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Squeezing and squeezing-like terms in the master equation for a two-level atom in strong fields

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Abstract
If a broadband squeezed vacuum is treated as a reservoir with respect to a two-level atom, the non-zero phase-dependent reservoir correlation functions characterizing the squeezed vacuum introduce 'squeezing terms' to the master equation. These terms are responsible, for example, for the well known narrowing of the spectral lines in the resonance fluorescence spectrum of the atom. For a squeezed vacuum reservoir with finite bandwidth it is possible to derive the master equation that is consistent with the Born–Markov approximation by a two-step procedure which consists in dressing the atom first and next coupling it to the finite-bandwidth reservoir. The master equation is valid whenever the bandwidth of the reservoir is much broader than the atomic linewidth but not necessarily broader than the Rabi frequency. This procedure can be applied not only for the squeezed vacuum reservoir but also for the ordinary vacuum with the structure of the mode density that is not flat, e.g., modelled by a Lorentzian function as in a cavity. The master equation for this case, in the operator form, shows some similarities to the master equation for the squeezed vacuum reservoir due to the presence of the 'squeezing-like terms'. The similarities and differences of the two master equations are discussed in this paper.

Keywords: Squeezing, master equation, structured reservoir

1. Introduction

Spontaneous emission of an excited two-level atom results as an effect of its interaction with a continuum of vacuum modes that play a role of a reservoir to the atom. The spontaneous emission rate is usually supposed to be an inherent property of the atom, but, in fact, it has been known for a long time that the atomic damping rates depend on the mode structure of the atomic environment [1–3]. However, when the density of modes of the reservoir is essentially flat, as for a vacuum, one can neglect any modifications of the atomic spontaneous emission rate and assume that it is equal to the Einstein A coefficient. When the atom is driven on resonance by a strong monochromatic laser beam the structure of atomic levels changes dramatically. For very strong fields, when the Rabi frequency becomes much bigger than the spontaneous emission rate, the dressed atom picture can be used to describe atomic dynamics [4, 5]. In many cases, the strong laser field can be considered as a classical field, and the 'semiclassical dressed states' can be used to describe atomic radiative properties [6–8]. Also, in the dressed atom description the damping rates are usually treated as constants that do not depend on the strength of the applied field and the structure of the reservoir. However, the situation is quite different when the driven atom is placed in an environment with the density of modes that appreciably depends on frequency [3, 9, 10]. Lewenstein and Mossberg [9] have analysed the spectral and statistical properties of atoms driven by a strong, single-mode light field and coupled to a reservoir of electromagnetic field modes with strong frequency dependence. They used a non-Markovian approach leading to a rather complicated set of equations describing the atomic

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dynamics. Their theory predicted a number of interesting features of the atomic spectra, one of them was an asymmetry of the fluorescence spectrum radiated to the background modes which has been measured by Lezama *et al* [11].

Recently, the master equation has been derived [12, 13] for the reduced atomic density matrix under the Born–Markov approximation which takes into account the dependence of the relaxation rates on the strength of the laser field. In this master equation, even for flat reservoirs such as an ordinary vacuum, the relaxation rates depend on the strength of the field through the ω^3 factor in the vacuum density of modes. Keitel *et al* [14, 15] have shown that in the secular limit the resonance fluorescence spectra should be symmetric even for tailored reservoirs with an asymmetric density of modes despite the fact that the dressed states populations are not equal. The reason for this is that the difference in populations is compensated for by the difference in the transition rates between the dressed states. They emphasized that it is important for strong fields to perform the dressing operation first and only after that consider the coupling of the dressed atom to the reservoir modes. The results obtained in this way differ from the results obtained in the conventional treatment.

Gardiner, in his seminal paper [16], has shown that when an atom is damped to a squeezed vacuum reservoir the atomic dipole moment can decay with two different rates, one much longer and the other much shorter than that in the ordinary vacuum. Consequently, a subnatural linewidth has been predicted in the spontaneous emission spectrum. The addition of a coherent driving field to the problem introduces a strong dependence of the atom dynamics and the fluorescence spectrum on the relative phase between the coherent field and the squeezed field. Carmichael *et al* [17] have shown that, depending on the phase, the central peak of the Mollow triplet [18] can either be much narrower or much broader than the natural linewidth of the atom. Thus, the spectrum can be modified quantitatively from the spectrum associated with the normal vacuum. Apart from the quantitative modifications, the qualitative changes of the fluorescence spectrum have also been predicted. Courty and Reynaud [19] have found that for a certain detuning of the driving field from the atomic resonance the central peak and one of the sidebands can be suppressed due to a population trapping in the dressed state. Smart and Swain [20–22] have found unusual features in the resonance fluorescence spectra, such as a hole burning and dispersive profiles. These features, however, appear for Rabi frequencies comparable to the atomic linewidth and are very sensitive to the various parameters involved.

Most of the studies dealing with the problem of a two-level atom in a squeezed vacuum assume that the squeezed vacuum is broadband, i.e. the bandwidth of the squeezed vacuum is much larger than the atomic linewidth and the Rabi frequency of the driving field. Experimental realizations of squeezed states [23–26], however, indicate that the bandwidth of the squeezed light is typically of the order of the atomic linewidth. The most popular schemes for generating squeezed light are those using a parametric oscillator operating below threshold, the output of which is a squeezed beam with a bandwidth of the order of the cavity bandwidth [27, 28].

First studies of the finite-bandwidth effects have been performed by Gardiner *et al* [27], Parkins and Gardiner [29] and Ritsch and Zoller [30]. The approaches were based on stochastic methods and numerical calculations, and were applied to analyse the narrowing of the spontaneous emission and absorption lines. The fundamental effect of narrowing has been confirmed, but the effect of finite bandwidth was to degrade the narrowing of the spectral lines rather than enhance it. Later, however, numerical simulations done by Parkins [31, 32] demonstrated that for strong driving fields a finite bandwidth of squeezing can have a positive effect on the narrowing of the Rabi sidebands.

It has recently been shown by Yeoman and Barnett [33] that it is possible to obtain a master equation consistent with the Born–Markov approximation by first including the interaction of the atom with the driving field exactly, and then considering the coupling of this combined dressed atom system with the finite-bandwidth squeezed vacuum. The advantage of this dressed-atom method over the more complex treatments based on adjoint equation or stochastic methods [31, 32, 34] is that simple analytical expressions for the spectra can be obtained, thus explicitly displaying the factors that determine the intensities of the spectral features and their widths. This idea has been extended by Ficek *et al* [35] to the case of a fully quantized dressed-atom model coupled to a finite bandwidth squeezed field inside an optical cavity and by Tanaś *et al* [36] to the case of a non-resonant classical driving field. It has been shown [36] that despite the complexity of the problem, it is possible to obtain a quite simple operator form of the master equation that is valid for arbitrary values of the Rabi frequency and the detuning but is restricted to the squeezing bandwidths much greater than the natural linewidth. The same approach has been used by Kowalewska-Kudłaszyk and Tanaś [37] to derive the master equation for a two-level atom driven by a strong laser field and damped to a reservoir with the non-flat density of modes, that is, the structured or tailored reservoir. It has been shown that despite the non-squeezed nature of the reservoir, the resulting master equation contains terms of the type known for the squeezed reservoirs that have been referred to as ‘squeezing-like’ terms.

The aim of this paper is to compare the two different models. The operator form of the master equation for either model is derived under the same approximations using the two-step procedure described above. Particular attention is paid to the role of the ‘squeezing-like’ terms that appear in the master equation whenever the reservoir has finite bandwidth. The Bloch equations based on the master equation for either model are derived and their solutions discussed shortly. Analytical expressions for the parameters governing the Bloch equations are found, in particular, the expressions for the two different damping rates of the quadrature components of the atomic dipole and for the steady-state values of the dressed atom population inversion. It is shown that these quantities depend strongly on the nature of the reservoir. The similarities and differences of the atomic evolution in both reservoirs are emphasized.

