Spin-squeezing suddenly vanishes under decoherence

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In order to witness multipartite correlations beyond pairwise entanglement, spin-squeezing parameters are analytically calculated for a spin ensemble in a collective initial state under three different decoherence channels. It is shown that, in analogy to pairwise entanglement, the spin squeezing described by different parameters can suddenly become zero at different vanishing times. This finding shows the universality of sudden vanishing phenomena of quantum correlations in many-body systems, which here is referred to as spin-squeezing sudden death (SSSD). It is shown that the SSSD usually occurs due to the decoherence, and that SSSD never occurs for some initial states in the amplitude-damping channel. We also analytically obtain the vanishing times of spin squeezing.

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I. INTRODUCTION

Quantum entanglement [1] plays an important role in both the foundations of quantum physics and quantum-information processing [2]. Moreover, various entangled states have been produced in many experiments for different goals, when studying various non-classical phenomena and their applications [3–11]. Thus, entanglement is a quantum resource, and how to measure and detect entanglement is very crucial for both theoretical investigations and potential practical applications.

For a system of two spin-1/2 particles or a composite system of a spin-1/2 and a spin-1, there are operationally-computable entanglement measures such as concurrence [12] and negativity [13, 14], but no universal measures have been found for general many-body systems. To overcome this difficulty, entanglement witnesses are presented to detect some kinds of entanglement in many-body systems [15]. Now it is believed that spin squeezing [16, 17] may be useful for this task [18–20]. In a general sense, spin-squeezing parameters are multiparticle entanglement witnesses. For a class of many-particle states, it has been proved that the concurrence is linearly related to some squeezing parameters [21]. In fact, spin-squeezing parameters [16–19] could be calculated also in a simple operational fashion, which characterizes multiparticle quantum correlations beyond the pairwise entanglement. Another important reason for choosing spin-squeezing parameters as measures of multiparticle correlations is that spin squeezing is relatively easy to generate [17, 22] and measure experimentally [23, 24].

Besides being a parameter characterizing multiparticle correlations, spin squeezing is physically natural for controlling many-body systems. It is difficult to control a quantum many-body system since its constituents cannot be individually addressed. In this sense, one needs to use collective operations, and spin squeezing is one of the most successful approaches for controlling such systems. For example, creating spin squeezing of an atomic ensemble could result in precision measurements based on many-atom spectroscopy [17]. Therefore, we can also regard spin squeezing as a quantum resource since for more than two particles it behaves as two-particle entanglement in controlling and detecting quantum correlations. On this quantum resource, we need to further consider the effects of decoherence [25, 26]. Thus, it is important to study the environment-induced decoherence effects on both spin squeezing and multipartite entanglement [27–37]. A decaying time-evolution of the spin squeezing under decoherence [27, 38–40] can be used to analyze whether or not this quantum resource is robust.

In this paper we address this problem by calculating three spin-squeezing parameters for a spin ensemble in a collective excited state. We study the time evolution of spin squeezing under local decoherence, acting independently and equally on each spin. Here, the irreversible processes are modelled as three decoherence channels: the amplitude-damping, pure-dephasing and depolarizing channels. It is our finding that, similar to the sudden death of pairwise entanglement [41], spin squeezing can also suddenly vanish with different lives for some decoherence channels, showing in general different vanishing times in multipartite correlations in quantum many-body systems. Thus, similar to the discovery of the pairwise entanglement sudden death (ESD) [41], the spin squeezing sudden death (SSSD) occurs due to the decoherence. We will see that for some initial states, the SSSD never occurs under the amplitude-damping channel. We also give analytical expressions for the vanishing time of spin squeezing and pairwise entanglement. The ESD has been tested experimentally [39, 42] and, we also expect that the SSSD can also be realized experimentally.
This paper is organized as follows. In Sec. II, we introduce the initial state from the one-axis twisting Hamiltonian and then the decoherence channels. In Sec. III, we give three parameters of spin squeezing and discuss the relations among them. For a necessary comparison, the concurrence is also calculated. We also study initial-state squeezing. In Sec. IV, we study three different types of spin squeezing and concurrence under three different decoherence channels. Both analytical and numerical results are given. We conclude in Sec. V.

