}}{2N+1}. \tag{28}$$

In this limit the concurrence is always positive, increases with N and approaches unity at a large N. This is in contrast to the case of identical atoms where values of concurrence are below 0.25 even for $\gamma_{12} = 1$ and approach zero for large N. This behaviour can be easily explained by the fact that in the case of nonidentical atoms and $\gamma_{12} = 1$ the population stored in the symmetric and antisymmetric states, $\rho_{ss} + \rho_{aa}$, is equal to zero. At the same time, ρ_u tends to unity as N increases, giving the maximum concurrence $\mathcal{C}_1 = 1$.

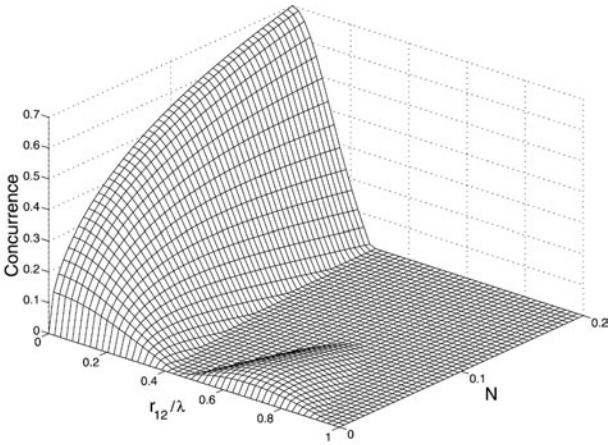

Figure 2. Concurrence $\mathcal{C}$ for two nonidentical atoms as a function of the interatomic distance r_{12}/λ and the mean number of photons N for $|M| = \sqrt{N(N+1)}$.

In real situations, we have $\gamma_{12} < 1$, and for large N the concurrence C_1, given by (27), goes to zero, similarly to the case of identical atoms. The maximum of C_1 for nonidentical atoms, nonetheless, is much more pronounced than that for identical atoms. The real scale of large number of photons is in this case given by $N(N+1)(1-\gamma_{12}^2) \gg 1$ rather than by $N(N+1)(1+\gamma_{12}^2) \gg 1$ as is the case for identical atoms.

The dependence of the concurrence $\mathcal{C} = \max(0, \mathcal{C}_1)$, with $\mathcal{C}_1$ given by formula (27), on the interatomic distance r_{12} and the mean number of photons of the squeezed field N is shown in figure 2. The dependence on the interatomic distance is similar to that seen for identical atoms with the revival of concurrence for $r_{12} \approx 3\lambda/4$ and not too large N. For $\gamma_{12} \neq 1$ the dependence on N is also similar to that for identical atoms, except that the maximum values of concurrence for given N are much higher than for identical atoms. Comparing the solutions for identical and nonidentical atoms indicates that one reason for higher values of concurrence for nonidentical as compared to identical atoms interacting with the squeezed vacuum is the fact that for nonidentical atoms less population remains in the lower block of the density matrix (16) represented by states $|3\rangle$ and $|4\rangle$ (or $|a\rangle$ and $|s\rangle$) as the atoms become different, i.e., $\Delta = (\omega_2 - \omega_1)/2$ becomes large. To confirm this fact we plot in figure 3 the concurrence as well as the populations, $\rho_{ss} + \rho_{aa}$ and $\rho_{gg} + \rho_{ee}$, which are stored in the two blocks of the density matrix (16), as a function of Δ. Since for identical atoms, according to (9), a considerable amount of population remains in the antisymmetric state in contrast to solutions (12) for nonidentical atoms, it is clear from figure 3 that as the transition frequencies of the two atoms become more and more different the population of the antisymmetric state goes down, reducing the total population $\rho_{ss} + \rho_{aa}$ of the lower block and increasing the total population $\rho_{gg} + \rho_{ee}$ of the upper block of (16), which means higher values of concurrence.

