We propose how to achieve quantum nonreciprocity via unconventional photon blockade (UPB) in a compound device consisting of an optical harmonic resonator and a spinning optomechanical resonator. We show that, even with very weak single-photon nonlinearity, nonreciprocal UPB can emerge in this system, i.e., strong photon antibunching can emerge only by driving the device from one side but not from the other side. This nonreciprocity results from the Fizeau drag, leading to different splitting of the resonance frequencies for the optical counter-circulating modes. Such quantum nonreciprocal devices can be particularly useful in achieving back-action-free quantum sensing or chiral photonic communications.

1. INTRODUCTION

Photon blockade (PB) [1–5], i.e., the generation of the first photon in a nonlinear cavity, diminishes to almost zero the probability of generating another photon in the cavity; it plays a key role in single-photon control for quantum technology applications nowadays [6–8]. In experiments, PB has been demonstrated in cavity-QED or circuit-QED systems [4,5,9–12]. It has also been predicted in various nonlinear optical systems [13–15] and optomechanical (OM) devices [16–20]. Conventional PB occurs under the stringent condition of strong single-photon nonlinearities, which is highly challenging in practice.

To overcome this obstacle, coupled-resonator systems, with destructive interferences of different dissipative pathways [21–24], have been proposed to achieve unconventional PB (UPB) even for arbitrarily weak nonlinearities [23–37]. UPB provides a powerful tool to generate optimally sub-Poissonian light and also a way to reveal quantum correlations in weakly nonlinear devices [33,34]. Recently, UPB was demonstrated experimentally in coupled optical [36] or superconducting resonators [37].

It should be stressed that PB and UPB are very different phenomena, and, thus, their nonreciprocal generalizations are different as well. Indeed PB refers to a process where a single photon is blocking the entry (or generation) of more photons in a strongly nonlinear cavity. Thus, PB refers to state truncation, also referred to as nonlinear quantum scissors [38,39]. PB can be used as a source of single photons, since the PB light is sub-Poissonian (or photon antibunched) in second and higher orders, as characterized by the correlation functions $g^{(n)}(0) < 1$ for $n = 2, 3, \ldots$. By contrast to PB, UPB refers to the light that is optimally sub-Poissonian in the second order, $g^{(2)}(0) \approx 0$, and is generated in a weakly nonlinear system allowing multi-path interference (e.g., two linearly coupled cavities, when one of them is also weakly coupled to a two-level atom). Thus, PB and UPB are induced by different effects: PB due to a large system nonlinearity and UPB via multipath interference even assuming extremely weak system nonlinearity. Note that light generated via UPB can exhibit higher-order super-Poissonian photon-number statistics, $g^{(n)}(0) > 1$ for some $n > 2$. Thus, UPB is, in general, not a good source of single photons. This short comparison of PB and UPB indicates that the term UPB, as coined in Ref. [40] and now commonly accepted, is fundamentally different from PB, concerning their physical mechanisms and the properties of the light generated in them.

Here, we propose achieving and controlling nonreciprocal UPB with spinning devices. Nonreciprocal devices allow the flow of light from one side but block it from the other. Thus, such devices can be applied in noise-free quantum information signal processing and quantum communication for canceling interfering signals [41]. Nonreciprocal optical devices have been realized in OM devices [41–43], Kerr resonators [44–46], thermo systems [47–49], devices with temporal modulation [50,51], and non-Hermitian systems [52–54].
In a very recent experiment [55], 99.6% optical isolation was achieved in a spinning resonator based on the optical Sagnac effect. By using the spinning resonators, optomechanically induced transparency [56] and ultrasensitive nanoparticle sensing [57] have also been studied. However, these studies have mainly focused on the classical regimes, i.e., unidirectional control of transmission rates instead of quantum noises. We also note that in recent works, single-photon diodes [58–60], unidirectional quantum amplifiers [61–65], and one-way quantum routers [66] have been explored. In particular, nonreciprocal PB was predicted in a Kerr resonator [67] or a quadratic OM system [68], which, however, relies on the conventional condition of strong single-photon nonlinearity. These quantum nonreciprocal devices have potential applications for quantum control of light chiral and topological quantum technologies [69].

We also note that coupled-cavity systems have been studied extensively in experiments [37,70–72], providing a unique way to achieve not only UPB, but also phonon antibunching [73] or unidirectional quantum devices [74–76], slow light [77], and force sensing [70,71,78]. Here, we study nonreciprocal UPB in a coupled system with an optical harmonic cavity and a spinning OM resonator. We find that, by the spinning of an OM resonator, UPB can emerge nonreciprocally even with weak single-photon nonlinearity; that is, strongly antibunched photons can emerge only when the device is driven from one side but not the other side. Our work opens up a new route to engineer quantum chiral UPB devices, which can have practical applications in achieving, for example, photonic diodes or circulators, and nonreciprocal quantum communications at the few-photon level.

2. MODEL AND SOLUTIONS

We consider a compound system consisting of an optical harmonic resonator (with a resonance frequency $\omega_0$ of the cavity field and a decay rate of $\kappa_L$) and a spinning anharmonic resonator (with a resonance frequency $\omega_R$ of the cavity field and a decay rate of $\kappa_R$), as shown in Fig. 1. External light is coupled into and out of the resonator through a tapered fiber of frequency $\omega_d$ and these two whispering-gallery-mode resonators are evanescently coupled to each other with a coupling strength of $J$ [79]. Note that the required strong Kerr nonlinearity, $K \approx 3\kappa$ (where $\kappa$ is the cavity linewidth), in the previous proposal [67] is challenging for the current experiments. Here, we can use an experimentally feasible Kerr-nonlinear strength to realize nonreciprocal PB, i.e., $K \approx 0.04\kappa$ [37], which is two orders of magnitude smaller than that in the former work [67]. Weak Kerr couplings can be achieved in cavity-atom systems [80], magnon devices [81], and OM systems [82], on which we focus here. We consider a weak OM coupling strength ($g \approx 0.63c$) in an auxiliary cavity that is well within the current experimental abilities [83–85]. In a spinning resonator, the refractive indices associated with the clockwise (+) and anticlockwise (−) optical modes are given as $n_{\pm} = n[1 \pm n\nu(n^2 - 1)/c]$, where $\nu = n\Omega$ is the tangential velocity with an angular velocity of $\Omega$ and radius $r$ [55]. For light propagating in the spinning resonator, the optical mode experiences a Fizeau shift $\Delta_F$ [86], that is, $\omega_R \rightarrow \omega_R + \Delta_F$, with

$$\Delta_F = \pm \frac{nr\Omega \omega_R}{c} \left(1 - \frac{1 - \nu^2}{n^2} \frac{\lambda}{n} \frac{dn}{d\lambda}\right) = \pm \eta \Omega, \quad (1)$$