2. An atom in a squeezed vacuum reservoir with finite bandwidth

2.1. Master equation

In this section we consider a two-level atom driven by a monochromatic laser field of frequency ω_L with the Rabi frequency Ω detuned by $\Delta = \omega_L - \omega_A$ from the atomic transition frequency ω_A, which is damped to a squeezed vacuum reservoir with finite bandwidth. The idea of the approach was proposed by Carmichael and Walls [38] and Cresser [39], and recently used by Yeoman and Barnett [33] and Tanaś *et al* [36] to derive the master equation for a two-level atom damped by a squeezed vacuum with finite bandwidth. In this approach, we first perform the dressing transformation to include the interaction of the atom with the driving field and then couple the resulting dressed atom to the reservoir. We derive the master equation under the Markov approximation which requires the reservoir bandwidth to be much greater than the atomic linewidth, but not necessarily greater than the Rabi frequency of the driving field and the detuning. For simplicity, we assume that the squeezing properties are symmetric about the central frequency of the squeezed field which, in turn, is exactly equal to the laser frequency.

We start from the Hamiltonian of the system which in the rotating-wave and electric-dipole approximations is given by

$$H = H_A + H_R + H_L + H_I \tag{1}$$

where

$$H_A = \tfrac{1}{2}\hbar\omega_A\sigma_z = -\tfrac{1}{2}\hbar\Delta\sigma_z + \tfrac{1}{2}\hbar\omega_L\sigma_z \tag{2}$$

is the Hamiltonian of the atom,

$$H_R = \hbar\int_0^\infty \omega b^+(\omega)b(\omega)\,\mathrm{d}\omega \tag{3}$$

is the Hamiltonian of the vacuum field,

$$H_L = \tfrac{1}{2}\hbar\Omega[\sigma_+\exp[-\mathrm{i}(\varphi_L+\omega_L t)]+\sigma_-\exp[\mathrm{i}(\varphi_L+\omega_L t)]] \tag{4}$$

is the interaction between the atom and the classical laser field, and

$$H_I = \mathrm{i}\hbar\int_0^\infty K(\omega)[\,\sigma_+b(\omega) - b^+(\omega)\sigma_-]\,\mathrm{d}\omega \tag{5}$$

is the interaction of the atom with the vacuum field. In (2)–(5), $K(\omega)$ is the coupling of the atom to the vacuum modes, $\Delta = \omega_L - \omega_A$ is the detuning of the driving laser field frequency ω_L from the atomic resonance ω_A, φ_L is the laser field phase, and σ_+, σ_-, and σ_z are the Pauli pseudo-spin operators describing the two-level atom. The laser driving field strength is given by the Rabi frequency Ω, while the operators $b(\omega)$ and $b^+(\omega)$ are the annihilation and creation operators for the vacuum modes satisfying the commutation relation

$$[b(\omega), b^+(\omega')] = \delta(\omega-\omega'). \tag{6}$$

In order to derive the master equation we perform the two-step unitary transformation. In the first step we use the second part of the atomic Hamiltonian (2) and the free field Hamiltonian (3) to transform to the frame rotating with the laser frequency ω_L and to the interaction picture with respect to the vacuum modes. The rotating frame is also shifted in phase by φ_L, i.e. we introduce new raising and lowering operators which absorb the phase factor according to the relations

$$\sigma_-\mathrm{e}^{\mathrm{i}\varphi_L} \to \sigma_-, \qquad \sigma_+\mathrm{e}^{-\mathrm{i}\varphi_L} \to \sigma_+. \tag{7}$$

After this transformation our system is described by the Hamiltonian

$$H_0 + H_I^r(t), \tag{8}$$

where

$$H_0 = -\tfrac{1}{2}\hbar\Delta\sigma_z + \tfrac{1}{2}\hbar\Omega(\sigma_+ + \sigma_-) \tag{9}$$

and

$$H_I^r(t) = \mathrm{i}\hbar\int_0^\infty K(\omega)\sigma_+b(\omega)\exp[\mathrm{i}\varphi_L + \mathrm{i}(\omega_L-\omega)t]\mathrm{d}\omega + \mathrm{H.c.} \tag{10}$$

The second step is the unitary dressing transformation performed with the Hamiltonian H_0, given by (9). The transformation

$$\sigma_\pm(t) = \exp\left[-\frac{\mathrm{i}}{\hbar}H_0 t\right]\sigma_\pm\exp\left[\frac{\mathrm{i}}{\hbar}H_0 t\right] \tag{11}$$

leads to the following time-dependent atomic raising and lowering operators:

$$\begin{aligned}\sigma_\pm(t) = \tfrac{1}{2}[&\mp(1\pm\tilde{\Delta})\tilde{\sigma}_-\exp(-\mathrm{i}\Omega' t)\\ &\pm(1\mp\tilde{\Delta})\tilde{\sigma}_+\exp(\mathrm{i}\Omega' t) + \tilde{\Omega}\tilde{\sigma}_z]\end{aligned} \tag{12}$$

where

$$\begin{aligned}\tilde{\sigma}_- &= \tfrac{1}{2}[(1-\tilde{\Delta})\sigma_- - (1+\tilde{\Delta})\sigma_+ - \tilde{\Omega}\sigma_z]\\ \tilde{\sigma}_+ &= \tfrac{1}{2}[-(1+\tilde{\Delta})\sigma_- + (1-\tilde{\Delta})\sigma_+ - \tilde{\Omega}\sigma_z]\\ \tilde{\sigma}_z &= \tilde{\Omega}(\sigma_- + \sigma_+) - \tilde{\Delta}\sigma_z\end{aligned} \tag{13}$$

are the dressed operators oscillating at frequencies $-\Omega'$, Ω' and 0, respectively, and

$$\tilde{\Omega} = \frac{\Omega}{\Omega'} \qquad \tilde{\Delta} = \frac{\Delta}{\Omega'} \qquad \Omega' = \sqrt{\Omega^2+\Delta^2}. \tag{14}$$

Since we assume $\Omega' > 0$, as $\Omega \to 0$, the dressed operators $\tilde{\sigma}_\pm \to \sigma_\pm$, $\tilde{\sigma}_z \to \sigma_z$ for $\Delta < 0$, and $\tilde{\sigma}_\pm \to -\sigma_\mp$, $\tilde{\sigma}_z \to -\sigma_z$ for $\Delta > 0$.

Under the transformation (12) the interaction Hamiltonian takes the form

$$\begin{aligned}H_I(t) = \mathrm{i}\hbar\int_0^\infty &\{K(\omega)\sigma_+(t)b(\omega)\exp[\mathrm{i}\varphi_L + \mathrm{i}(\omega_L-\omega)t]\\ &-K^*(\omega)b^+(\omega)\sigma_-(t)\exp[-\mathrm{i}\varphi_L - \mathrm{i}(\omega_L-\omega)t]\}\mathrm{d}\omega.\end{aligned} \tag{15}$$

The master equation for the reduced density operator ρ of the system can be derived using standard methods [40]. In the Born approximation the equation of motion for the reduced density operator is given by [40]

$$\frac{\partial\rho^D}{\partial t} = -\frac{1}{\hbar^2}\int_0^t \mathrm{Tr}_R\{[H_I(t),[H_I(t-\tau),\rho_R(0)\rho^D(t-\tau)]]\}\mathrm{d}\tau \tag{16}$$

where the superscript D stands for the dressed picture, $\rho_R(0)$ is the density operator for the field reservoir, Tr_R is the trace over the reservoir states and the Hamiltonian $H_I(t)$ is given by (15). We next make the Markov approximation [40] by replacing $\rho^D(t-\tau)$ in (16) by $\rho^D(t)$, substitute the Hamiltonian (15) and take the trace over the reservoir variables.

In the case of a squeezed vacuum reservoir, the trace over the reservoir operators gives the non-zero values for the two 'diagonal' correlation functions

$$\begin{aligned}\mathrm{Tr}_R[\rho_R(0)b(\omega)b^+(\omega')] &= [N(\omega)+1]\delta(\omega-\omega')\\ \mathrm{Tr}_R[\rho_R(0)b^+(\omega)b(\omega')] &= N(\omega)\delta(\omega-\omega')\end{aligned} \tag{17}$$

where $N(\omega)$ is the mean number of photons at frequency ω, and the two 'nondiagonal' correlation functions, which are specific for the squeezed vacuum reservoir

$$\begin{aligned}\mathrm{Tr}_R[\rho_R(0)b(\omega)b(\omega')] &= M(\omega)\exp(\mathrm{i}\varphi_s)\delta(2\omega_L-\omega-\omega') \\ \mathrm{Tr}_R[\rho_R(0)b^+(\omega)b^+(\omega')] &= M(\omega)\exp(-\mathrm{i}\varphi_s)\delta(2\omega_L-\omega-\omega')\end{aligned} \tag{18}$$

where $M(\omega)$ describes the value of squeezing and φ_s is the phase of squeezed light, and we have assumed that the carrier frequency of the squeezed field ω_s is equal to the laser frequency ω_L. The phase-dependent two-photon correlation functions (18) introduce 'squeezing terms' proportional to $M(\omega)$ that will appear in the master equation.