Another physical explanation of the origin of the better entanglement for nonidentical atoms is provided by the observation that the stationary state of nonidentical atoms, for small N for which concurrence is maximal, is close to a pure state, whilst the stationary state of identical atoms is already far from a pure state. This is illustrated in figure 4, where we plot the purity measure $P = \mathrm{Tr}(\rho^2) = \rho_{gg}^2 + \rho_{ee}^2 + \rho_{ss}^2 + \rho_{aa}^2 + |\rho_u|^2/2$

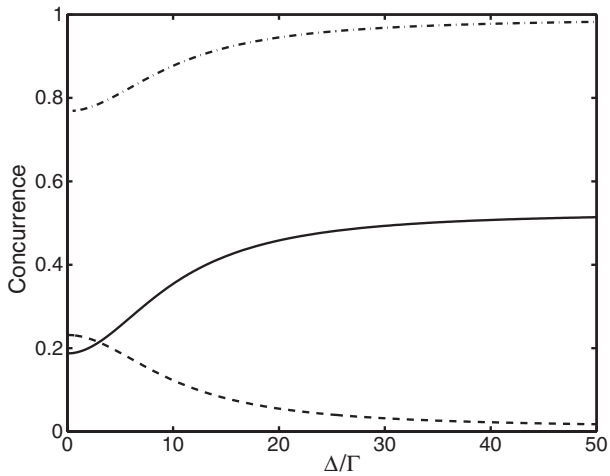

Figure 3. Concurrence $\mathcal{C}$ (solid curve), populations $\rho_{ss} + \rho_{aa}$ (dashed curve) and populations $\rho_{gg} + \rho_{ee}$ (dashed–dotted curve) for two atoms as a function of Δ for $r_{12}/\lambda = 0.05$ and $N = 0.1$ for $|M| = \sqrt{N(N+1)}$.

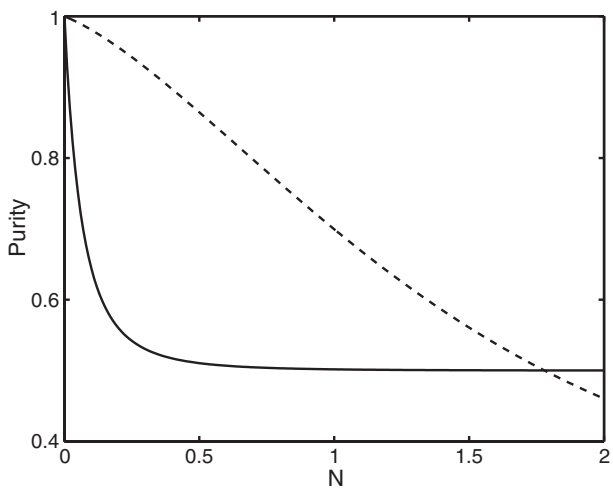

Figure 4. Purity $P = \mathrm{Tr}(\rho^2)$ as a function of the mean number of photons N for $r_{12}/\lambda = 0.05$ and $|M| = \sqrt{N(N+1)}$: identical atoms (solid curve); nonidentical atoms (dashed curve).

as a function of N for the steady state of two identical as well as nonidentical atoms. It is seen that in both cases the purity decreases as the number of photons increases, but in the case of identical atoms the purity goes down much faster.

It should be emphasized, however, that the main source of entanglement in the system is the nonclassical two-photon correlations that create two-photon coherences between the states $|g\rangle$ and $|e\rangle$. The two-photon coherences are nonzero only when the squeezing parameter $|M|$ is nonzero. In fact, to have entanglement in the system the squeezed field must represent quantum correlations with $|M| > N$. There is, moreover, one more necessary condition to have nonzero ρ_u, which is a nonzero value of the collective damping parameter γ_{12}. The two-photon coherences cause the system to decay into entangled states involving the ground state $|g\rangle$ and the upper state $|e\rangle$ without any involvement of the entangled states $|s\rangle$ and $|a\rangle$. Unfortunately, the spontaneous emission from the state $|e\rangle$ redistributes some of the atomic population over the states $|s\rangle$ and $|a\rangle$, limiting in this way the degree of entanglement.

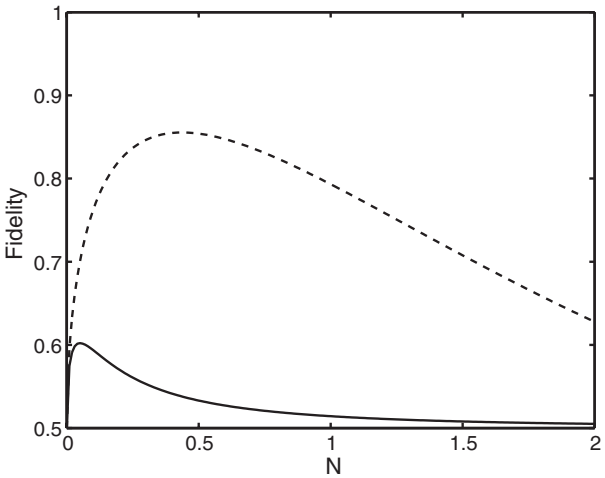

Figure 5. Fidelity F_+ as a function of the mean number of photons N for $r_{12}/\lambda = 0.05$ and $|M| = \sqrt{N(N+1)}$: identical atoms (solid curve); nonidentical atoms (dashed curve).