II. INITIAL STATE AND DECOHERENCE CHANNELS

We consider an ensemble of \( N \) spin-1/2 particles with ground state \( |1\rangle \) and excited state \( |0\rangle \). This system has the exchange symmetry, and its dynamical properties can be described by the collective operators

\[
J_\alpha = \sum_{k=1}^{N} j_{k\alpha} = \frac{1}{2} \sum_{k=1}^{N} \sigma_{k\alpha}
\]

for \( \alpha = x, y, z \). Here, \( \sigma_{k\alpha} \) are the Pauli matrices for the \( k \)th qubit. To study the decoherence of spin squeezing, we choose a state which is initially squeezed. One typical class of such spin squeezed states is the one-axis twisting collective spin state [16],

\[
|\Psi(\theta_0)\rangle_0 = e^{-i\theta_0 J_z^2/2} |1\rangle^\otimes N = e^{-i\theta_0 J_z^2/2} |1\rangle,
\]

which could be prepared by the one-axis twisting Hamiltonian

\[
H = \chi J_z^2,
\]

where

\[
\theta_0 = 2\chi t
\]

is the one-axis twist angle and \( \chi \) is the coupling constant. For this state, it was proved [21] that the spin squeezing \( \xi_1^2 \) [16] and the concurrence \( C_0 \) [12] are equivalent since there exists a linear relation \( 1 - (N-1)C_0 \) between them. Physically, they occur and disappear simultaneously. The spin squeezing of this state can be generated and stored in, e.g., a two-component Bose-Einstein condensate [43].

A. Initial-state symmetry

The initial state has an obvious symmetry resulting from Eq. (2), the so-called even-parity symmetry, which means that only even excitations of spins occur in the state. Since \( J_\alpha \) define an angular momentum spinor representation of \( SO(3) \), the general definitions of spin squeezing for abstract operators \( J_x, J_y, \) and \( J_z \) can work well by identifying \( N/2 \) with the highest weight \( J \), which corresponds to the collective ground state

\[
|J, -J\rangle = |1\rangle^\otimes N \equiv |1\rangle
\]

indicating that all spins are in the ground state. The symmetric space is generated by the collective operator \( J_+ = \sum_{k=1}^{N} \sigma_{k+} / 2 \) acting on the collective ground state. Here, \( \sigma_{k\pm} = (\sigma_{kx} \pm i\sigma_{ky})/2 \). In others words, the state is in the maximally-symmetric space spanned by the Dicke states. So, the \( N \) spin-1/2 system behaves like a larger spin-\( N/2 \) system. It can be proved that any pure state with exchange symmetry belongs to the above-mentioned symmetric space, but for mixed states, the state space can be extended to include a space beyond the symmetric one [44]. In the following discussions, we focus on such an extended space.

In fact, after decoherence, not only the symmetric Dicke states are populated, but also states with lower symmetry. So, it is not sufficient to describe the system in only \( (N+1) \)-dimensional space. Although the maximal symmetry is broken, the exchange symmetry is not affected by the decoherence as each local decoherence equally acts on each spin. In other words, a state with exchange symmetry does not necessarily belong to the maximally-symmetric space.

With only the exchange symmetry, from Eq. (1), the global expectations or correlations of collective operators are obtained as

\[
\langle J_\alpha^2 \rangle = \frac{N}{4} + \frac{N(N-1)}{4} \langle \sigma_{1\alpha} \sigma_{2\alpha} \rangle,
\]

\[
\langle J_\alpha J_\beta \rangle = \frac{N(N-1)}{4} \langle \sigma_{1-} \sigma_{2-} \rangle,
\]

\[
\langle [J_x, J_y]_+ \rangle = \frac{N(N-1)}{4} \langle [\sigma_{1+} \sigma_{2+}]_+ \rangle.
\]

Furthermore, it follows from Eq. (6) that

\[
\langle J_x^2 + J_y^2 \rangle = \frac{N}{2} + \frac{N(N-1)}{2} \langle \sigma_{1+} \sigma_{2-} + \sigma_{1-} \sigma_{2+} \rangle,
\]

\[
\langle J_x^2 + J_y^2 + J_z^2 \rangle = \frac{N^2}{4} \left[ \frac{3}{N} + \left( 1 - \frac{1}{N} \right) \langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle \right].
\]

These equations show the relations between the global and local expectations and correlations, which are useful in the following calculations.

