Elektrodynamika

Część 7

Zasady zachowania

Ryszard Tanaś

Zakład Optyki Nieliniowej, UAM

`http://zon8.physd.amu.edu.pl/\~tanas`

Spis treści

8 Zasady zachowania

8.1 Ładunek i energia

8.1.1 Równanie ciągłości

$Q(t) = \int\limits_{\mathcal{V}} \rho(\boldsymbol{r}, t)\, \mathrm{d}\tau$ ładunek w obszarze $\mathcal{V}$

$\dfrac{\mathrm{d}Q}{\mathrm{d}t} = -\int\limits_{\mathcal{S}} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a}$ zasada zachowania ładunku

$$\int\limits_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, \mathrm{d}\tau = -\int\limits_{\mathcal{V}} \nabla \cdot \boldsymbol{J}\, \mathrm{d}\tau$$

$\boxed{\dfrac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{J}}$ **równanie ciągłości**

8.1.2 Twierdzenie Poyntinga

$$W_{\mathrm{e}} = \frac{\epsilon_0}{2} \int E^2\, \mathrm{d}\tau$$

$$W_{\mathrm{m}} = \frac{1}{2\mu_0} \int B^2\, \mathrm{d}\tau$$

$U_{\mathrm{em}} = \dfrac{1}{2} \int \left(\epsilon_0 E^2 + \dfrac{1}{\mu_0} B^2 \right) \mathrm{d}\tau$ całkowita energia zmagazynowana w polu elektromagnetycznym

Ogólniejsze wyprowadzenie:

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}\, \mathrm{d}t = q\boldsymbol{E} \cdot \boldsymbol{v}\, \mathrm{d}t$$

$$q = \rho\, \mathrm{d}\tau, \qquad \rho \boldsymbol{v} = \boldsymbol{J}$$

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int_{\mathcal{V}} (\boldsymbol{E} \cdot \boldsymbol{J})\, \mathrm{d}\tau$$

$$\boldsymbol{E} \cdot \boldsymbol{J} = \frac{1}{\mu_0} \boxed{\boldsymbol{E} \cdot (\nabla \times \boldsymbol{B})} - \epsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t}$$ z prawa Ampère'a-Maxwella

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{E}) - \boxed{\boldsymbol{E} \cdot (\nabla \times \boldsymbol{B})}$$ pochodne iloczynów

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ prawo Faradaya

$$\boxed{\boldsymbol{E} \cdot (\nabla \times \boldsymbol{B})} = -\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B})$$

$$\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (B^2), \qquad \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (E^2)$$

$$\boldsymbol{E} \cdot \boldsymbol{J} = -\frac{1}{2} \frac{\partial}{\partial t} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \frac{1}{\mu_0} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B})$$

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \mathrm{d}\tau - \frac{1}{\mu_0} \oint_{\mathcal{S}} (\boldsymbol{E} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{a}$$

Twierdzenie Poyntinga: praca wykonana nad ładunkami przez siły elektromagnetyczne jest równa ubytkowi enegii zmagazynowanej w polach, pomniejszonemu o energię, która wypłynęła przez powierzchnię ograniczającą obszar

$$\boxed{\boldsymbol{S} \equiv \frac{1}{\mu_0} (\boldsymbol{E} \times \boldsymbol{B})}$$ **wektor Poyntinga**

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{\mathrm{d}U_{\mathrm{em}}}{\mathrm{d}t} - \oint_{\mathcal{S}} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a}$$

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} u_{\mathrm{mech}} \, \mathrm{d}\tau$$

praca wykonana nad ładunkami
zwiększa ich energię mechaniczną

$$\boxed{u_{\mathrm{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)}$$

gęstość energii pola

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} (u_{\mathrm{mech}} + u_{\mathrm{em}}) \, \mathrm{d}\tau = -\oint_{\mathcal{S}} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} = -\int_{\mathcal{V}} (\nabla \cdot \boldsymbol{S}) \, \mathrm{d}\tau$$

$$\boxed{\frac{\partial}{\partial t}(u_{\mathrm{mech}} + u_{\mathrm{em}}) = -\nabla \cdot \boldsymbol{S}}$$

8.2 Pęd

8.2.2 Tensor napięć Maxwella

$$\boldsymbol{F} = \int_{\mathcal{V}} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \rho \, \mathrm{d}\tau = \int_{\mathcal{V}} (\rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}) \, \mathrm{d}\tau$$

$$\boldsymbol{f} = \boxed{\rho} \boldsymbol{E} + \boxed{\boldsymbol{J}} \times \boldsymbol{B}$$

gęstość siły

$$\boldsymbol{f} = \boxed{\epsilon_0 (\nabla \cdot \boldsymbol{E})} \boldsymbol{E} + \left(\boxed{\frac{1}{\mu_0} \nabla \times \boldsymbol{B} - \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}} \right) \times \boldsymbol{B}$$

$$\frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}) = \left(\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B} \right) + \left(\boldsymbol{E} \times \frac{\partial \boldsymbol{B}}{\partial t} \right)$$

tożsamość

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \quad \text{prawo Faradaya}$$

$$\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B} = \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}) + \boldsymbol{E} \times (\nabla \times \boldsymbol{E})$$