The explicit form of $N(\omega)$ and $M(\omega)$ depend on the source of squeezed light. The most popular sources of the squeezed vacuum are the degenerate parametric oscillator (DPO) and nondegenerate parametric oscillator (NDPO), which generate two different types of squeezed field. For a degenerate parametric oscillator (DPO) the squeezing properties are described by [27]

$$N(x) = \frac{\lambda^2-\mu^2}{4}\left[\frac{1}{x^2+\mu^2} - \frac{1}{x^2+\lambda^2}\right] \tag{19}$$

$$M(x) = s\frac{\lambda^2-\mu^2}{4}\left[\frac{1}{x^2+\mu^2} + \frac{1}{x^2+\lambda^2}\right] \tag{20}$$

while for a non-degenerate parametric oscillator (NDPO) the frequency dependence is given by [28]

$$\begin{aligned}N(x) = \frac{\lambda^2-\mu^2}{8}\bigg[&\frac{1}{(x-\alpha)^2+\mu^2} + \frac{1}{(x+\alpha)^2+\mu^2} \\ &- \frac{1}{(x-\alpha)^2+\lambda^2} - \frac{1}{(x+\alpha)^2+\lambda^2}\bigg]\end{aligned} \tag{21}$$

$$\begin{aligned}M(x) = \frac{\lambda^2-\mu^2}{8}\bigg[&\frac{1}{(x-\alpha)^2+\mu^2} + \frac{1}{(x+\alpha)^2+\mu^2} \\ &+ \frac{1}{(x-\alpha)^2+\lambda^2} + \frac{1}{(x+\alpha)^2+\lambda^2}\bigg]\end{aligned} \tag{22}$$

where $x = \omega - \omega_L$, and λ and μ are related to the cavity damping rate, γ_c, and the real amplification constant, ϵ, of the parametric oscillator according to

$$\begin{aligned}\lambda &= \gamma_c + \epsilon \\ \mu &= \gamma_c - \epsilon.\end{aligned} \tag{23}$$

The parameter α is characteristic of a two-mode squeezed field generated by the non-degenerate parametric oscillator and represents the displacement from the central frequency of the squeezing at which the two-mode squeezed vacuum is maximally squeezed. Thus, for a realistic source of squeezed vacuum $N(\omega)$ and $M(\omega)$ are combinations of Lorentzians (positive and negative) with the widths λ and μ and they have important frequency dependence. In order to validate the Markov approximation used here, we assume that the widths of the Lorentzians are much larger than the atomic linewidth.

In the Markov approximation we can extend the upper limit of the integration over τ in (16) to infinity and next perform necessary integrations using the formula

$$\int_0^\infty \exp(\pm\mathrm{i}\epsilon\tau)\mathrm{d}\tau = \pi\delta(\epsilon) \pm \mathrm{i}\mathcal{P}\frac{1}{\epsilon} \tag{24}$$

where $\mathcal{P}$ means the Cauchy principal value. After lengthy but simple operator algebra, which can be performed, for example, with the computer algebra program Form [41], we obtain the master equation which, in the frame rotating with the laser frequency ω_L and shifted in phase by φ_L, can be written as [36]

$$\begin{aligned}\dot\rho = &\tfrac{1}{2}\mathrm{i}\delta[\sigma_z,\rho] - \tfrac{1}{2}\mathrm{i}\Omega[\sigma_+ + \sigma_-,\rho] \\ &+ \tfrac{1}{2}\tilde N(2\sigma_+\rho\sigma_- - \sigma_-\sigma_+\rho - \rho\sigma_-\sigma_+) \\ &+ \tfrac{1}{2}(\tilde N+1)(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-) \\ &- \tilde M\sigma_+\rho\sigma_+ - \tilde M^*\sigma_-\rho\sigma_- \\ &+ \tfrac{1}{4}(\beta[\sigma_+,[\sigma_z,\rho]] + \beta^*[\sigma_-,[\sigma_z,\rho]])\end{aligned} \tag{25}$$

where γ is the natural atomic linewidth, and the other parameters are defined by

$$\begin{aligned}\tilde N &= \gamma N_+ + \tfrac{1}{2}(1-\tilde\Delta^2)\mathrm{Re}\Gamma_- \\ \tilde M &= \gamma(M_+ + \mathrm{i}\tilde\Delta\delta_M)\mathrm{e}^{\mathrm{i}\phi} - \tfrac{1}{2}(1-\tilde\Delta^2)\Gamma_- \\ \delta &= \Delta + \gamma\tilde\Delta\delta_N - \tfrac{1}{2}(1-\tilde\Delta^2)\mathrm{Im}\,\Gamma_- \\ \beta &= \gamma\tilde\Omega[\mathrm{i}(\delta_N + \delta_M\mathrm{e}^{\mathrm{i}\phi}) + \tilde\Delta\Gamma_-] \\ \Gamma_- &= \gamma(N_0 - N_+) - \gamma(M_0 - M_+)\mathrm{e}^{\mathrm{i}\phi}\end{aligned} \tag{26}$$

where

$$\begin{aligned}N_0 &= N(\omega_L) \qquad N_+ = N(\omega_L + \Omega') \\ M_0 &= M(\omega_L) \qquad M_+ = M(\omega_L + \Omega') \\ \phi &= 2\varphi_L + \varphi_s \\ \delta_N &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{N(x)}{x+\Omega'}\mathrm{d}x \\ \delta_M &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{M(x)}{x+\Omega'}\,\mathrm{d}x.\end{aligned} \tag{27}$$

In the derivation of equation (25) we have assumed that the phase φ_s does not depend on frequency [42], and we have included the divergent frequency shifts (the Lamb shift) to the redefinition of the atomic transition frequency [40]. We have also assumed that the squeezed vacuum is symmetric about the central frequency ω_L, so that $N(\omega_L - \Omega') = N(\omega_L + \Omega')$ ($N_- = N_+$), $M(\omega_L - \Omega') = M(\omega_L + \Omega')$ ($M_- = M_+$). Moreover, the coupling $K(\omega)$ is assumed to be a slowly varying function of ω with respect to the squeezing parameters $N(\omega)$ and $M(\omega)$, so we can consider it as being constant, $K(\omega) = \sqrt{\gamma/2\pi}$.

The Cauchy principal values of the integrals in (27) can be evaluated using the contour integration which gives

$$\begin{aligned}\delta_N &= \delta_\mu - \delta_\lambda \\ \delta_M &= \delta_\mu + \delta_\lambda\end{aligned} \tag{28}$$

where the form of δ_μ and δ_λ depends on the type of squeezing being considered and is explicitly given by:

(i) for the degenerate case

$$\delta_\mu = \gamma\Omega'\frac{\lambda^2-\mu^2}{4}\frac{1}{\mu(\Omega'^2+\mu^2)} \tag{29}$$

$$\delta_\lambda = \gamma\Omega'\frac{\lambda^2-\mu^2}{4}\frac{1}{\lambda(\Omega'^2+\lambda^2)} \tag{30}$$

and (ii) for the non-degenerate case with $\alpha = \Omega'$

$$\delta_\mu = \gamma\Omega' \frac{\lambda^2 - \mu^2}{4} \frac{1}{\mu(4\Omega'^2 + \mu^2)} \tag{31}$$

$$\delta_\lambda = \gamma\Omega' \frac{\lambda^2 - \mu^2}{4} \frac{1}{\lambda(4\Omega'^2 + \lambda^2)}. \tag{32}$$

The master equation (25) has the form known from the broadband squeezing approaches with the new effective squeezing parameters $\tilde{N}$ and $\tilde{M}$ given by (26). Beside the known terms, there are also new terms (proportional to β), which are essentially narrow bandwidth modifications to the master equation. All the narrow bandwidth modifications are determined by the parameters Γ_-, δ_N and δ_M defined in (26) and (27). The parameter Γ_- describes the asymmetry of the squeezing properties at the centre ($\omega = \omega_L$) and at the sidebands ($\omega = \omega_L \pm \Omega'$), and the parameters δ_N and δ_M are the shifts associated with the non-zero Cauchy principal values appearing in (24). All of them become zero when the squeezing bandwidth goes to infinity ($N_0 = N_+$, $M_0 = M_+$). In the broadband squeezing reservoir the characteristic squeezing terms, $\sigma_+\rho\sigma_+$ and $\sigma_-\rho\sigma_-$, in the master equation (25) are proportional to the constant squeezing parameter $M = M_0 = M_\pm$. The squeezing terms come from the squeezing properties of the reservoir expressed exclusively by the correlation functions (18), and in this case they are really squeezing terms in the sense that they disappear if there is no squeezing in the reservoir ($M = 0$). However, it is also clear from (26) that for finite bandwidth of the reservoir $\tilde{M}$ can be non-zero even for $M_0 = M_+ = 0$ and $\Delta = 0$ if $N_0 \neq N_+$, i.e. if the mean number of photons of the reservoir is different at the centre and at the sidebands. Thus, we can distinguish between the contributions from (18) and (17) to $\tilde{M}$. We will refer to the latter as 'squeezing-like' contributions. The role of particular contributions becomes more clear when looking at the Bloch equations generated by the master equation.