B. Decoherence channels

Having introduced the initial state, now we discuss three typical decoherence channels: the amplitude-damping channel (ADC), the phase-damping channel (PDC), and the depolarizing channel (DPC).

1. Amplitude-damping channel

The ADC is defined as

\[
\mathcal{E}_{ADC}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger,
\]

where

\[
E_0 = \sum_{k=1}^{N} \sigma_{k-} / 2
\]

and

\[
E_1 = \sum_{k=1}^{N} \sigma_{k\pm} / 2
\]
where
\[ E_0 = \sqrt{s} |0\rangle \langle 0 | + |1\rangle \langle 1 |, \quad E_1 = \sqrt{p} |1\rangle \langle 0 | \]
are the Kraus operators, \( p = 1 - s, \quad s = \exp(-\gamma t/2), \) and \( \gamma \) is the damping rate.

2. Phase-damping channel

The PDC is described by the map
\[ E_{PDC}(\rho) = s\rho + p (\rho_{00} |0\rangle \langle 0 | + \rho_{11} |1\rangle \langle 1 |), \]
and obviously the three Kraus operators are given by
\[ E_0 = \sqrt{s} \mathbb{1}, \quad E_1 = \sqrt{p} |1\rangle \langle 0 |, \quad E_2 = \sqrt{p} |1\rangle \langle 1 |, \]
where \( \mathbb{1} \) is the identity operator. For the PDC, there is no energy change and a loss of decoherence occurs with probability \( p \).

3. Depolarizing channel

The definition of the DPC is given via the map
\[ E_{DPC}(\rho) = \sum_{i=0}^{3} E_k \rho E_k^\dagger, \]

\[ = (1 - p')\rho + \frac{p'}{3}(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z), \]

where
\[ E_0 = \sqrt{1 - p'} \mathbb{1}, \quad E_1 = \sqrt{\frac{p'}{3}} \sigma_x, \]
\[ E_2 = \sqrt{\frac{p'}{3}} \sigma_y, \quad E_3 = \sqrt{\frac{p'}{3}} \sigma_z, \]
are the Kraus operators. By using the following identity
\[ \sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z + \rho = 2 \mathbb{1}, \]
we obtain
\[ E_{DPC}(\rho) = s\rho + p \frac{\mathbb{1}}{2}, \]
where \( p = 4p'/3 \). We see that for the DPC, the spin is unchanged with probability \( s = 1 - p \) or is depolarized to the maximally mixed state \( \mathbb{1}/2 \) with probability \( p \).

III. SPIN-SQUEEZING DEFINITIONS AND CONCURRENCE

A. Spin-squeezing parameters and their relations

1. Definitions of spin squeezing

There are several spin-squeezing parameters, but we only list three typical and related ones as follows [16–19]:

\[ \xi_1^2 = \frac{4(\Delta J_{\perp})^2_{\text{min}}}{N}, \]
\[ \xi_2^2 = \frac{N^2}{4\langle J^2 \rangle} \xi_1^2, \]
\[ \xi_3^2 = \frac{\lambda_{\text{min}}}{\langle J^2 \rangle} - \frac{N}{2}. \]

Here, the minimization in the first equation is over all directions denoted by \( \vec{n}_{\perp} \), perpendicular to the mean spin direction \( \langle J \rangle / \langle J^2 \rangle \); \( \lambda_{\text{min}} \) is the minimum eigenvalue of the matrix [19]
\[ \Gamma = (N - 1)\gamma + C, \]
where
\[ \gamma_{kl} = C_{kl} - \langle J_k \rangle \langle J_l \rangle \text{ for } k, l \in \{x, y, z\} = \{1, 2, 3\}, \]
is the correlation matrix and \( C = [C_{kl}] \) with
\[ C_{kl} = \frac{1}{2} \langle J_l J_k + J_k J_l \rangle \]
is the global correlation matrix. The parameters \( \xi_1^2, \xi_2^2, \) and \( \xi_3^2 \) were defined by Kitagawa and Ueda [16], Wineland et al. [17], and Tóth et al. [19], respectively. If \( \xi_2^2 < 1 \) (\( \xi_3^2 < 1 \)), spin squeezing occurs, and we can safely say that the multipartite state is entangled [18, 19]. Although we cannot say that the squeezed state via parameter \( \xi_1^2 \) is entangled, it is indeed closely related to quantum entanglement [21].