Fig. 1. Nonreciprocal UPB in a coupled-resonator system. Spinning the OM (Kerr-type) resonator results in different Fizeau drag $\Delta_F$ for the counter-circulating whispering-gallery modes of the resonator. (a) By driving the system from the left-hand side, the direct excitation from state $|1,0\rangle$ to state $|2,0\rangle$ (red dotted arrow) will be forbidden by destructive quantum interference with the other paths drawn by green arrows, leading to photon antibunching. (b) Photon bunching occurs when the system is driven from the right side, due to lack of complete destructive quantum interference between the indicated levels (drawn by crossed green dotted arrows). Here, $\delta = \delta^2/\omega_n$ is the energy shift induced by the OM nonlinearity.
where \( \omega_R = 2\pi c/\lambda \) is the optical resonance frequency of the nonspinning OM resonator, \( c \) (\( \lambda \)) is the speed (wavelength) of light in vacuum, and \( n \) is the refractive index of the cavity. The dispersion term \( d\gamma/d\omega \), characterizing the relativistic origin of the Sagnac effect, is relatively small in typical materials (~1%) \([55,86]\). For convenience, we always assume counterclockwise rotation of the resonator. Hence, \( \pm \Delta \epsilon_f \) denote light propagating against (\( \Delta \epsilon_f > 0 \)) and along (\( \Delta \epsilon_f < 0 \)) the direction of the spinning OM resonator, respectively.

In a rotating frame with respect to \( H_0 = \omega_d (a_L^\dagger a_L + a_R^\dagger a_R) \), the effective Hamiltonian of the system can be written as (see Appendix A for more details)

\[
\mathcal{H} = \hbar \Delta \epsilon_f a_L^\dagger a_L + \hbar (\Delta \epsilon_f + \Delta \epsilon) a_R^\dagger a_R + \hbar \omega_m b^\dagger b \\
+ \hbar J (a_L^\dagger a_R + a_R^\dagger a_L) + \hbar g a_R^\dagger a_R (b^\dagger + b) \\
+ i\hbar \epsilon_R (a_L^\dagger - a_L),
\]  

(2)

where \( a_L \) (\( a_L^\dagger \)) and \( a_R \) (\( a_R^\dagger \)) are the photon annihilation (creation) operators for the cavity modes of the optical cavity (denoted with the subscript \( L \)) and the OM cavity (denoted with the subscript \( R \)), respectively. \( b \) (\( b^\dagger \)) is the annihilation (creation) operator for the mechanical mode of the OM cavity. The frequency detuning between the cavity field in the left (right) cavity and the driving field is denoted as \( \Delta \epsilon_f = \omega_f - \omega_d \), where \( K = L, R \). The parameter \( J \) denotes the strength of the photon hopping interaction between the two cavity modes, and \( g = \gamma_m / [\hbar / (2 \omega_0 m)] \) describes the radiation-pressure coupling between the optical and vibrational modes in the OM resonator with frequency \( \omega_m \) and effective mass \( m \). \( \epsilon_R = \sqrt{\kappa_f P_{in}/(\hbar \omega_R)} \) denotes the driving strength that is coupled into the compound system through the optical fiber waveguide with a cavity loss rate of \( \kappa_L \) and driving power \( P_{in} \).

The Heisenberg equations of motion of the system are then written as

\[
\frac{d}{dt} q = \omega_m p, \\
\frac{d}{dt} p = -\omega_m q - \bar{g}_d a_L^\dagger a_R - \frac{\gamma_m}{2} p + \xi, \\
\frac{d}{dt} a_L = -\left( \frac{\kappa_f}{2} + i\Delta \epsilon_f \right) a_L - if a_R + e_d + \sqrt{\kappa_f} a_{L, in}, \\
\frac{d}{dt} a_R = -\left( \frac{\kappa_R}{2} + i\Delta \epsilon_f \right) a_R - if a_L - ig_d q a_R + \sqrt{\kappa_R} a_{R, in},
\]

(3)

where \( p \) and \( q \) are dimensionless canonical position and momentum, with \( p = i(b^\dagger - b)/\sqrt{2} \) and \( q = (b + b^\dagger)/\sqrt{2} \), respectively. \( \Delta \epsilon_f = \Delta \epsilon + \Delta \epsilon_f \) and \( g_d = \sqrt{2} \bar{g}_d \), and \( \kappa_f = \omega_f / Q_L \) (\( \kappa_R = \omega_R / Q_R \)) is the dissipation rate and \( Q_L \) (\( Q_R \)) is the quality factor of the left (right) cavity. \( \gamma_m = \omega_m / Q_{th} \) is the damping rate with \( Q_{th} \) the quality factor of the mechanical mode. Moreover, \( \xi \) is the zero-mean Brownian stochastic operator, \( \langle \xi(t) \rangle = 0 \), resulting from the coupling of the mechanical resonator with the corresponding thermal environment and satisfying the correlation function \([87]\)

\[
\langle \xi(t) \xi(t') \rangle = \frac{1}{2\pi} \int d\omega e^{-i\omega(t-t')} \Gamma_m(\omega),
\]

(4)

where

\[
\Gamma_m(\omega) = \frac{\omega_m}{\omega_m} \frac{1}{e^\frac{\hbar \omega}{2k_B T} - 1}.
\]

(5)

\( T \) is effective temperature of the environment of the mechanical resonator, and \( k_B \) is the Boltzmann constant. The annihilation operators \( a_{L, in} \) and \( a_{R, in} \) are, respectively, the input vacuum noise operators of the optical cavity and the OM cavity with zero mean value, i.e., \( \langle a_{L, in} \rangle = \langle a_{R, in} \rangle = 0 \), and they comply with the time-domain correlation functions \([88,89]\)

\[
\langle a_{K, in}^\dagger(t) a_{K, in}(t') \rangle = 0, \\
\langle a_{K, in}^\dagger(t) a_{K, in}(t') \rangle = \delta(t - t'),
\]