2.2. Bloch equations

From the master equation (25) we easily derive the optical Bloch equations for the mean values of the atomic operators [36]

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \langle\sigma_-(t)\rangle \\ \langle\sigma_+(t)\rangle \\ \langle\sigma_z(t)\rangle \end{pmatrix} = -\boldsymbol{A}\begin{pmatrix} \langle\sigma_-(t)\rangle \\ \langle\sigma_+(t)\rangle \\ \langle\sigma_z(t)\rangle \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} \tag{33}$$

where

$$\boldsymbol{A} = \begin{pmatrix} \frac{\gamma}{2} + \tilde{N} - \mathrm{i}\delta & \tilde{M} & -\frac{\mathrm{i}}{2}\Omega \\ \tilde{M}^\star & \frac{\gamma}{2} + \tilde{N} + \mathrm{i}\delta & \frac{\mathrm{i}}{2}\Omega \\ -\mathrm{i}\Omega + \beta^\star & \mathrm{i}\Omega + \beta & \gamma + 2\tilde{N} \end{pmatrix}. \tag{34}$$

Defining the Hermitian operators σ_x and σ_y as

$$\sigma_x = \frac{1}{2}(\sigma_- + \sigma_+) \qquad \sigma_y = \frac{1}{2\mathrm{i}}(\sigma_- - \sigma_+) \tag{35}$$

we get from (33) the following equations of motion for the atomic polarization quadratures:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \langle\sigma_x(t)\rangle \\ \langle\sigma_y(t)\rangle \\ \langle\sigma_z(t)\rangle \end{pmatrix} = -\boldsymbol{B}\begin{pmatrix} \langle\sigma_x(t)\rangle \\ \langle\sigma_y(t)\rangle \\ \langle\sigma_z(t)\rangle \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} \tag{36}$$

with the matrix $\boldsymbol{B}$ given by

$$\boldsymbol{B} = \begin{pmatrix} \gamma_x & \delta_y & 0 \\ -\delta_x & \gamma_y & -\frac{1}{2}\Omega \\ -2\Omega_x & 2\Omega_y & \gamma_x + \gamma_y \end{pmatrix} \tag{37}$$

where we have introduced the notation

$$\gamma_x = \frac{\gamma}{2} + \tilde{N} + \mathrm{Re}\tilde{M} \qquad \gamma_y = \frac{\gamma}{2} + \tilde{N} - \mathrm{Re}\,\tilde{M} \tag{38}$$

$$\delta_x = \delta - \mathrm{Im}\,\tilde{M} \qquad \delta_y = \delta + \mathrm{Im}\,\tilde{M} \tag{39}$$

$$\Omega_x = \mathrm{Re}\,\beta \qquad \Omega_y = \Omega + \mathrm{Im}\,\beta. \tag{40}$$

The quantities γ_x and γ_y given by (38) describe two different damping rates for the two quadrature components of the atomic dipole moment. They are modified with respect to the values known for the broadband squeezing [16] because the values of $\tilde{N}$ and $\tilde{M}$ are different from their broadband squeezing counterparts, although the form of the damping rates remains the same. Similarly, δ_x and δ_y are modified detunings.

According to (26) the damping rates γ_x and γ_y can be rewritten in the form

$$\gamma_x = \gamma[\tfrac{1}{2} + N_+ + M_+\cos\phi - \tilde{\Delta}\delta_M \sin\phi] \tag{41}$$

$$\begin{aligned}\gamma_y = \gamma[\tfrac{1}{2} &+ N_+ - M_+\cos\phi + \tilde{\Delta}\delta_M\sin\phi \\ &+ (1-\tilde{\Delta}^2)[N_0 - N_+ - (M_0 - M_+)\cos\phi]]\end{aligned} \tag{42}$$

and on resonance ($\tilde{\Delta} = 0$) we have

$$\gamma_x = \gamma[\tfrac{1}{2} + N_+ + M_+\cos\phi] \tag{43}$$

$$\gamma_y = \gamma[\tfrac{1}{2} + N_0 - M_0\cos\phi] \tag{44}$$

which shows that on resonance γ_x is determined by the squeezing properties on the sidebands while γ_y is determined by the squeezing properties at the central line [33]. Both of them depend on the squeezing phase. For broadband squeezing they go over into the well known form [16].

The Bloch equations (36) can be easily solved for the steady-state values of the atomic variables, and the result is given by

$$\begin{aligned}\langle\sigma_x\rangle_{ss} &= \frac{1}{2}\frac{\gamma\Omega\delta_y}{d} \\ \langle\sigma_y\rangle_{ss} &= -\frac{1}{2}\frac{\gamma\Omega\gamma_x}{d} \\ \langle\sigma_z\rangle_{ss} &= -\frac{\gamma(\gamma_x\gamma_y + \delta_x\delta_y)}{d}\end{aligned} \tag{45}$$

where

$$d = (\gamma_x + \gamma_y)(\gamma_x\gamma_y + \delta_x\delta_y) + \Omega(\gamma_x\Omega_y + \delta_y\Omega_x). \tag{46}$$

An interesting feature of the steady-state solutions (45) is the fact that the dispersion component $\langle\sigma_x\rangle_{ss}$ of the atomic dipole can be non-zero even for a resonant driving field ($\Delta = 0$). We find from (26), (27) and (39) that

$$\delta_y = \delta + \mathrm{Im}\,\tilde{M} = \gamma M(\omega_A)\sin\phi \tag{47}$$

indicating that even for $\Delta = 0$ the $\langle\sigma_x\rangle_{ss}$ component of the Bloch vector can have a non-zero steady-state solution provided the phase ϕ is different from 0 or π and there is a non-zero squeezing at the atomic resonance. This effect can lead to unequal populations of the dressed states of the system [43], which has important physical consequences [36]. The steady-state dressed population inversion $\tilde{\sigma}_z$, according to the transformation (13), is related to the steady-state solutions (45) by

$$\langle\tilde{\sigma}_z\rangle_{ss} = 2\tilde{\Omega}\langle\sigma_x\rangle_{ss} - \tilde{\Delta}\langle\sigma_z\rangle_{ss} \tag{48}$$

and on resonance ($\tilde{\Delta} = 0$) it is expressed solely by $\langle\sigma_x\rangle_{ss}$, which is different from zero when δ_y is non-zero. Thus, the squeezed vacuum reservoir makes it possible to produce population inversion between the dressed atomic states even on resonance provided the squeezing phase is different from 0 or π.

The different damping rates γ_x and γ_y given by (38) have important consequences in the fluorescence and absorption spectra of the atom in a squeezed vacuum leading to narrowing or broadening of the spectral lines [17, 33, 36]. The phase-dependent narrowing of the spectral lines is a characteristic feature of the atomic resonance fluorescence when the atom is damped to a squeezed vacuum reservoir which has been widely discussed in the literature (see, for example, [44–46] and the papers cited therein). Here, we restrict our considerations to the master equation and the Bloch equations, and we are going to compare their forms for two different reservoirs: a squeezed vacuum with finite bandwidth and a structured but non-squeezed reservoir.