2. Squeezing parameters for states with parity

We know from Sec. II. A that the initial state has an even parity and that the mean spin direction is along the z-direction. During the transmission through all the decoherence channels discussed here, the mean spin direction does not change. For states with a well-defined parity (even or odd), the spin squeezing parameter \( \xi_1^2 \) was found to be [21]
\[ \xi_1^2 = \frac{2}{N} \left( \langle J_z^2 \rangle + \langle J_0^2 \rangle - |\langle J^2 \rangle| \right). \]

Then, the parameter \( \xi_2^2 \) given by Eq. (19) then becomes
\[ \xi_2^2 = \frac{N^2 \xi_1^2}{4\langle J_z \rangle^2} = \frac{N}{2\langle J_z \rangle^2} \langle (J_x^2 + J_y^2) - |\langle J^2 \rangle| \rangle. \]
For the third squeezing parameter (see Appendix A for the derivation), we have
\[
\xi_3^2 = \frac{\min \{\xi_1^2, \xi_2^2\}}{4N^{-2}\langle \hat{J}^2 \rangle - 2N^{-1}}, \tag{26}
\]
where
\[
\xi^2 = \frac{4}{N^2} \left[ N(\Delta J_z)^2 + \langle J_z^2 \rangle \right]. \tag{27}
\]
Note that the first parameter \(\xi_3^2\) becomes a key ingredient for the latter two squeezing parameters (\(\xi_2^2\) and \(\xi_3^2\)).

3. Spin-squeezing parameters in terms of local expectations

For later applications, we now express the squeezing parameters in terms of local expectations and correlations, and also examine the meaning of \(\zeta^2\), which will be clear by substituting Eqs. (1) and (6) into Eq. (27).

Thus, the parameter \(\zeta^2\) is simply related to the correlation \(C_{zz}\) along the z-direction. A negative correlation \(C_{zz} < 0\) is equivalent to \(\zeta^2 < 1\). It is already known that the spin squeezing parameter \(\xi_1^2\) can be written as \[46\]
\[
\xi_1^2 = 1 + (N - 1)C_{\hat{n}_\perp, \hat{n}_\perp}, \tag{29}
\]
where \(C_{\hat{n}_\perp, \hat{n}_\perp}\) is the correlation function in the direction perpendicular to the mean spin direction. So, the spin squeezing \(\xi_1^2 < 1\) is equivalent to the negative pairwise correlations \(C_{\hat{n}_\perp, \hat{n}_\perp} < 0\).

Thus, from the above analysis, spin squeezing and negative correlations are closely connected to each other. The parameter \(\zeta^2 < 1\) indicates that spin squeezing occurs along the z-direction, and \(\xi_1^2 < 1\) implies spin squeezing along the direction perpendicular to the mean spin direction. Furthermore, from Eq. (26), a competition between the transversal and longitudinal correlations is evident.

By substituting Eqs. (7) and (9) to Eq. (24), one can obtain the expression of \(\xi_1^2\) in terms of local correlations \(\langle \sigma_1^+\sigma_2^- \rangle\) and \(\langle \sigma_1^-\sigma_2^+ \rangle\) as follows:
\[
\xi_1^2 = 1 + (N - 1)\langle \sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+ \rangle
- 2(N - 1)|\langle \sigma_1^+\sigma_2^- \rangle|,
= 1 + 2(N - 1)|\langle \sigma_1^+\sigma_2^- \rangle| - |\langle \sigma_1^-\sigma_2^+ \rangle|. \tag{30}
\]
The second equality in Eq. (30) results from the exchange symmetry. From Eqs. (1), (10), and (28), one finds
\[
\xi_2^2 = \frac{\xi_1^2}{\langle \sigma_{1z} \rangle^2}, \tag{31}
\]
\[
\xi_3^2 = \frac{\min \{\xi_1^2, 1 + C_{zz}\}}{(1 - N^{-1})\langle \sigma_1^+ \sigma_2^- \rangle + N^{-1}}. \tag{32}
\]
Thus, we have reexpressed the squeezing parameters in terms of local correlations and expectations.