(6)

for \( K = L, R \). Because the whole system interacts with a low-temperature environment (here we consider 0.1 mK), we neglect the mean thermal photon numbers at optical frequencies in the two cavities. In order to linearize the dynamics around the steady state of the system, we expend the operators as the sum of its steady-state mean values and a small fluctuation with zero mean value around it; that is, \( a_L = \alpha + \delta a_L \), \( a_R = \beta + \delta a_R \), \( q = q_s + \delta q \), and \( p = p_s + \delta p \). By neglecting higher-order terms, \( \delta a_L^\dagger \delta a_L \), the linearized equations of the fluctuation terms can be written as

\[
\frac{d}{dt} \delta q = \omega_m \delta p, \\
\frac{d}{dt} \delta p = -\omega_m \delta q - \bar{g}_d (\beta^* \delta a_R + \beta \delta a_R^\dagger) - \frac{\gamma_m}{2} \delta p + \xi, \\
\frac{d}{dt} \delta a_L = -\left( \frac{\kappa_f}{2} + i\Delta \epsilon_f \right) \delta a_L - if \delta a_R + \sqrt{\kappa_f} \delta a_{L, in}, \\
\frac{d}{dt} \delta a_R = -\left( \frac{\kappa_R}{2} + i\Delta \epsilon_f \right) \delta a_R - if \delta a_L - ig_d q \delta a_R \\
- ig_d \beta \delta q + \sqrt{\kappa_R} \delta a_{R, in}.
\]

(7)

These equations can be solved in the frequency domain (see Appendix B). In particular, we find

\[
\delta a_L(a) = E(a) a_{L, in}(a) + F(a) a_{L, in}^\dagger(a) + G(a) a_{R, in}(a) \\
+ H(a) a_{R, in}^\dagger(a) + Q(a) \xi(a),
\]

(8)

where

\[
E(a) = \sqrt{\kappa_f} A_1(a)/A_3(a), \\
F(a) = -\sqrt{\kappa_f} A_2(a)/A_3(a), \\
G(a) = \sqrt{\kappa_R} A_1(a)/A_3(a), \\
H(a) = -\sqrt{\kappa_R} A_2(a)/A_3(a), \\
Q(a) = -ig_d \bar{g}_d(a) \omega_m A_3(a) [\beta A_3(a) + \beta^* A_4(a)],
\]

(9)

and
where $R_h$ can be calculated as $g_{l}^2|\beta|^2\chi(\omega)\left(\frac{\omega^2_m}{\omega^2 - \omega^2 + \frac{i\omega\gamma_m}{2}}\right),$ $V_{1}^\pm(\omega) = \frac{\kappa_l}{2} \pm i(\Delta_L - \omega),$ $V_{2}^\pm(\omega) = \frac{\kappa_R}{2} \pm i(\Delta_R - \omega).$

\[ \Delta''_l = \Delta'_R + g_{l}\beta - g_{l}\beta|\beta|^2\chi(\omega), \]
\[ \chi(\omega) = \omega^2_m\left(\frac{\omega^2_m}{2} + \frac{i\omega\gamma_m}{2}\right), \]
\[ V_{1}^\pm = A_1(\omega) + ijA_3(\omega), \]
\[ V_{2}^\pm = \frac{\kappa_R}{2} \pm i(\Delta_R - \omega). \]

3. NONRECIPROCAL OPTICAL CORRELATIONS

Now, we focus on the statistical properties of photons in an optical cavity, which are described quantitatively via the normalized zero-time-delay second-order correlation function $g_{l}^2(0) = \langle a_L^2 a_L^\dagger \rangle / \langle a_L^\dagger a_L \rangle^2$ [29,89]. By taking the semi-classical approximation, i.e., $a_L = \alpha + \delta a_L$, the correlation function $g_{l}^2(0)$ can be given by [29]

\[ g_{l}^2(0) = \left|\alpha\right|^4 + 4\left|\alpha\right|^2 \frac{R_1 + 2\text{Re}[\alpha^2 R_2] + R_3}{\left(\left|\alpha\right|^2 + R_1\right)^2}, \]

where $R_1 = \langle \delta a_L^\dagger(t)\delta a_L(t) \rangle,$ $R_2 = \langle \delta a_L^\dagger(t)\delta a_L(t) \rangle,$ and $R_3 = \langle \delta a_L^\dagger(t)\delta a_L^\dagger(t)\delta a_L(t)\delta a_L(t) \rangle = 2R_1 + \left| R_2 \right|^2.$

From Eq. (8), the correlation between $\delta a_L(t)$ and $\delta a_L^\dagger(t)$ can be calculated as

\[ \langle \delta a_L^\dagger(t)\delta a_L(t) \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_{\alpha}\, d\omega, \]

\[ \Delta \alpha^\dagger(t) = \Delta alpha^\dagger(t), \Delta \alpha^\dagger = |Q(\omega)|^2 \Omega_m(\omega) + |F(\omega)|^2 + |H(\omega)|^2, \]
\[ X_{\alpha} = Q(\omega)Q(\omega)\Omega_m(\omega) + E(\omega)F(\omega) + G(\omega)H(\omega). \]

To obtain more accurate results, we introduce the density operator $\rho(t)$ and numerically calculate the normalized zero-time-delay second-order correlation by the Lindblad master equation [90]:

\[ \dot{\rho} = \frac{1}{i\hbar}[\mathcal{H}, \rho] + \frac{\kappa_l}{2} \mathcal{L}[a_L](\rho) + \frac{\kappa_R}{2} \mathcal{L}[a_R](\rho) + \frac{\gamma_m}{2}(\bar{n}_m + 1)\mathcal{L}[b](\rho) + \frac{\gamma_m}{2}\bar{n}_m\mathcal{L}[b^\dagger](\rho), \]

where $\mathcal{L}[\rho](\rho) = 2\rho\mathcal{O} - \mathcal{O}\rho - \rho\mathcal{O}^\dagger$ are the Lindblad super-operators [89], for $\mathcal{O} = a_L, a_R, b,$ and $b^\dagger,$ and $\bar{n}_m = 1/\exp(h\omega_m/k_BT) - 1$ is the mean thermal photon number of the mechanical mode at temperature $T.$