3. An atom in a structured reservoir

3.1. Master equation

In section 2.1 we have presented the derivation of the master equation for the case when the atom is driven by a strong laser field and is damped to a squeezed vacuum reservoir. The crucial element of the derivation was to perform dressing transformation before the atom was coupled to the reservoir and next to make the Markov approximation despite the finite bandwidth of the squeezed vacuum reservoir. Strictly speaking, the reservoir with finite bandwidth is non-Markovian, but if the bandwidth of the reservoir is much larger than the natural atomic linewidth, or to be more specific, if the linewidths of the Lorentzians appearing in $N(\omega)$ and $M(\omega)$ are much larger than the atomic linewidth, we can safely assume that the Markovian approximation still works sufficiently well. Performing the dressing transformation before the atom is coupled to the reservoir allows us to lift the requirement that the bandwidth of the reservoir must also be larger than the Rabi frequency describing the coupling of the atom to the laser field. Under these assumptions it turned out to be possible to obtain a simple master equation, in the operator form quite similar to that known for the broadband squeezing reservoir, in which modifications coming from the finite bandwidth of the reservoir have been explicitly accounted for. The important feature of the master equation for the atom in a squeezed vacuum are 'squeezing terms', $\sigma_+\rho\sigma_+$ and $\sigma_-\rho\sigma_-$, that stem from the correlation functions (18). Such terms do not appear in the master equation in the case of the thermal reservoir, which is characterized by the correlations (17) with $N(\omega)$ being a slowly varying function of ω. It is interesting to see, however, that for the reservoir with the mode density having a non-trivial structure, e.g., a Lorentzian peak as in the cavity situation, the terms $\sigma_+\rho\sigma_+$ and $\sigma_-\rho\sigma_-$ do appear in the master equation even if the reservoir is described solely by the correlations (17). We refer to such terms as 'squeezing-like' terms because they have a different origin, but yet they have some properties of the squeezing terms. To show this explicitly, we present here the master equation for the atom driven by a strong laser field and damped to a structured or tailored reservoir, in the form derived by Kowalewska-Kudłaszyk and Tanaś [37].

Again, we consider a two-level atom driven by a strong monochromatic laser field of frequency ω_L with the Rabi frequency Ω and detuned by $\Delta = \omega_L - \omega_A$ from the atomic transition frequency ω_A. We derive the master equation that takes into account explicitly the dependence of atomic relaxation rates on the strength of the field as well as the structure of the reservoir. The derivation proceeds along the same lines to those presented in section 2.1, i.e. we first perform the dressing transformation (11) to include the interaction of the atom with the driving field and then couple the resulting dressed atom to the reservoir. We make the Markov approximation which requires the reservoir bandwidth to be much greater than the atomic linewidth, but not necessarily greater than the Rabi frequency of the driving field and the detuning. Since the reservoir is assumed to be thermal, it is characterized by the correlations (17), where $N(\omega)$ is the mean number of photons at frequency ω and its dependence on ω can be ignored. However, the density of modes has a structure, which is accounted for by assuming that the coupling parameter $K(\omega)$ depends essentially on frequency. To take this dependence into account we put

$$K^2(\omega) = \frac{\gamma}{2\pi}\left(\frac{\omega}{\omega_A}\right)^3 \eta(\omega) \tag{49}$$

where $\eta(\omega)$ describes the deviation of the reservoir density of modes from the vacuum density of modes, for the vacuum $\eta(\omega) = 1$ and $K(\omega_A) = \sqrt{\gamma/2\pi}$, as assumed before. In the calculations we take into account only the frequency dependence assuming that the integration over angular variables has already been performed. The ω^3 dependence stems from the ordinary vacuum density of modes and γ is the natural atomic linewidth. With these assumptions, following the same steps as in section 2.1, we arrive at the following master equation [37]:

$$\begin{aligned}\dot{\rho} = {} & \frac{\mathrm{i}}{2}\delta[\sigma_z, \rho] - \frac{\mathrm{i}}{2}\Omega\,[\sigma_+ + \sigma_-, \rho] \\ & + \tfrac{1}{2}\tilde{N}\,(2\sigma_+\rho\sigma_- - \sigma_-\sigma_+\rho - \rho\sigma_-\sigma_+) \\ & + \tfrac{1}{2}(\tilde{N} + a)(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-) \\ & - \tilde{M}\sigma_+\rho\sigma_+ - \tilde{M}^\star\sigma_-\rho\sigma_- \\ & + \tfrac{1}{2}L[\sigma_+, \rho\sigma_z] - \tfrac{1}{2}L^\star[\sigma_-, \sigma_z\rho] \\ & + \tfrac{1}{2}(L + b)[\sigma_-, \rho\sigma_z] - \tfrac{1}{2}(L + b)^\star[\sigma_+, \sigma_z\rho]\end{aligned} \tag{50}$$

and now the parameters appearing in the master equation are defined by

$$\begin{aligned}\delta &= \Delta + \Delta_p \\ \Delta_p &= \frac{\gamma}{8}[(1 + \tilde{\Delta})^2(1 + 2N_-)b_- + (1 - \tilde{\Delta})^2(1 + 2N_+)b_+ \\ &\quad + 2(1 - \tilde{\Delta}^2)(1 + 2N_0)b_0]\end{aligned}$$

$$
\begin{aligned}
\tilde{N} &= \frac{\gamma}{4}[(1+\tilde{\Delta})^2 N_- a_- + (1-\tilde{\Delta})^2 N_+ a_+ + 2(1-\tilde{\Delta}^2) N_0 a_0] \\
a &= \frac{\gamma}{4}[(1+\tilde{\Delta})^2 a_- + (1-\tilde{\Delta})^2 a_+ + 2(1-\tilde{\Delta}^2) a_0] \\
\tilde{M} &= \frac{\gamma}{8}(1-\tilde{\Delta}^2)[(1+2N_-)(a_- - \mathrm{i}b_-) \\
&\quad + (1+2N_+)(a_+ - \mathrm{i}b_+) - 2(1+2N_0)(a_0 - \mathrm{i}b_0)] \\
L &= \frac{\gamma}{4}\tilde{\Omega}[(1+\tilde{\Delta})N_-(a_- + \mathrm{i}b_-) - (1-\tilde{\Delta})N_+(a_+ + \mathrm{i}b_+) \\
&\quad -2\tilde{\Delta} N_0(a_0 + \mathrm{i}b_0)] \\
b &= \frac{\gamma}{4}\tilde{\Omega}[(1+\tilde{\Delta})(a_- + \mathrm{i}b_-) - (1-\tilde{\Delta})(a_+ + \mathrm{i}b_+) \\
&\quad -2\tilde{\Delta}(a_0 + \mathrm{i}b_0)]
\end{aligned}
\tag{51}
$$

and

$$
\begin{aligned}
N_0 &= N(\omega_L) \qquad N_\pm = N(\omega_L \pm \Omega') \\
a_0 &= \left(\frac{\omega_L}{\omega_A}\right)^3 \eta(\omega_L) \qquad a_\pm = \left(\frac{\omega_L \pm \Omega'}{\omega_A}\right)^3 \eta(\omega_L \pm \Omega') \\
b_0 &= -\frac{1}{\gamma}\mathcal{P}\int_0^\infty \frac{K(\omega)^2}{\omega_L - \omega}\mathrm{d}\omega \\
b_\pm &= -\frac{1}{\gamma}\mathcal{P}\int_0^\infty \frac{K(\omega)^2}{\omega_L - \omega \pm \Omega'}\mathrm{d}\omega
\end{aligned}
\tag{52}
$$

where $N(\omega)$ is the mean number of the reservoir photons at frequency ω. As before, in the derivation of equation (50) we have included the divergent frequency shifts (the Lamb shift) in the redefinition of the atomic transition frequency, and we have explicitly calculated the shifts that come from the principal value terms in (52). These shifts can give contributions to the master equation in cases when the atom is placed in a cavity with frequency-dependent density of modes and $\eta(\omega)$ has essential ω dependence.