4. New spin-squeezing parameters

In order to characterize spin squeezing more conveniently, we define the following squeezing parameters:
\[
\zeta_k^2 = \max(0, 1 - \xi_k^2), \quad k \in \{1, 2, 3\}. \tag{33}
\]
This definition is similar to the expression of the concurrence given below. Spin squeezing appears when \(\zeta_k^2 > 0\), and there is no squeezing when \(\zeta_k^2\) vanishes. Thus, the definition of the first parameter \(\zeta_1^2\) has a clear meaning, namely, it is the strength of the negative correlations as seen from Eq. (29). The larger \(\zeta_1^2\), the larger is the strength of the negative correlation, and the larger of is the squeezing. More explicitly, for the initial state, we have \(\xi_1^2 = 1 - (N - 1)C_0\) [21], so \(\zeta_1^2\) is just the rescaled concurrence \(\xi_1^2 = C_r(0) = (N - 1)C_0\) [48].

Here, we give a few comments on the spin squeezing parameter \(\zeta_3^2\), which represents a competition between \(\xi_1^2\) and \(\langle \sigma_{1z} \rangle^2\): the state is squeezed according to definition of \(\xi_3^2\) if \(\xi_3^2 < \langle \sigma_{1z} \rangle^2\). We further note that [49]
\[
\langle \sigma_{1z} \rangle^2 = 1 - 2E_L, \tag{34}
\]
where \(E_L\) is the linear entropy of one spin and it can be used to quantify the entanglement of pure states [14]. So, there is a competition between the strength of negative correlations and the linear entropy \(2E_L\) in the parameter \(\zeta_3^2\), and \(\zeta_3^2 > 2E_L\) implies the appearance of squeezing.

B. Concurrence for pairwise entanglement

It has been found that the concurrence is closely related to spin squeezing [21]. Here, we consider its behavior under various decoherence channels. The concurrence quantifying the entanglement of a pair of spin-1/2 can be calculated from the reduced density matrix. It is defined as [12]
\[
C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{35}
\]
where the quantities \(\lambda_i\) are the square roots of the eigenvalues in descending order of the matrix product
\[
\varrho_{12} = \rho_{12}(\sigma_{1y} \otimes \sigma_{2y})\rho_{12}^*(\sigma_{1y} \otimes \sigma_{2y}). \tag{36}
\]
In (36), \(\rho_{12}^*\) denotes the complex conjugate of \(\rho_{12}\).

The two-spin reduced density matrix for a parity state with the exchange symmetry can be written in a block-diagonal form [47]
\[
\rho_{12} = \begin{pmatrix} v_+ & w^* \\ u & y \end{pmatrix} \oplus \begin{pmatrix} w & y \\ w^* & v_- \end{pmatrix}, \tag{37}
\]
in the basis \([00], [11], [01], [10]\), where
\[
v_\pm = \frac{1}{4} (1 \pm 2\langle \sigma_{1z} \rangle + \langle \sigma_{1z}\sigma_{2z} \rangle), \tag{38}
\]
\[
w = \frac{1}{4} (1 - \langle \sigma_{1z}\sigma_{2z} \rangle), \tag{39}
\]
\[
u = \langle \sigma_{1z}\sigma_{2z} \rangle, \tag{40}
\]
\[
y = \langle \sigma_{1z}\sigma_{2z} \rangle. \tag{41}
\]
The concurrence is then given by [50]

\[ C = \max \left\{ 0, 2 \left( |u| - w \right), 2(y - \sqrt{v^2 - w^2}) \right\}. \] (42)

From the above expressions of the spin-squeezing parameters and concurrence, we notice that if we know the expectation \( \langle \sigma_{1z} \rangle \), and the correlations \( \langle \sigma_{1+}\sigma_{2-} \rangle \), \( \langle \sigma_{1-}\sigma_{2+} \rangle \), and \( \langle \sigma_{1+}\sigma_{2+} \rangle \), all the squeezing parameters and concurrence can be determined. Below, we will give explicit analytical expressions for them subject to three decoherence channels.