The second-order correlation function $g_{l}^2(2)(0)$ is shown in Fig. 2 as a function of optical detuning $\Delta/k$ and angular velocity $\Omega.$ We assume $\Delta_L = \Delta_R = \delta = \Delta$, and $\kappa_L = \kappa_R = \kappa$ and use experimentally feasible parameters [53,83,91–95], that is, $\lambda = 1550 \text{ nm}, Q_L = 3 \times 10^7, r = 0.3 \text{ mm}, n = 1.44, m = 5 \times 10^{-11} \text{ kg},$ and $P_{in} = 2 \times 10^{-17} \text{ W}$. $Q_L$ is typically $10^6–10^{12}$ [92,94,95], $g$ is typically $10^3–10^6$ Hz [83,91,92] in optical microresonators, and $g_{l}^2(2)(0) \sim 0.37$ [36,37] was achieved experimentally. $J$ can be adjusted by changing the distance of the double resonators [72]. In a recent experiment, autocorrelation measurements ranging from $g_{l}^2(0) = 6 \times 10^{-3}$ to 2 were achieved with an average fidelity of 0.998 in a photon-number-resolving detector [96]. Moreover, we set $\Omega = 12 \text{ kHz}$, which is experimentally feasible. The resonator with a radius of $r = 1.1$ mm can spin at an angular velocity of $\Omega = 6.6 \text{ kHz}$ [55]. Using a levitated OM system [97,98], $\Omega$ can be increased even up to GHZ values.

Our analytical results agree well with the numerical one. In the case of a nonspinning resonator, as shown in Fig. 2(a), $g_{l}^2(2)(0)$ is reciprocal regardless of the direction of the driving light, and always has a dip at $\Delta/k \approx -0.29$ and a peak at $\Delta/k \approx 0.166,$ corresponding to strong photon antibunching and photon bunching, respectively [29]. The physical origin of the strong photon antibunching is the destructive interference between the direct and indirect paths of two-photon excitations, i.e.,

Fig. 2. Correlation function $g_{l}^2(2)(0)$ versus optical detuning $\Delta/k$ (in units of cavity loss rate $\kappa_L = \kappa_R = \kappa$) with (a) $\Omega = 0$ and (b) $\Omega = 12 \text{ kHz},$ which is found numerically (solid curves) and analytically (dotted curve). The PB can be generated (red curves) or suppressed (blue curves) for different driving directions, which can be seen more clearly in panel (c). The other parameters are $g/k = 0.63, \omega_m/k = 10$ [91], $J/k = 3, T = 0.1 \text{ mK}$ (case 1), and $g/k = 0.1$ [28], $\omega_m/k = 30$ [92], $J/k = 20, T = 1 \text{ mK}$ (case 2).
In contrast, for a spinning device, $g^{(2)}_{L}(0)$ exhibits giant nonreciprocity, which can be seen in Fig. 2(b). The PB can be generated, i.e., $g^{(2)}_{L}(0) \sim 0.06$, for $\Delta_F < 0$, whereas it is significantly suppressed, i.e., $g^{(2)}_{L}(0) \sim 4.72$, for $\Delta_F > 0$; this can be seen more clearly in Fig. 2(c). Nonreciprocal UPB induced by the Fizeau light-dragging effect, with difference in $g^{(2)}_{L}(0)$ up to two orders of magnitude for opposite directions, can be achieved even with weak nonlinearity and, to our knowledge, has not been studied previously. Furthermore, in Fig. 2(b), we use two sets of parameters for solid (case 1) and dashed curves (case 2), respectively. It can be seen that nonreciprocity still exists in a parameter range closer to that in the experiment.

Since the anharmonicity of the system is very small, destructive quantum interference (rather than anharmonicity) is responsible for observing strong photon antibunching (referred to as UPB) and photon bunching (referred to as photon-induced tunneling) in the spinning devices, as shown in Fig. 1 and confirmed by our analytical calculations. Note that the role of complete (incomplete) destructive quantum interference is the same in both spinning and non-spinning UPB systems, and, thus, we refer to Ref. [24] where this interference-based mechanism was first explained in detail. Spinning the OM resonator results in different Fizeau drag $\Delta_F$ for the counter-circulating whispering-gallery modes of the resonator. By driving the system from the left-hand side, direct excitation from state $|1, 0\rangle$ to state $|2, 0\rangle$ will be forbidden by destructive quantum interference with the indirect paths of two-photon excitations, leading to photon antibunching. In contrast, photon bunching occurs when the system is driven from the right side, due to lack of complete destructive quantum interference between the indicated levels [99]. Increasing the angular velocity results in an opposing frequency shift of $\hbar \Omega$ for light coming from opposite directions. $g^{(2)}_{L}(0)$ also shifts linearly with $\Omega$, but with different directions for $\Delta_F < 0$ and $\Delta_F > 0$; that is, we observe either a blue shift [see Fig. 3(a)] or a red shift [see Fig. 3(b)] with $\Delta_F > 0$ or $\Delta_F < 0$, respectively. A highly tunable nonreciprocal UPB device is thus achievable, by flexible tuning of $\Omega$ and $\Delta/\kappa$. In addition, since $g^{(2)}_{L}(0)$ is sensitive to $\Omega$, this may also indicate a way for accurate measurements of velocity.

### 4. OPTIMAL PARAMETERS FOR STRONG ANTIBUNCHING

As discussed above, UPB can be generated nonreciprocally. In this section, we analytically derive the optimal conditions for

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**Fig. 3.** Correlation function $g^{(2)}_{L}(0)$ versus optical detuning $\Delta/\kappa$ (in units of cavity loss rate $\kappa_L = \kappa_R = \kappa$) at various angular velocities $\Omega$ upon driving the device from (a) the right-hand side or (b) the left-hand side. The dashed curves show our approximate analytical results, given in Eq. (12), whereas the solid curves are our numerical solutions. The other parameters are the same as those in Fig. 2 (case 1).