Since the two-photon coherences create superposition (entangled) states involving the states $|g\rangle$ and $|e\rangle$, one can ask how close the entangled stationary state of the system is to one of the maximally entangled Bell states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle + |e\rangle\right), \qquad |\Phi^-\rangle = -\frac{1}{\sqrt{2}}\left(|g\rangle - |e\rangle\right). \tag{29}$$

To answer this question we calculate the fidelities

$$F_+ = \langle\Phi^+|\rho|\Phi^+\rangle = \tfrac{1}{2}\left(\rho_{gg} + \rho_{ee} + \rho_u\right), \qquad F_- = \langle\Phi^-|\rho|\Phi^-\rangle = \tfrac{1}{2}\left(\rho_{gg} + \rho_{ee} - \rho_u\right). \tag{30}$$

The fidelities depend on whether ρ_u is positive or negative. For small interatomic distances γ_{12} is positive, and then the coherence ρ_u is positive. Hence, the fidelity F_+ becomes large while the fidelity F_- is small. Thus, we can conclude that the stationary state of the system is close to the maximally entangled Bell state $|\Phi_+\rangle$. In figure 5 we plot the fidelity F_+ as a function of N for identical as well as nonidentical atoms. Comparing the dependence on N of the fidelity F_+ and concurrence C gives us clear evidence that the entanglement in the system can be related to the Bell state $|\Phi_+\rangle$. The state of the system is of course mixed, but it is closer to the pure Bell state $|\Phi_+\rangle$ for nonidentical atoms than for identical atoms. The entanglement created by the two-photon correlations present in the squeezed light is limited by the population stored in the other states. As the number of photons increases, for $\gamma_{12} < 1$, more and more population goes to the states $|s\rangle$ and $|a\rangle$, and eventually entanglement disappears for both identical and nonidentical atoms. There are optimal values of the mean number of photons for which the highest possible stationary entanglement can be obtained.

4. Conclusions

In this paper, we have studied analytically the entanglement creation in a system of two atoms interacting with a squeezed vacuum field. We have demonstrated that nonclassical two-photon correlations characteristic of the squeezed field can create a large steady-state entanglement in the system.

In our approach we have used a master equation to describe a system of two two-level atoms subjected to a squeezed vacuum field. The two atoms are coupled to each other via the vacuum field which leads to collective damping and collective dipole–dipole type interaction between the atoms. We have assumed the two atoms to be separated by a distance r_{12}, so the collective parameters depend explicitly on this distance. Steady-state solutions for the atomic density matrix have been found for two cases: (i) identical atoms; (ii) nonidentical atoms.

We have derived analytical expressions for concurrence which is used to quantify the amount of entanglement created in the system. Our results show that the necessary condition for entanglement is nonclassical two-photon correlations of the squeezed field. The entanglement also depends on the interatomic separation and the mean number of photons of the squeezed vacuum. The necessary condition for entanglement is quantum correlations of the squeezed field. There is no entanglement for a classically correlated field. We have found that the degree of entanglement created in the system is a result of competition between the coherent process of transferring two-photon coherences from the squeezed vacuum to the atomic system and the incoherent process of spontaneous emission redistributing atomic population over the states not involved in the former process. In particular, we have shown that there is an optimum value of the mean number of photons for which the concurrence takes its maximum, and this happens for a small number of photons. This is important from the point of view of practical applications. Moreover, we have also found that the degree of entanglement obtainable in this way is much higher when the two atoms are not identical. We have discussed in detail physical reasons for such behaviour of the two-atom system.