C. Initial-state squeezing and concurrence

We will now investigate initial spin squeezing and pairwise entanglement by using our results for the spin-squeezing parameters and concurrence obtained in the last subsections. We find that the third squeezing parameter \( \xi^2_3 \) is equal to the first one \( \xi^2_1 \). The squeezing parameter \( \xi^2_1 \) is given by (see Appendix B):

\[
\xi^2_1(0) = 1 - C_r(0) = 1 - (N - 1)C_0, = 1 - 2(N - 1)(|u_0| - y_0),
\] (43)

where

\[
C_0 = \frac{1}{4} \left\{ [(1 - \cos^2 \theta_0)^2 + 16 \sin^2 (\theta_0/2) \cos^2 (\theta_0/2)]^{1/2} - 1 \right\} \cos^{N-2} \theta_0
\] (44)

is the concurrence [21].

The parameter \( \xi^2_2(0) \) is easily obtained, as we know both \( \xi^2_1(0) \) and \( \langle \sigma_{1+}\sigma_{2-} \rangle \) (B6). For this state, following from Eq. (10), \( \langle \hat{\sigma}_1 \cdot \hat{\sigma}_2 \rangle = 1 \), and thus the third parameter given by Eq. (32) becomes

\[
\xi^2_3(0) = \min[\xi^2_1(0), \xi^2_2(0)] = \min[\xi^2_1(0), 1 + C_{zz}(0)],
\] (45)

where the correlation function is

\[
C_{zz}(0) = \frac{1}{2} \left( 1 + \cos^{N-2} \theta_0 \right) - \cos^{2N-2} \theta_0/2 \geq 0.
\] (46)

The proof of the above inequality is given in Appendix C.

As the correlation function \( C_{zz}(0) \) and the concurrence \( C_r(0) \) are always \( \geq 0 \), Eq. (45) reduces to

\[
\xi^2_3(0) = \xi^2_1(0) = 1 - C_r(0).
\] (47)

So, for the initial state, the spin-squeezing parameters \( \xi^2_2(0) \) and \( \xi^2_1(0) \) are equal or equivalently, we can write \( \xi^2_2(0) = \xi^2_3(0) = C_r(0) \) according to the definition of parameter \( \xi^2_1 \) given by Eq. (33). Below we made a summary of results of this section in Table I.

IV. SPIN SQUEEZING UNDER DECOHERENCE

Now we begin to study spin squeezing under three different decoherence channels. From the previous analysis, all the spin-squeezing parameters and the concurrence are determined by some correlation functions and expectations. So, if we know the evolution of them under decoherence, the evolution of any squeezing parameters and pairwise entanglement can be calculated.

A. Heisenberg Approach

We now use the Heisenberg picture to calculate the correlation functions and the relevant expectations. A decoherence channel with Kraus operators \( K_\mu \) is defined via the map

\[
\mathcal{E}(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger.
\] (48)

Then, an expectation value of the operator \( A \) can be calculated as \( \langle A \rangle = \text{Tr}[A \mathcal{E}(\rho)] \). Alternatively, we can define the following map,

\[
\mathcal{E}^\dagger(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger.
\] (49)

It is easy to check that

\[
\langle A \rangle = \text{Tr} [A \mathcal{E}(\rho)] = \text{Tr} [\mathcal{E}^\dagger (A) \rho].
\] (50)

So, one can calculate the expectation value via the above equation (50). This is very similar to the standard Heisenberg picture.