**Fig. 4.** Correlation function $g^{(2)}_{L}(0)$ in logarithmic scale [i.e., $\log_{10} g^{(2)}_{L}(0)$] versus (a) radiation-pressure coupling $g/\kappa$ (in units of cavity loss rate $\kappa_L = \kappa_R = \kappa$) and optical detuning $\Delta/\kappa$, and (b) coupling strength of the resonators $J/\kappa$ and radiation-pressure coupling $g/\kappa$ for optical detuning of $\Delta/\kappa = -0.05$. The angular velocity is $\Omega = 12$ kHz and the white dashed curve corresponds to $g^{(2)}_{L}(0) = 1$. The other parameters are the same as those in Fig. 3.
Fig. 5. Correlation function \( g_{\ell}^{(2)}(0) \) versus optical detuning \( \Delta / \kappa \) (in units of cavity loss rate \( \kappa_L = \kappa_R = \kappa \)) with varied mean thermal phonon numbers \( n_{\text{th}} \) for various angular velocities \( \Omega \), and the resulting Fizeau shifts \( \Delta_F \). The other parameters are the same as those in Fig. 4.

strong antibunching. Here we apply the method described in Ref. [24], which is based on the evolution of a complex non-Hermitian Hamiltonian, as given in Appendix C. Thus, our solution corresponds to only a semi-classical approximation of the solution of the quantum master equation, given in Eq. (15), where the terms corresponding to quantum jumps are ignored.

Since the phonon states can be decoupled from the photon states by using the unitary operator \( U = \exp[-g(b^\dagger - b)/\omega_m] \), the states of the system can be expressed as \( |\psi\rangle = |\psi\rangle|\phi_m\rangle \), where \( |\psi\rangle \) and \( |\phi_m\rangle \) are the photon states and phonon states, respectively. Under the weak-driving condition, we make the ansatz [24]

\[
|\psi\rangle = C_{00}|0,0\rangle + C_{10}|1,0\rangle + C_{01}|0,1\rangle + C_{20}|2,0\rangle \\
+ C_{11}|1,1\rangle + C_{21}|0,2\rangle,
\]

and consider that \( C_{mn} \ll C_{m'n'} \ll C_{00} \) for \( m + n = 2 \), \( m' + n' = 1 \), and the condition of \( C_{20} = 0 \); the optimal conditions are given by fixing \( J \) and \( \kappa \) (see Appendix C):

\[
\Delta_{\text{opt}} \approx -a_3 + \text{sgn}(E) \sqrt{\lambda_1 - \lambda_2},
\]

\[
ge_{\text{opt}} = \sqrt{-\frac{a_m(4a_{\text{opt}}^2 + 5\lambda^2) + \Delta_F^2}{2(2J^2 - \lambda^2) + 2\Delta_F^2}},
\]

the signal function \( \text{sgn}(E) \), \( a_3 = -96\Delta_F \kappa \), and \( \lambda_{1,2} \), which are defined in Appendix C, are related to the Fizeau drag \( \Delta_F \).

Fig. 6. (a) Correlation function \( g_{\ell}^{(2)}(0) \) versus effective temperature \( T \) of the environment of the mechanical resonator for three values of Fizeau shift \( \Delta_F (\Delta_F > 0, \Delta_F = 0, \text{and} \Delta_F < 0) \) for optimal values of \( \Delta_{\text{opt}} \) and \( e_{\text{opt}} \). The other parameters are set the same as in case 2 in Fig. 2. Also shown is the correlation function \( g_{\ell}^{(2)}(0) \) versus \( T \) for various values of (b) spinning frequency, (c) mechanical decay, and (d) cavity decay, assuming the device is driven from the left-hand side and optical detuning is fixed at the optimal values.
In UPB, Hilbert space larger than $g$ effect) with vanishing (or almost vanishing) second-order phonon noises. We note that thermal phonons greatly affect the correlation $g^{(2)}(0)$ of photons and tend to destroy PB. Thus, to show this effect, in Fig. 6(a) we plot the correlation $g^{(2)}(0)$ as a function of temperature $T$ for various Fizeau shifts. We see that nonreciprocal UPB can be observed below the critical temperature $T_o \approx 4 \text{ mK}$ for the spinning frequency of $\Omega = 12 \text{ kHz}$ ($\Omega = 50 \text{ kHz}$) [see Fig. 6(b)]. By further increasing the optical dissipation of the OM cavity, as shown in Fig. 6(d), the critical temperature $T_0$ can be made to reach a value of 10 mK.

Finally, we note that a state (generated via UPB or another effect) with vanishing or almost vanishing second-order photon-number correlations, $g^{(2)}(0) \approx 0$, is not necessarily a good single-photon source, i.e., the state might not be a (partially incoherent) superposition of only the vacuum and single-photon states. A good single-photon source is characterized not only by $g^{(2)}(0) \approx 0$, but also by vanishing higher-order photon-number correlation functions, $g^{(n)}(0) \approx 0$ for $n > 2$. In UPB, $g^{(n)}(0)$ for $n > 2$ can be greater than $g^{(2)}(0) \approx 0$, or even greater than 1 [100]. Indeed, a standard analytical method for analyzing UPB, as proposed by Bamba et al. [24] and applied here, is based on expanding the wave function $|\varphi\rangle$ of a two-resonator system in the power series $|\varphi\rangle = \sum C_{n}\,|n\rangle\,|m\rangle$ up to the terms $C_{n}\,|n\rangle\,|m\rangle$ (see Appendix B) and solving for the optimal system parameters, which minimize $g^{(2)}(0)$ in UPB, this method requires setting $C_{2.0} = 0$ as set in Appendix C. Actually, the same expansion of $|\varphi\rangle$ and the same ansatz are made in Ref. [24]. These assumptions imply that higher-order correlation functions $g^{(n)}(0)$ with $n = 3, 4, \ldots$ vanish too. However, the truncation of the above expansion at the terms $C_{2.0}$ is often not justified for a system exhibiting UPB. Indeed, we find parameters for our system for which $g^{(2)}(0) \approx 0$ and, simultaneously, $g^{(3)}(0) > 1$. We have confirmed this by precise numerical calculation of the steady states of our system based on the non-Hermitian Hamiltonian, given in Eq. (C1), in a Hilbert space larger than $4 \times 4$.

5. CONCLUSIONS

In summary, we studied nonreciprocal UPB in a system consisting of a purely optical resonator and a spinning OM resonator. Due to interference between two-photon excitation paths and the Sagnac effect, UPB can be generated nonreciprocally in our system; that is, UPB can occur when the system is driven from one direction but not from the other, even under weak OM interactions. The optimal conditions for one-way UPB were presented analytically. Moreover, we found that this quantum nonreciprocity can still exist by considering thermal phonon noises.

Concerning a possible experimental implementation of nonreciprocal UPB, it is worth noting that UPB for non-spinning devices has already been demonstrated experimentally in two recent works [36,37]. A number of experiments (including a very recent work [55]) have shown nonreciprocal quantum effects in spinning devices. So the main experimental task for achieving nonreciprocal UPB in a spinning device would be to combine the experimental setups of, e.g., Refs. [36,37,55] into a single spinning UPB setup.