The principal value terms in (52) can be evaluated when $\eta(\omega)$ is known. In the calculations we model the mode structure by the dimensionless Lorentzian functions. Let us assume that $\eta(\omega)$ is a Lorentzian

$$
\eta(\omega) = \frac{\gamma_c^2}{(\omega-\omega_c)^2 + \gamma_c^2}
\tag{53}
$$

with the width γ_c ($\gamma_c \gg \gamma$) and centred at some frequency ω_c (for $\gamma_c \to \infty$ $\eta(\omega) \to 1$). Physically, this can be considered, for example, as a cavity situation. More realistic modelling of the cavity introduces flat background modes and cavity modes with a Lorentzian peak at the cavity resonance [3, 9]. In such a case, instead of being just a Lorentzian, $\eta(\omega)$ would be a constant independent of ω representing the background modes plus a Lorentzian describing the cavity modes. Since we are mainly interested in structured reservoirs, here we use only the Lorentzian function to describe the non-flat reservoir, although adding a constant part would be straightforward (the constant part does not contribute to the shifts). The width γ_c should be much greater than the atomic linewidth γ in order not to violate the Markovian approximation made in the derivation of the master equation. From the definitions (52), using (53), we can calculate the parameters b_0 and $b_\pm$ in the following way:

$$
\begin{aligned}
b_0 &= -\frac{1}{\gamma}\mathcal{P}\int_0^\infty \frac{K(\omega)^2}{\omega_L - \omega}\mathrm{d}\omega \\
&= -\frac{1}{2\pi}\mathcal{P}\int_0^\infty \left(\frac{\omega}{\omega_A}\right)^3 \frac{\gamma_c^2}{(\omega-\omega_c)^2+\gamma_c^2}\frac{1}{\omega_L-\omega}\mathrm{d}\omega \\
&\approx -\left(\frac{\omega_L}{\omega_A}\right)^3 \frac{1}{2\pi}\mathcal{P}\int_0^\infty \frac{\gamma_c^2}{(\omega-\omega_c)^2+\gamma_c^2}\frac{1}{\omega_L-\omega}\mathrm{d}\omega \\
&= \left(\frac{\omega_L}{\omega_A}\right)^3 \frac{1}{2\pi}\mathcal{P}\int_{-\infty}^\infty \frac{\gamma_c^2}{(x^2+\gamma_c^2)(x-\delta_c)}\mathrm{d}x \\
&= -\frac{1}{2}\left(\frac{\omega_L}{\omega_A}\right)^3 \frac{\delta_c\gamma_c}{\delta_c^2+\gamma_c^2}
\end{aligned}
\tag{54}
$$

where $\delta_c = \omega_L - \omega_c$, and after changing variables we have extended the integration from $-\omega_c$ to $-\infty$. Proceeding in the same manner we get

$$
b_\pm = -\frac{1}{2}\left(\frac{\omega_L \pm \Omega'}{\omega_A}\right)^3 \frac{(\delta_c \pm \Omega')\gamma_c}{(\delta_c \pm \Omega')^2 + \gamma_c^2}.
\tag{55}
$$

The values of the shifts depend on the width γ_c and the position of the mode density peak. The most interesting cases are when the peak is centred at the laser frequency ($\delta_c = 0$), or at the Rabi sidebands ($\delta_c = \pm\Omega'$).

The master equation (50), in operator form, is a generalization of the standard master equation known for the two-level atom. The generalization takes into account the dependence of the relaxation rates on the strength of the driving field, described by the dependence of $a_\pm$ on the Rabi frequency Ω' through the ω^3 terms as well as the difference of the reservoir mode density $\eta(\omega)$ from the ordinary vacuum mode density. N_0 and $N_\pm$ are the mean number of reservoir photons at the laser frequency ω_L and at the sidebands $\omega_L \pm \Omega'$, respectively. On neglecting the shift terms, master equation (50), although different in form, is equivalent to the generalized Bloch equations introduced by Kocharovskaya *et al* [12]. The difference is that we have performed the dressing transformation on the operators rather than on the atomic states. As we believe, the advantage of this approach is a strikingly simple and transparent form of the master equation (50) which allows for easy identification of the standard terms known for the ordinary vacuum and recognizing the new, non-standard terms that appear due to the strong-field modification of the damping rates and/or tailoring of the reservoir. It is also easy to compare master equations obtained for squeezed vacuum and structured reservoirs, which is the main goal of this paper.

For weak driving fields and thermal reservoirs ($\eta(\omega) = 1$), we have $a_0 = a_\pm = 1$ and $N_0 = N_\pm$ is the mean number of photons of the reservoir, which means that $\tilde{N} = \gamma N_0$ and $a = \gamma$ while $\tilde{M} = L = b = 0$, and master equation (50) takes the well known standard form. For non-thermal or tailored reservoirs, however, for which $\eta(\omega)$ is different from unity, the new terms become important, and the atomic evolution is changed in an essential way. It is particularly interesting that the new terms, proportional to $\tilde{M}$, that are well known for the atom damped to the squeezed vacuum reservoir, appear in the master equation (50) despite the fact that the reservoir does not exhibit non-diagonal, phase-dependent correlations. These terms appear for the ordinary vacuum because of the asymmetry introduced to the system by the strong field and/or the non-flat mode structure of the reservoir. Other non-standard terms are those proportional to b and L.

Since the atomic operators $\sigma_\pm$ contain, according to (7), the phase factors $\exp(\mp \mathrm{i}\varphi)$, the terms proportional to Ω, $\tilde{M}$, b, and L in the master equation (50) are phase dependent. Their phase dependence, however, stems solely from the phase of the driving field in contrast to the squeezed vacuum reservoir, where the squeezing terms come from the phase-dependent reservoir correlation functions (18). Therefore, the phase will appear in the steady-state mean values of the atomic dipole moment $\langle\sigma_\pm\rangle_{ss}$, for example, but not in the resonance fluorescence and absorption spectra, in which the phase factors cancel. This makes an important difference between the 'squeezing-like' terms, proportional to $\tilde{M}$ in the master equation (50) and the real squeezing terms coming from the squeezed vacuum reservoir appearing in the master equation (25). In the case of squeezing reservoir the phase dependence of these terms is $\exp(\pm\mathrm{i}\phi)$, where $\phi = 2\varphi + \varphi_s$, and even if the phase factors stemming from the driving field cancel in the resonance fluorescence spectrum, the dependence on the squeezing phase remains, and the fluorescence spectrum is sensitive to the squeezing phase φ_s. However, as it will become clear later, the phase-sensitive terms that appear in the master equation (50) lead to some effects that are known for squeezing reservoirs, e.g., the difference in the damping rates of the two quadrature components of the atomic dipole.

For strong laser fields and flat reservoirs, assuming that $\eta(\omega) = 1$ and $\Omega'/\omega_A \ll 1$, we can expand a_0 and $a_\pm$ in a power series with respect to this small quantity. Keeping only the linear terms we get the approximate relations

$$a_0 \approx 1 + 3\tilde{\Delta}\frac{\Omega'}{\omega_A} \qquad a_\pm \approx 1 + 3(\tilde{\Delta} \pm \tilde{\Omega})\frac{\Omega'}{\omega_A}. \tag{56}$$

Moreover, if the reservoir is the ordinary vacuum (thermal field at $T = 0$), which means that the mean number of photons $N_0 = N_\pm = 0$ and the mode structure is flat ($\eta(\omega) = 1$), we have $\tilde{N} = L = 0$ and $b_0 = b_\pm = 0$. This gives us the following approximate expressions:

$$\begin{aligned} a &\approx \gamma\left[1 + 3\tilde{\Delta}(1 - \tilde{\Omega})\frac{\Omega'}{\omega_A}\right] \\ b &\approx -\frac{3}{2}\gamma(1 - \tilde{\Delta}^2)\frac{\Omega'}{\omega_A} \\ \tilde{M} &\approx 0 \end{aligned} \tag{57}$$

which shows that when the asymmetry introduced to the system comes from the dependence of the relaxation rates on the strength of the field only, the 'squeezing-like' terms proportional to $\tilde{M}$, in the first approximation, are zero. This is not true, however, if the density of modes of the reservoir differs considerably from the free space density, i.e. $\eta(\omega)$ appreciably depends on frequency, or the mean number of photons $N(\omega)$ is not zero and essentially depends on frequency. In this case the full form of the coefficients (51) and (52) should be used in the master equation (50). For the structured vacuum, when the mean number of photons is zero ($N_+ = N_- = N_0 = 0$) and $\eta(\omega)$ is given by (53), we have $\tilde{N} = 0$, but $\tilde{M}$ is non-zero and it is given by the following simple formula:

$$\tilde{M} = \frac{\gamma}{8}(1 - \tilde{\Delta}^2)[a_- + a_+ - 2a_0 - \mathrm{i}(b_- + b_+ - 2b_0)] \tag{58}$$

with a and b defined in (52). If the maximum of the density of modes given by the Lorentzian (53) is tuned to a particular frequency (the central line at ω_L or one of the sidebands at $\omega_L \pm \Omega'$) $\tilde{M}$ take values significantly different from zero, which has an important effect on the evolution of the atomic dipole. The real part of $\tilde{M}$ affects the damping rates for the two quadrature components of the atomic dipole, the imaginary part of $\tilde{M}$ modifies the detuning. This is better seen from the Bloch equations.