B. Amplitude-damping channel

1. Squeezing parameters

Based on the above approach and the Kraus operators for the ADC given by Eq. (12), we now find the evolutions of the following expectations under decoherence (see appendix D for details)

\[
\langle \sigma_{1z} \rangle = s \langle \sigma_{1z} \rangle_0 - p,
\] (51a)

\[
\langle \sigma_{1-}\sigma_{2-} \rangle = s \langle \sigma_{1-}\sigma_{2-} \rangle_0,
\] (51b)

\[
\langle \sigma_{1+}\sigma_{2-} \rangle = s \langle \sigma_{1+}\sigma_{2-} \rangle_0 - 2sp \langle \sigma_{1z} \rangle_0 + p^2.
\] (51c)

To determine the squeezing parameters and the concurrence, it is convenient to know the correlation function \( C_{zz} \) and the expectation \( \langle \hat{\sigma}_1 \cdot \hat{\sigma}_2 \rangle \), which can be determined from the above expectations as follows:

\[
\langle \hat{\sigma}_1 \cdot \hat{\sigma}_2 \rangle = 1 - sp x_0,
\] (52)

\[
C_{zz} = s^2 (\langle \sigma_{1+}\sigma_{2+} \rangle_0 - \langle \sigma_{1z} \rangle_0 \langle \sigma_{2z} \rangle_0) = s^2 C_{zz}(0),
\] (53)
where

\[ x_0 = 1 + 2\langle\sigma_z\rangle_0 + \langle\sigma_{1z}\sigma_{2z}\rangle_0. \]  

(54)

Substituting the relevant expectation values and the correlation function into Eqs. (30), (31), and (32) leads to the explicit expression of the spin-squeezing parameters

\[ \xi_1^2 = 1 - sC_r(0), \]  

(55)

\[ \xi_2^2 = \frac{\xi_1^2}{(s\langle\sigma_{1z}\rangle_0 - p)^2}, \]  

(56)

\[ \xi_3^2 = \frac{\min\{\xi_1^2, 1 + s^2C_{zz}(0)\}}{1 + (1 + N^{-1})sp x_0}. \]  

(57)

As the correlation function \( C_{zz}(0) \geq 0 \), given by Eq. (46), the third parameter can be simplified as

\[ \xi_3^2 = \frac{1 - sC_r(0)}{1 + (1 + N^{-1})sp x_0}. \]  

(58)

Initially, the state is spin squeezed, i.e., \( \xi_1^2(0) < 1 \) or \( C_r(0) > 0 \). From Eq. (55), one can find that \( \xi_1^2 < 1 \), except in the asymptotic limit of \( p = 1 \). As we will see below, for the PDC and DPC, \( \xi_1^2 = 1 - s^2C_r(0) \). Thus, we conclude that according to \( \xi_1^2 \), the initially spin-squeezed state is always squeezed for \( p \neq 1 \), irrespective of both the decoherence strength and decoherence models. In other words, there exists no SSSD if we quantify spin squeezing by the first parameter \( \xi_1^2 \).

2. Concurrence

In the expression (42) of the concurrence, there are three terms inside the max function. The expression can be simplified to (see Appendix E for details):

\[ C_r = 2(N - 1)\max(0, |u| - w). \]  

(59)

By using Eqs. (39) and (51c), one finds

\[ 2(|u| - w) \]

\[ = 2s|u_0| + \frac{s}{2}|s - 2 + s(\sigma_{1z}\sigma_{2z})_0 - 2p(\sigma_{1z})_0| \]

\[ = sC_0 - \frac{sp x_0}{2}. \]  

(61)

So, we obtain the evolution of the re-scaled concurrence as

\[ C_r = \max\left[0, sC_r(0) - 2^{-1}(N - 1)sp x_0\right], \]  

(62)

which depends on the initial concurrence, expectation \( \langle\sigma_{1z}\rangle_0 \), and correlation \( \langle\sigma_{1z}\sigma_{2z}\rangle_0 \).

3. Numerical results

The numerical results for the squeezing parameters and concurrence are illustrated in Fig. 1 for different initial values of the twist angle \( \theta_0 \), defined in Eq. (4). For the smaller value of \( \theta_0 \), e.g., \( \theta_0 = \pi/10 \), we see that there is no ESD and SSSD. All the spin squeezing and the pairwise entanglement are completely robust against decoherence. Intuitively, the larger is the squeezing, the larger is the vanishing time. However, here, in contrast to this, no matter how small are the squeezing parameters and concurrence, they vanish only in the asymptotic limit. This results from the complex correlations in the initial state and the special characteristics of the ADC.