References

[1] Beige A, Braun D, Tregenna B and Knight P L 2000 *Phys. Rev. Lett.* **85** 1762
[2] Kozhekin A E, Molmer K and Polzik E 2000 *Phys. Rev.* A **62** 033809
[3] Hald J, Sorensen J L, Schori C and Polzik E S 1999 *Phys. Rev. Lett.* **83** 1319
[4] Fleischhauer M, Yelin S F and Lukin M D 2000 *Opt. Commun.* **179** 395
[5] Schneider S and Milburn G J 2002 *Phys. Rev.* A **65** 042107
[6] Sackett C A *et al* 2000 *Nature* **404** 256
[7] Lukin M D, Fleischhauer M, Cote R, Duan L M, Jaksch D, Cirac J I and Zoller P 2001 *Phys. Rev. Lett.* **87** 037901
[8] Wang X and Molmer K 2002 *Eur. Phys. J.* D **18** 385
[9] Akram U, Ficek Z and Swain S 2000 *Phys. Rev.* A **62** 013413
[10] Kim M S, Lee J, Ahn D and Knight P L 2002 *Phys. Rev.* A **65** 040101(R)
[11] Zhou L, Song H S and Li C 2002 *J. Opt. B: Quantum Semiclass. Opt.* **4** 425
[12] Messikh A, Ficek Z and Wahiddin M R B 2003 *J. Opt. B: Quantum Semiclass. Opt.* **5** L4
[13] Plenio M B, Huelga S F, Beige A and Knight P L 1999 *Phys. Rev.* A **59** 2468
[14] Beige A, Bose S, Braun D, Huelga S F, Knight P L, Plenio M B and Verdal V 2000 *J. Mod. Opt.* **47** 2583
[15] Cabrillo C, Cirac J I, Garcia-Fernandez P and Zoller P 1999 *Phys. Rev.* A **59** 1025
[16] Guo G C and Yang C P 1998 *Physica* A **260** 173

[17] Yang C P and Guo G C 1999 *Physica* A **273** 352
[18] Basharov A M 2002 *JETP Lett.* **75** 123
[19] Jakóbczyk L 2002 *J. Phys. A: Math. Gen.* **35** 6383
[20] Dicke R H 1954 *Phys. Rev.* **93** 99
[21] Eichmann U, Bergquist J C, Bollinger J J, Gilligan J M, Itano W M, Wineland D J and Raizen M G 1993 *Phys. Rev. Lett.* **70** 2359
[22] DeVoe R G and Brewer R G 1996 *Phys. Rev. Lett.* **76** 2049
[23] Riesch D, Abich K, Neuhauser W, Wunderlich Ch and Toschek P E 2002 *Phys. Rev.* A **65** 053401
[24] Ficek Z and Tanaś R 2003 *J. Mod. Opt.* **50** 2765
[25] Tanaś R and Ficek Z 2004 *J. Opt. B: Quantum Semiclass. Opt.* **6** S90
(Tanaś R and Ficek Z 2003 *Preprint* quant-ph/0307045)
[26] Ficek Z and Tanaś R 2002 *Phys. Rep.* **372** 369
[27] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245
Hill S and Wootters W K 1997 *Phys. Rev. Lett.* **78** 5022
[28] Bennett C H, Brassard G, Popescu S, Schumacher B, Smolin J and Wootters W K 1996 *Phys. Rev. Lett.* **76** 722
[29] Vedral V, Plenio M B, Jacobs K and Knight P L 1997 *Phys. Rev.* A **56** 4452
[30] Peres A 1996 *Phys. Rev. Lett.* **77** 1413
[31] Horodecki P 1997 *Phys. Lett.* A **232** 333
[32] Życzkowski K, Horodecki P, Sanpera A and Lewenstein M 1998 *Phys. Rev.* A **58** 883
[33] Vidal G and Werner R F 2002 *Phys. Rev.* A **65** 032314
[34] Ishizaka S and Hiroshima T 2000 *Phys. Rev.* A **62** 022310
[35] Wei T C, Nemoto K, Goldbart P M, Kwiat P G, Munro W J and Verstraete F 2003 *Phys. Rev.* A **67** 022110
[36] Lehmberg R H 1970 *Phys. Rev.* A **2** 883
Lehmberg R H 1970 *Phys. Rev.* A **2** 889
[37] Agarwal G S 1974 *Quantum Statistical Theories of Spontaneous Emission and their Relation to other Approaches* (*Springer Tracts in Modern Physics* vol 70) ed G Höhler (Berlin: Springer)
[38] Ficek Z, Tanaś R and Kielich S 1987 *Physica* A **146** 452
[39] Ficek Z, Tanaś R and Kielich S 1981 *Opt. Commun.* **36** 121
Ficek Z, Tanaś R and Kielich S 1983 *Opt. Acta* **30** 713