Our proposal provides a feasible method to control the behavior of one-way photons, with potential applications in achieving, e.g., photonic diodes or circulators, quantum chiral communications, and nonreciprocal light engineering in the deep quantum regime.

APPENDIX A: DERIVATION OF EFFECTIVE HAMILTONIAN

The coupled system can be described by the Hamiltonian

$$H = H_0 + H_{in} + H_{dr},$$

$$H_0 = \hbar \omega_L a_L^\dagger a_L + \hbar (\omega_R + \Delta \epsilon) a_R^\dagger a_R + \hbar \omega_m b^\dagger b,$$

$$H_{in} = \hbar f (a_L^\dagger a_R + a_R^\dagger a_L) + \hbar g (a_R^\dagger a_R (b^\dagger + b),$$

$$H_{dr} = i \hbar \epsilon_{d} (a_L^\dagger e^{i\omega_d t} - a_L e^{-i\omega_d t}),$$

where $a_L$ ($a_R$) and $a_R^\dagger$ ($a_L^\dagger$) are the photon annihilation (creation) operators for the cavity modes of the optical cavity (denoted with the subscript $L$) and the OM cavity (denoted with the subscript $R$), respectively. $b$ ($b^\dagger$) is the annihilation (creation) operator for the mechanical mode of the OM cavity. The frequencies of the cavity fields are denoted with $\omega_L$ and $\omega_R$. $f$ is the coupling strength between the two resonators, and $g = \omega_R / r [\hbar f (2 \omega_{m})]^{1/2}$ is the OM coupling strength between the optical mode and the mechanical mode in the OM cavity. $\epsilon_{d} = \sqrt{\kappa_{L} P_{in} / (\hbar \omega_d)}$ denotes the driving strength that is coupled into the compound system through the optical fiber waveguide.

Using the unitary operator $U = \exp[-g(b^\dagger - b) / \omega_m]$ for the Hamiltonian (A1), we obtain a Kerr-type Hamiltonian [82]

$$H_{eff} = U^\dagger H U = \hbar \omega_L a_L^\dagger a_L + \hbar (\omega_R + \Delta \epsilon) a_R^\dagger a_R - \hbar \delta (a_R^\dagger a_R)^2 + \hbar f (a_L^\dagger a_R e^{-i\omega_d t} + a_R^\dagger a_L e^{i\omega_d t}) + i \hbar \epsilon_{d} (a_L^\dagger e^{i\omega_d t} - a_L e^{-i\omega_d t}),$$

where $\delta = g^2 / \omega_m$. Under the conditions $g / \omega_m \ll 1$ and $f / \omega_m / 2$, the Hamiltonian (A2) can be read as
\[ H'_{\text{eff}} = \hbar \omega_i a_L^\dagger a_L + \hbar (\omega_R + \Delta)_R a_R^\dagger a_R - \hbar \delta(a_R^\dagger a_R)^2 + \hbar \beta (a_R^\dagger a_L + a_L^\dagger a_R) + i \hbar \epsilon_R (a_R^\dagger e^{i \omega t} - a_L e^{i \omega t}) \]

**APPENDIX B: FOURIER ANALYSIS OF FLUCTUATION TERMS**

According to the Heisenberg equations of motion of Hamiltonian (2), and using the semi-classical approximation method, i.e., \( a_L = \alpha + \delta a_L \), \( a_R = \beta + \delta a_R \), \( q = q_i + \delta q \), and \( p = p_i + \delta p \), the steady-state values of the system satisfy the equations

\[
\begin{align*}
0 &= \left( \frac{K_L}{2} + i \Delta_L \right) \alpha + i \beta - c_d, \\
0 &= \frac{K_R}{2} + i (\Delta_R^\prime + \delta q_i) \beta - if \alpha, \\
0 &= \omega_m q_i - g_i |\beta|^2.
\end{align*}
\]

where

\[
\begin{align*}
b_0 &= g_i^2, \\
b_1 &= \omega_m \left( \frac{K_L K_R}{4} + J^2 \right)^2 + \omega_m \left( \frac{K_L \Delta_L^\prime}{2} + \frac{K_R \Delta_R^\prime}{2} \right)^2 - \omega_m \Delta_L \Delta_R \left( \frac{K_L K_R}{2} + 2J^2 - \Delta_L \Delta_R^\prime \right), \\
b_2 &= 2 \omega_m g_i \left[ \frac{K_L^2 \Delta_R^\prime}{4} \right. \\
b_3 &= \omega_m \left( \frac{K_R^2}{4} + \Delta_R^2 \right).
\end{align*}
\]

The fluctuation terms of the system can be written as

\[
\begin{align*}
\frac{d}{dt} \delta q &= \omega_m \delta p, \\
\frac{d}{dt} \delta p &= -\omega_m \delta q - g_i (\beta^* \delta a_R + \beta \delta a_R^\dagger) - \frac{\gamma_m}{2} \delta p + \xi, \\
\frac{d}{dt} \delta a_L &= \left( \frac{K_L}{2} + i \Delta_L \right) \delta a_L - i \delta a_R + \sqrt{K_L} \delta a_{\text{Lin}}, \\
\frac{d}{dt} \delta a_R &= \left( \frac{K_R}{2} + i \Delta_R^\dagger \right) \delta a_R - i \delta a_L - ig_i \delta q \delta a_R - ig_i \delta q_i \delta a_R^\dagger + i \delta a_{\text{Lin}},
\end{align*}
\]

where we have neglected higher-order terms, \( \delta a_L^\dagger \delta a_L \). Here, the steady-state mean value \( q_i \) is numerically solved from Eqs. (B2) and (B3).