$$\boldsymbol{f} = \epsilon_0[(\nabla \cdot \boldsymbol{E})\boldsymbol{E} - \boldsymbol{E} \times (\nabla \times \boldsymbol{E})]$$

$$- \frac{1}{\mu_0}[\boldsymbol{B} \times (\nabla \times \boldsymbol{B})] - \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}) + \frac{1}{\mu_0} \underbrace{(\nabla \cdot \boldsymbol{B})}_{=0} \boldsymbol{B}$$

$$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$$

$$+ (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} \quad \text{pochodne iloczynów}$$

$$\nabla(E^2) = \nabla(\boldsymbol{E} \cdot \boldsymbol{E}) = 2(\boldsymbol{E} \cdot \nabla)\boldsymbol{E} + 2\boldsymbol{E} \times (\nabla \times \boldsymbol{E})$$

$$\boldsymbol{E} \times (\nabla \times \boldsymbol{E}) = \frac{1}{2}\nabla(E^2) - (\boldsymbol{E} \cdot \nabla)\boldsymbol{E}$$

$$\boldsymbol{B} \times (\nabla \times \boldsymbol{B}) = \frac{1}{2}\nabla(B^2) - (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}$$

$$\boldsymbol{f} = \epsilon_0[(\nabla \cdot \boldsymbol{E})\boldsymbol{E} + (\boldsymbol{E} \cdot \nabla)\boldsymbol{E}] + \frac{1}{\mu_0}[(\nabla \cdot \boldsymbol{B})\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}]$$

$$- \frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B})$$

$$\boxed{T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)}$$ **tensor napięć Maxwella**

$$\delta_{ij} = \begin{cases} 1 & \text{dla } i = j \\ 0 & \text{dla } i \neq j \end{cases}$$ **delta Kroneckera**

$$T_{xx} = \frac{1}{2}\epsilon_0(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2\mu_0}(B_x^2 - B_y^2 - B_z^2)$$

$$T_{xy} = \epsilon_0(E_xE_y) + \frac{1}{\mu_0}(B_xB_y) \quad \text{itd.}$$

$$(\boldsymbol{a} \cdot \overleftrightarrow{\boldsymbol{T}})_j = \sum_{i=x,y,z} a_i T_{ij} \quad \text{iloczyn wektora } \boldsymbol{a} \text{ z tensorem } \overleftrightarrow{\boldsymbol{T}}$$

$$(\nabla \cdot \overleftrightarrow{\boldsymbol{T}})_j = \epsilon_0 \left[(\nabla \cdot \boldsymbol{E})E_j + (\boldsymbol{E} \cdot \nabla)E_j - \frac{1}{2}\nabla_j E^2\right] + \frac{1}{2\mu_0}\left[(\nabla \cdot \boldsymbol{B})B_j + (\boldsymbol{B} \cdot \nabla)B_j - \frac{1}{2}\nabla_j B^2\right]$$

$$\boxed{\boldsymbol{f} = \nabla \cdot \overleftrightarrow{\boldsymbol{T}} - \epsilon_0\mu_0 \frac{\partial \boldsymbol{S}}{\partial t}}$$

$$\boxed{\boldsymbol{F} = \oint_{\mathcal{S}} \overleftrightarrow{\boldsymbol{T}} \cdot \mathrm{d}\boldsymbol{a} - \epsilon_0\mu_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \boldsymbol{S}\, \mathrm{d}\tau}$$

8.2.3 Zasada zachowania pędu

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}_{\mathrm{mech}}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\boldsymbol{p}_{\mathrm{mech}}}{\mathrm{d}t} = -\epsilon_0\mu_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \boldsymbol{S}\, \mathrm{d}\tau + \oint_{\mathcal{S}} \overleftrightarrow{\boldsymbol{T}} \cdot \mathrm{d}\boldsymbol{a} \quad \text{zasada zachowania pędu}$$

$$\boldsymbol{p}_{\mathrm{em}} = \epsilon_0\mu_0 \int_{\mathcal{V}} \boldsymbol{S}\, \mathrm{d}\tau \quad \text{pęd pola elektromagnetycznego}$$

$$\boxed{\wp_{\mathrm{em}} = \epsilon_0 \mu_0 \boldsymbol{S}}$$ gęstość pędu pola

$$\boxed{\frac{\partial}{\partial t}(\wp_{\mathrm{mech}} + \wp_{\mathrm{em}}) = \nabla \cdot \overleftrightarrow{\boldsymbol{T}}}$$

$\overleftrightarrow{\boldsymbol{T}}$ — gęstość strumienia pędu

8.2.4 Moment pędu

$$u_{\mathrm{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$ gęstość energii

$$\wp_{\mathrm{em}} = \epsilon_0 \mu_0 \boldsymbol{S} = \epsilon_0 (\boldsymbol{E} \times \boldsymbol{B})$$ gęstość pędu

$$\boldsymbol{\ell}_{\mathrm{em}} = \boldsymbol{r} \times \wp_{\mathrm{em}} = \epsilon_0 [\boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B})]$$ gęstość momentu pędu

Nawet statyczne pola mogą mieć pęd i moment pędu!