3.2. Bloch equations

As before, from the master equation (50) one can derive the generalized Bloch equations describing the time evolution of the expectation values of the atomic operators, which take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \langle\sigma_-(t)\rangle \\ \langle\sigma_+(t)\rangle \\ \langle\sigma_z(t)\rangle \end{pmatrix} = -\boldsymbol{A}\begin{pmatrix} \langle\sigma_-(t)\rangle \\ \langle\sigma_+(t)\rangle \\ \langle\sigma_z(t)\rangle \end{pmatrix} - \begin{pmatrix} \frac{1}{2}(L - L^\star - b^\star) \\ \frac{1}{2}(L^\star - L - b) \\ a \end{pmatrix} \tag{59}$$

$$\boldsymbol{A} = \begin{pmatrix} \frac{a}{2} + \tilde{N} - \mathrm{i}\delta & \tilde{M} & -\frac{\mathrm{i}}{2}\Omega \\ \tilde{M}^\star & \frac{a}{2} + \tilde{N} + \mathrm{i}\delta & \frac{\mathrm{i}}{2}\Omega \\ -\mathrm{i}\Omega - b - L - L^\star & \mathrm{i}\Omega - b^\star - L - L^\star & a + 2\tilde{N} \end{pmatrix}. \tag{60}$$

Introducing the Hermitian operators (35), we get from (59) the following equations of motion for the atomic polarization quadratures:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \langle\sigma_x(t)\rangle \\ \langle\sigma_y(t)\rangle \\ \langle\sigma_z(t)\rangle \end{pmatrix} = -\boldsymbol{B}\begin{pmatrix} \langle\sigma_x(t)\rangle \\ \langle\sigma_y(t)\rangle \\ \langle\sigma_z(t)\rangle \end{pmatrix} - \begin{pmatrix} -\frac{1}{2}r_x \\ \frac{1}{2}r_y \\ a \end{pmatrix} \tag{61}$$

with the matrix $\boldsymbol{B}$ of the form given by (37), but now with the parameters constituting the matrix that are defined as follows:

$$\gamma_x = \frac{a}{2} + \tilde{N} + \mathrm{Re}\,\tilde{M} \qquad \gamma_y = \frac{a}{2} + \tilde{N} - \mathrm{Re}\,\tilde{M} \tag{62}$$

$$\delta_x = \delta - \mathrm{Im}\,\tilde{M} \qquad \delta_y = \delta + \mathrm{Im}\,\tilde{M} \tag{63}$$

$$\Omega_x = \mathrm{Re}\,(b + 2L) \qquad \Omega_y = \Omega + \mathrm{Im}\,b \tag{64}$$

$$r_x = \mathrm{Re}\,b \qquad r_y = \mathrm{Im}\,(b + 2L). \tag{65}$$

The generalized Bloch equations (59) and (61) are different from the standard Bloch equations. The relaxation rates have been obtained by coupling the dressed atom rather than the bare atom to the reservoir, so they take into account the dependence of the relaxation rates on the strength of the laser field and the structure of the reservoir modes including the shifts which are non-zero when the density of modes is not flat. If we ignore the shift terms coming from the principal value contributions, our Bloch equations are equivalent to the Bloch equations obtained earlier by Kocharovskaya *et al* [12].

It is interesting to notice in the generalized Bloch equations (61) the presence of different damping rates γ_x and γ_y for the two quadratures of the atomic dipole and the modified detunings δ_x and δ_y given by (63). They contain the $\tilde{M}$ terms which play a similar role to the $\tilde{M}$ terms in the case of the squeezed vacuum reservoir, as seen from (38) and (39). These terms introduce coupling between $\langle\sigma_+\rangle$ and $\langle\sigma_-\rangle$ in the

Bloch equations (59), and according to (37) and (62) the two quadrature components of the atomic dipole have different damping rates similarly to the squeezed vacuum reservoir, for which the rates are given by (38). The physical origin of this effect, however, is quite different in both cases. Assuming that the reservoir is the structured vacuum ($N_0 = N_- = N_+ = 0$ and $\eta(\omega)$ given by (53)) and using (58), we find

$$\gamma_x = \frac{\gamma}{4}[(1+\tilde{\Delta})a_- + (1-\tilde{\Delta})a_+] \tag{66}$$

$$\gamma_y = \frac{\gamma}{4}[\tilde{\Delta}(1+\tilde{\Delta})a_- - \tilde{\Delta}(1-\tilde{\Delta})a_+ + 2(1-\tilde{\Delta}^2)a_0] \tag{67}$$

and, on resonance, we have very simple formulae

$$\gamma_x = \frac{\gamma}{4}(a_- + a_+) \tag{68}$$

$$\gamma_y = \frac{\gamma}{2}a_0. \tag{69}$$

It is interesting to compare (66)–(69) with the corresponding formulae (41)–(44) for the case of the squeezed vacuum reservoir. Similarly as before, on resonance, the rate γ_x is determined by the properties of the reservoir at the sidebands while the rate γ_y is determined by the properties at the central line. In contrast to the squeezed vacuum, the rates do not depend on phase, but similarly to the squeezed vacuum there are two different rates for the two quadrature components of the atomic dipole if the properties of the reservoir are different at the sidebands and at the central line.

To make the physical interpretation of the asymmetry in the damping rates (68) and (69) more transparent it is essential to realize that $\gamma a_0 = \gamma_0$, $\gamma a_- = \gamma_-$ and $\gamma a_+ = \gamma_+$ are the damping rates for the transitions between the dressed states of the atom with frequencies ω_L, $\omega_L - \Omega$ and $\omega_L + \Omega$, respectively. These rates are different because the transition frequencies ω_L and $\omega_L \pm \Omega$ are different and the density of modes is different at these frequencies. The differences are important, however, when $\Omega \geqslant \gamma_c$, i.e. the Rabi frequency is sufficiently large and the width of the density of modes is sufficiently narrow as to make the three damping rates noticeably different. The asymmetry of the damping rates for the two components of the bare atomic dipole appears because of the splitting of the atomic levels in the strong field. Thus, in contrast to the squeezed vacuum reservoir for which the squeezing terms come from the non-zero value of the two-photon correlation function of the reservoir, the origin of squeezing-like terms is related to the change of atomic level structure when the atom is subjected to a resonant laser field. It is also important to realize that despite the asymmetry in the two damping rates the fluorescence spectrum from such an atom can still be symmetric [14, 15, 37] if the fluorescence goes to the structured reservoir modes. Asymmetry can be observed for the fluorescence that goes to the flat background modes [11].

Another interesting feature of the Bloch equations (61) is the presence of the free terms r_x and r_y, which give, for example, a non-zero steady state solution for $\langle\sigma_x\rangle$. The differences are important in the atomic spectra [37], which we are not going to discuss here. One more important feature of the Bloch equations (61) is their dependence on b_0 and $b_\pm$, which are additional parameters arising from the principal value contributions. They should manifest themselves in situations of moderately intense laser fields and atoms in reservoirs with frequency-dependent density of modes.

The steady state solutions to equations (61) are the following:

$$\begin{aligned}
\langle\sigma_x\rangle_{ss} &= \frac{1}{2d}\{(a\Omega\delta_y + r_x[\gamma_y(\gamma_x+\gamma_y) + \Omega\Omega_y] \\
&\quad + r_y\delta_y(\gamma_x+\gamma_y)\} \\
\langle\sigma_y\rangle_{ss} &= -\frac{1}{2d}\{(a\Omega\gamma_x - r_x[\delta_y(\gamma_x+\gamma_y) + \Omega\Omega_x] \\
&\quad + r_y\gamma_x(\gamma_x+\gamma_y)\} \\
\langle\sigma_z\rangle_{ss} &= -\frac{1}{d}\{a(\delta_x\delta_y + \gamma_x\gamma_y) + r_x(\delta_x\Omega_y - \gamma_y\Omega_x) \\
&\quad - r_y(\gamma_x\Omega_y + \delta_y\Omega_x)\}
\end{aligned} \tag{70}$$

where the denominator d has the form given by (46), but with the values of the parameters defined by (51)–(52) and (62)–(65).