By introducing the Fourier transform to the fluctuation equations, we find

\[
\begin{align*}
i \omega \delta a_{L}(\omega) &= -\left( \frac{K_L}{2} + i \Delta_L \right) \delta a_{L}(\omega) - i \delta a_R(\omega) + \frac{i}{\sqrt{K_L}} \delta a_{\text{Lin}}, \\
i \omega \delta a_{R}(\omega) &= -\left( \frac{K_R}{2} + i \Delta_R^\prime \right) \delta a_{R}(\omega) - i \delta a_L(\omega) - ig_i \delta q(\omega) + \frac{i}{\sqrt{K_R}} \delta a_{\text{Lin}},
\end{align*}
\]

From Eq. (B4), we have

\[ V(\omega) \delta a_{L}(\omega) = i \omega \delta a_{L}(\omega) + \frac{i}{\sqrt{K_L}} \delta a_{\text{Lin}}, \]

where \( V(\omega) = K_L/2 + i \omega - i \Delta_L \). Substituting Eq. (B13) into Eq. (B11), we find

\[ N(\omega) = \frac{K_R}{2} + i \omega - i \Delta_R^\prime + \frac{i}{\sqrt{K_R}} \delta a_{\text{Lin}}(\omega) \]

According to Eq. (B5), we obtain

\[ M(\omega) = \frac{K_R}{2} + i \omega + i \Delta_R^\prime - \frac{i}{\sqrt{K_R}} \delta a_{\text{Lin}}(\omega) \]
\[ T(\omega)\delta a_R^\dagger(\omega) = -i\chi(\omega)g_\beta^2 \beta^2 V(\omega)\delta a_L(\omega) \]
\[ + i\chi(\omega)g_\beta^2 \beta^2 V(\omega)\hat{\xi}_\beta(\omega) \]
\[ + if\sqrt{\kappa_L a_{L,\text{in}}}(\omega) + \sqrt{\kappa_R V(\omega)}a_{R,\text{in}}(\omega), \] (B14)
where \[ T(\omega) = N(\omega) V(\omega) + f^2. \] Substituting Eq. (B14) into Eq. (B8), we obtain
\[ F_R(\omega)\delta a_R(\omega) = -\chi^2(\omega)g_\beta^3 \beta^2 V(\omega)\xi^\dagger(\omega) \]
\[ - i\chi(\omega)g_\beta^3 \beta^2 V(\omega)\eta(\omega) \]
\[ - g_\beta^3 \beta^2 \chi(\omega)\sqrt{\kappa_L a_{L,\text{in}}} \]
\[ + i\chi(\omega)g_\beta^3 \beta^2 \chi(\omega)\sqrt{\kappa_R V(\omega)}a_{R,\text{in}}(\omega) \]
\[ - i\sqrt{\kappa_R[T(\omega)a_{R,\text{in}}] - \sqrt{\kappa_R} T(\omega)a_{R,\text{in}},} \] (B15)
where the auxiliary function is \[ F_R(\omega) = M(\omega) T(\omega) - \chi^2(\omega) V(\omega)g_\beta^3 \beta^2 \xi(\omega) \]
\[ - i\chi(\omega)g_\beta^3 \beta^2 V(\omega)\eta(\omega) \]
\[ - g_\beta^3 \beta^2 \chi(\omega)\sqrt{\kappa_L a_{L,\text{in}}} \]
\[ + i\chi(\omega)g_\beta^3 \beta^2 \chi(\omega)\sqrt{\kappa_R V(\omega)}a_{R,\text{in}}(\omega) \]
\[ - i\sqrt{\kappa_L[T(\omega)a_{L,\text{in}}] - \sqrt{\kappa_L} T(\omega)a_{L,\text{in}},} \] (B16)
where
\[ F_L(\omega) = [M(\omega) T(\omega) - U(\omega)] V(\omega) + f^2 T(\omega), \]
\[ U(\omega) = -\chi^2(\omega)g_\beta^4 |\beta|^4 \left( i\omega + \frac{k_L}{2} - i\Delta_L \right), \]
\[ V_1(\omega) = \frac{k_L}{2} + i\omega + i\Delta_L. \] (B17)
Then we find
\[ \delta a_L(\omega) = E(\omega)a_{L,\text{in}}(\omega) + F(\omega)a_{L,\text{in}}^\dagger(\omega) + G(\omega)a_{R,\text{in}}(\omega) \]
\[ + H(\omega)a_{R,\text{in}}^\dagger(\omega) + Q(\omega)\xi(\omega). \] (B18)
According to similar calculations, we find
\[ \delta a_R^\dagger(\omega) = E^*(\omega)a_{L,\text{in}}(\omega) + F^*(\omega)a_{L,\text{in}}^\dagger(\omega) \]
\[ + G^*(\omega)a_{R,\text{in}}(\omega) + H^*(\omega)a_{R,\text{in}}^\dagger(\omega) \]
\[ + Q^*(\omega)\xi(\omega). \] (B19)
Using the Fourier transform, we obtain
\[ \langle a_{L,\text{in}}(\omega)a_{L,\text{in}}^\dagger(\omega') \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \left[ a_{L,\text{in}}(t)e^{-i\omega t} \right] \]
\[ \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt' \left[ a_{L,\text{in}}^\dagger(t')e^{-i\omega' t'} \right] \]
\[ = \delta(\omega + \omega'), \] (B20)
and
\[ \langle a_{R,\text{in}}(\omega)a_{R,\text{in}}^\dagger(\omega') \rangle = \delta(\omega + \omega'). \] (B21)

**APPENDIX C: DERIVATION OF OPTIMAL PARAMETERS**

According to the quantum-trajectory method [101], the non-Hermitian Hamiltonian of the system containing the optical decay and mechanical damping terms is given by [101]
\[ H' = \hbar \left( \Delta_L - i\frac{\kappa_L}{2} \right) a_L^\dagger a_L + \hbar \left( \Delta_R - i\frac{\kappa_R}{2} \right) a_R^\dagger a_R \]
\[ + \hbar \left( \omega_m - i\frac{\omega_m}{2} \right) b^\dagger b + \hbar f (a_L^\dagger a_R + a_R^\dagger a_L) \]
\[ - \hbar \delta(a_R^\dagger a_R)^2 + i\hbar \epsilon d (a_L^\dagger - a_L), \] (C1)
where \[ \Delta_R' = \Delta_R + \Delta_F. \]

Under the weak-driving conditions, we can make the ansatz [24]
\[ |\varphi\rangle = C_{00}|0,0\rangle + C_{10}|1,0\rangle + C_{01}|0,1\rangle + C_{20}|2,0\rangle \]
\[ + C_{11}|1,1\rangle + C_{02}|0,2\rangle. \] (C2)
Then we substitute the Hamiltonian [Eq. (C1)] and the general state [Eq. (C2)] into the Schrödinger equation
\[ i\hbar \frac{d|\varphi\rangle}{dt} = H'|\varphi\rangle, \] (C3)
and then we have
\[ H' C_{00}|0,0\rangle = i\hbar \epsilon d C_{00}|1,0\rangle, \]
\[ H' C_{10}|1,0\rangle = \hbar \delta_L C_{10}|1,0\rangle + \hbar f C_{10}|0,1\rangle \]
\[ + i\hbar \epsilon d C_{10}|1,1\rangle, \]
\[ H' C_{01}|0,1\rangle = \hbar \delta_R C_{01}|0,1\rangle + \hbar f C_{01}|1,0\rangle \]
\[ + i\hbar \epsilon d C_{01}|1,1\rangle, \]
\[ H' C_{20}|2,0\rangle = 2\hbar \delta_L C_{20}|2,0\rangle + \sqrt{2}\hbar f C_{20}|1,1\rangle \]
\[ + i\hbar \epsilon d C_{20}|0,2\rangle \]
\[ H' C_{11}|1,1\rangle = \hbar \delta_L C_{11}|1,1\rangle + \hbar \delta_R C_{11}|1,1\rangle \]
\[ + \sqrt{2}\hbar f C_{11}|2,0\rangle + |0,2\rangle \]
\[ + i\hbar \epsilon d C_{11}|2,1\rangle \]
\[ H' C_{02}|0,2\rangle = 2\hbar \delta_R C_{02}|0,2\rangle - 2\delta C_{02}|0,2\rangle \]
\[ + \sqrt{2}\hbar f C_{02}|1,1\rangle + i\hbar \epsilon d C_{02}|1,2\rangle, \] (C4)
where the auxiliary functions are \[ \delta_L = \Delta_L - i\kappa_L/2 \] and \[ \delta_R = \Delta_R' - i\kappa_R/2, \] and we have ignored the effects of the mechanical model because the phonon states are decoupled from the photon states [see Eq. (C1)]. By comparing the coefficients, we have
Then the steady-state coefficients of the one- and two-particle states are given as
\[ 0 = \delta_r C_{10} + J C_{01} + i e_d C_{00}, \]
\[ 0 = \delta_r C_{01} + J C_{10}, \] (C6)
and
\[ 0 = 2\delta_r C_{20} + \sqrt{2} J C_{11} + i \sqrt{2} e_d C_{10}, \]
\[ 0 = (\delta_L + \delta_R) C_{11} + \sqrt{2} J C_{20} + \sqrt{2} J C_{02} + i e_d C_{01}, \]
\[ 0 = 2(\delta_r - \delta) C_{02} + \sqrt{2} J C_{11}, \] (C7)
where we have introduced the dissipative terms (proportional to \(\kappa_L\) and \(\kappa_R\)) and neglected the higher-order terms, as justified under the weak-driving conditions.

When we consider \(\Delta_L = \Delta_R - \delta = \Delta, \ \delta = g^2/\omega_m, \ \kappa_L = \kappa_R = \kappa, \) and the condition of \(C_{20} = 0,\) we have
\[ 0 = \kappa^2 (2\delta - 6\Delta - 5\Delta_F^2) + 4\Delta^2 (2\Delta - 2\delta - 5\delta\Delta_F^2) \]
\[- + 4\Delta_F (4\Delta_F - 3\Delta - 6\Delta_F + \Delta_F^2) - 4\Delta_F^2 \delta, \]
\[ 0 = 8\Delta - 12\Delta^2 + \kappa^2 + \Delta_F (6\delta - 20\Delta - 8\Delta_F). \] (C8)
By eliminating \(\delta,\) we obtain
\[ a_4 \Delta_F^4 + a_3 \Delta_F^3 + a_2 \Delta_F^2 + a_1 \Delta_F + a_0 = 0, \] (C9)
where
\[ a_0 = \kappa (4 \Delta_F - 10 \Delta_F^2) (\kappa^2 - 8 \Delta_F^2) - 2 \kappa (\kappa^4 - 44 \Delta_F^4), \]
\[ a_1 = -8 \Delta_F (6 \Delta_F^2 \kappa + 10 \Delta^2 + 3), \]
\[ a_2 = -8 \kappa (2 \kappa^2 + 6 \Delta_F^2 + 13 \Delta_F^2), \]
\[ a_3 = -96 \Delta_F \kappa, \]
\[ a_4 = -32 \kappa, \] (C10)
then we find the optimal conditions
\[ \Delta_{\text{opt}} \approx -a_3 + \text{sgn}(E) \sqrt{\lambda_1 - \sqrt{\lambda_2}}, \]
\[ \epsilon_{\text{opt}} = \sqrt{\frac{\omega_m (4 \Delta_{\text{opt}}^2 + 5 \kappa^2) + \Delta_F \lambda_3}{2 (2 \Delta_F^2 - \kappa^2) + 2 \Delta_F \lambda_4}}, \] (C11)
where
\[ \lambda_1 = \frac{D + \sqrt{z_1 + \sqrt{z_2}}}{3}, \]
\[ \lambda_2 = \frac{2D - \sqrt{z_1 - \sqrt{z_2}} + \sqrt{z_3}}{3}, \]
\[ \lambda_3 = 20 \Delta_{\text{opt}}^2 - 8 \Delta_{\text{opt}} \Delta_F - 4 \Delta_F^2 + 5 \kappa^2, \]
\[ \lambda_4 = 10 \Delta_{\text{opt}}^2 + 3 \Delta_{\text{opt}} + 2 \Delta_F, \] (C12)
and
\[ \text{sgn}(E) = \begin{cases} 1 & (E > 0), \\ -1 & (E < 0), \end{cases} \]
\[ x_{1,2} = AD + 3 \frac{-B \pm \sqrt{B^2 - 4AC}}{2}, \]
\[ z_3 = D^2 - D (\sqrt{z_1} + \sqrt{z_2}) + (\sqrt{z_1} + \sqrt{z_2})^2 - 3A, \]
\[ A = D^2 - 3F, \]
\[ B = DF - 9E^2, \]
\[ C = F^2 - 3D^2, \]
\[ D = 3a_3^2 - 8a_4 a_2, \]
\[ E = -a_3^4 + 4a_4 a_3 a_2 - 8a_4^2 a_1, \]
\[ F = 3a_3^4 + 16a_4^2 a_2^2 - 16a_4 a_3 a_2 + 16a_4^2 a_3 a_1 - 64a_4^3 a_0. \] (C13)

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**REFERENCES**

34. H. Flayac and V. Savona, “