In the strong field limit when Ω' is much greater than all the damping rates, the steady-state solutions (70) take a much simpler, approximate form (we keep only the lowest non-vanishing terms)

$$\begin{aligned}
\langle\sigma_x\rangle_{ss} &= \frac{1}{2}\frac{\tilde{\Delta}\tilde{\Omega}a + (1-\tilde{\Delta}^2)r_x}{(1-\tilde{\Delta}^2)\gamma_x + \tilde{\Delta}[\tilde{\Delta}(\gamma_x+\gamma_y) + \tilde{\Omega}\Omega_x]} \\
\langle\sigma_y\rangle_{ss} &= -\frac{1}{2\Omega'}\frac{\tilde{\Omega}\gamma_x a - [\tilde{\Delta}(\gamma_x+\gamma_y) + \tilde{\Omega}\Omega_x]r_x}{(1-\tilde{\Delta}^2)\gamma_x + \tilde{\Delta}[\tilde{\Delta}(\gamma_x+\gamma_y) + \tilde{\Omega}\Omega_x]} \\
\langle\sigma_z\rangle_{ss} &= -\frac{\tilde{\Delta}(\tilde{\Delta}a + \tilde{\Omega}r_x)}{(1-\tilde{\Delta}^2)\gamma_x + \tilde{\Delta}[\tilde{\Delta}(\gamma_x+\gamma_y) + \tilde{\Omega}\Omega_x]}.
\end{aligned} \tag{71}$$

For thermal reservoirs for which the mean number of photons does not depend appreciably on frequency, $N(\omega) = N_0 = N_\pm$, equations (71) go over into

$$\begin{aligned}
\langle\sigma_x\rangle_{ss} &= \frac{\tilde{\Omega}}{2(1+2N_0)}\frac{(1+\tilde{\Delta})^2a_- - (1-\tilde{\Delta})^2a_+}{(1+\tilde{\Delta})^2a_- + (1-\tilde{\Delta})^2a_+} \\
\langle\sigma_y\rangle_{ss} &= -\frac{\tilde{\Omega}}{4\Omega'}\frac{2(1-\tilde{\Delta}^2)a_-a_+ + [(1+\tilde{\Delta})^2a_- + (1-\tilde{\Delta})^2a_+]a_0}{(1+\tilde{\Delta})^2a_- + (1-\tilde{\Delta})^2a_+} \\
\langle\sigma_z\rangle_{ss} &= -\frac{\tilde{\Delta}}{1+2N_0}\frac{(1+\tilde{\Delta})^2a_- - (1-\tilde{\Delta})^2a_+}{(1+\tilde{\Delta})^2a_- + (1-\tilde{\Delta})^2a_+}.
\end{aligned} \tag{72}$$

From equations (72) it is evident that $\langle\sigma_y\rangle_{ss}$ is of the order of $1/\Omega'$ and becomes zero in the secular limit. It is also clear that the steady-state values of $\langle\sigma_x\rangle_{ss}$ and $\langle\sigma_z\rangle_{ss}$ depend on the density of photon modes at the sidebands only. Moreover, upon making an appropriate choice of the detuning $\tilde{\Delta}$ and choosing different mode densities at the two sidebands ($a_- \neq a_+$), steady-state atomic inversion can be realized. This effect, called vacuum-field dressed-state pumping, has been predicted by Lewenstein and Mossberg [9] and observed by Zhu *et al* [10]. On resonance, $\tilde{\Delta} = 0$, $\tilde{\Omega} = 1$, the steady-state solutions simplify even further, and the steady state value of $\langle\sigma_z\rangle_{ss}$ becomes zero meaning an equal population of the two atomic levels.

Another important feature of the solutions (72) is that the dispersion component of the atomic dipole $\langle\sigma_x\rangle_{ss}$ is non-zero if $a_- \neq a_+$. This can happen because of the difference in the mode density at the two sidebands and/or the dependence of the damping rate on the field intensity through the $((\omega \pm \Omega')/\omega_A)^3$ factor. The non-zero solution for $\langle\sigma_x\rangle_{ss}$ means the non-zero steady-state atomic dipole moment which has dramatic effect on the resonance fluorescence spectrum [9] in the frequency-dependent photon reservoirs. Similarly to the squeezed vacuum reservoir discussed in section 2.2 we can rewrite equation (48) for the population inversion of the dressed atom

$$\langle\tilde{\sigma}_z\rangle_{ss} = 2\tilde{\Omega}\langle\sigma_x\rangle_{ss} - \tilde{\Delta}\langle\sigma_z\rangle_{ss} \tag{73}$$

and now the steady-state solutions for $\langle\sigma_x\rangle_{ss}$ and $\langle\sigma_z\rangle_{ss}$ are given by (70) or for strong fields by (72). Again, on resonance, the dressed atom population inversion is determined by $\langle\sigma_x\rangle_{ss}$. For strong field and structured vacuum, according to (72), we have

$$\langle\tilde{\sigma}_z\rangle_{ss} = \frac{a_- - a_+}{a_- + a_+} \approx \frac{\eta(\omega_A - \Omega) - \eta(\omega_a + \Omega)}{\eta(\omega_A - \Omega) + \eta(\omega_A + \Omega)} \tag{74}$$

which shows that $\langle\tilde{\sigma}_z\rangle_{ss}$ can be close to unity when the reservoir density of modes has its peak at one of the Rabi sidebands and is close to zero at the other sideband. This means that only one of the dressed atomic states could be populated by appropriately tailoring the reservoir [9]. Populations of the dressed states determine the weights of the spectral components of the Mollow triplet in the strong-field resonance fluorescence. Details of the resonance fluorescence and absorption spectra for such a model have been discussed in [37].

The non-zero values of the dispersion component of the atomic dipole moment $\langle\sigma_x\rangle_{ss}$ is a common feature of both models considered in this paper. The physical origin of this feature, however, is quite different. In the case of the squeezed reservoir it is squeezing at the atomic frequency with the phase which is different from 0 or π, while in the case of the structured vacuum reservoir it is a difference of the density of modes at the Rabi sidebands. Despite the physical origin, nevertheless, the physical consequences are very similar.

4. Conclusion

This paper has been devoted to comparison of two different models describing the interaction of a two-level atom that is driven by a strong classical field and is damped to two different reservoirs: one—the squeezed vacuum with finite bandwidth, and the other—the structured but non-squeezed reservoir. The comparison has become easy since both models have been treated in the same way. The master equation has been obtained by applying the two-step procedure in which the first step is the dressing transformation that couples the atom to the strong laser field and the second step is the coupling of the dressed atom to the reservoir. All calculations have been performed on the operator level leading to the operator form of the master equations for both cases. The operator form of the master equations allows for easy comparison of the resulting master equations to each other and to the standard form of the master equation. We have shown that for reservoirs with finite bandwidth there are 'squeezing-like' terms appearing in the master equation even for reservoirs with zero phase-dependent two-photon correlation functions. In this respect both reservoirs considered here are similar—they both have finite bandwidth. When the bandwidth of the reservoir goes to infinity, the 'squeezing-like' terms disappear, and in the case of the squeezed vacuum reservoir the real squeezing terms remain.

Optical Bloch equations generated by the master equations reveal some physical consequences of the non-standard terms that occurred in the master equations. For example, it has been shown that the two quadrature components of the atomic dipole moment have different damping rates, the effect well known for the broadband squeezed vacuum reservoirs. A common feature of the two models appears to be the non-zero value of the steady-state solution for the dispersion component of the atomic dipole moment, although the origin of this feature is different in each model. This feature, in turn, leads to the non-zero population inversion of the atomic dressed states, which has important influence on the atomic spectra.

The master equations discussed in this paper have been obtained under the Born–Markov approximation. The Markov approximation made for the reservoir with finite bandwidth may, of course, be questionable because, strictly speaking, such a reservoir is non-Markovian. However, when the reservoir bandwidth is much larger than the atomic linewidth it is safe to assume that the Markov approximation gives reasonable results. The dressing transformation performed before coupling of the atom to the reservoir has the advantage that it lifts the requirement, otherwise necessary, that the reservoir bandwidth must also be larger than the Rabi frequency. This is important for strong fields. The master equation obtained in this way has broader applicability. However, one has to remember that the master equation is only valid for sufficiently broad bandwidth of the reservoir, and analytical results derived from it that explicitly depend on the reservoir bandwidth must be treated with care. Taking too small values for the reservoir bandwidth can even lead to unphysical results, but this is the price we pay for the simplicity of the master equation.

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