For larger values of \( \theta_0 \), as the decoherence strength \( p \) increases, the spin squeezing decreases until it suddenly vanishes, so the phenomenon of SSSD occurs. There exists a critical value \( p_c \), after which there is no spin squeezing. The vanishing time of \( \xi_3^2 \) is always larger than those of \( \xi_2^2 \) and the concurrence. We note that depending on the initial state, the concurrence can vanish before or after \( \xi_2^2 \). This means that in our model, the parameter
FIG. 1: Spin-squeezing parameters $\zeta_2^2$ (red curve with squares), $\zeta_3^2$ (top green curve with circles), and the concurrence $C$ (solid curve) versus the decoherence strength $p = 1 - \exp(-\gamma t)$ for the amplitude-damping channel, where $\gamma$ is the damping rate. Here, $\theta_0$ is the initial twist angle given by Eq. (4). In all figures, we consider an ensemble of $N = 12$ spins. Note that for small initial twist angle $\theta_0$ (e.g., $\theta_0 = 0.1\pi$), the two squeezing parameters and the concurrence all concur. For larger values of $\theta_0$, $\zeta_2^2$, $\zeta_3^2$, and $C$ become quite different and all vanish for sufficiently large values of the decoherence strength.

$\zeta_3^2 < 1$ implies the existence of pairwise entanglement, while $\zeta_2^2$ does not.

4. Decoherence strength $p_c$ corresponding to the SSSD

From Eqs. (56), (57), and (62), the critical value $p_c$ can be analytically obtained as

$$p_c^{(k)} = \frac{x_k C_r(0)}{(N - 1)x_0}, \quad (k = 1, 3)$$  \hspace{1cm} (63)

$$p_c^{(2)} = \frac{\langle \sigma_{1z} \rangle_0^2 + C_r(0) - 1}{1 + 2\langle \sigma_{1z} \rangle_0 + \langle \sigma_z \rangle_0^2},$$  \hspace{1cm} (64)

where $x_1 = 2$ for the concurrence and $x_3 = N$ for the squeezing parameter $\zeta_3^2$. The critical value $p_c^{(2)}$ is for the second squeezing parameter $\zeta_2^2$. Here, $p_c$ is related to the vanishing time $t_v$ via $p_c = 1 - \exp(-\gamma t_v)$.

In Fig. 2, we plot the critical values $p_c$ of the decoherence strength versus $\theta_0$. The initial-state squeezing parameter $\zeta_1^2$ is also plotted for comparison. For a range of small values of $\theta_0$, the entanglement and squeezing are robust to decoherence. The concurrence and parameter $\zeta_2^2$ intersect. However, we do not see the intersections between $\zeta_2^2$ and $\zeta_3^2$ or between $\zeta_2^2$ and the concurrence. We also see that for the same degree of squeezing, the vanishing times are quite different, which implies that except for the spin-squeezing correlations, other type of correlations exist. For large enough initial twist angles $\pi \leq \theta_0 \leq 2\pi$, the behavior of the squeezing parameter $\zeta_2^2$ is similar to those corresponding to $p_c^{(1)}$ and $p_c^{(3)}$.

C. Phase-damping channel

1. Squeezing parameters and concurrence

Now, we study the spin squeezing and pairwise entanglement under the PDC. For this channel, the expectation values $\langle \sigma_z^{(n)} \rangle$ are unchanged and the two correlations $\langle \sigma_{1-}\sigma_{2-} \rangle$ and $\langle \sigma_{1+}\sigma_{2-} \rangle$ evolve as (see Appendix D for details)

$$\langle \sigma_{1-}\sigma_{2-} \rangle = s^2 \langle \sigma_{1-}\sigma_{2-} \rangle,$$

$$\langle \sigma_{1+}\sigma_{2-} \rangle = s^2 \langle \sigma_{1+}\sigma_{2-} \rangle.$$  \hspace{1cm} (65)
From the above equations and the fact \( \langle \hat{\sigma}_1 \cdot \hat{\sigma}_2 \rangle_0 = 1 \), one finds

\[
\langle \hat{\sigma}_1 \cdot \hat{\sigma}_2 \rangle = s^2 \langle \sigma_{1x} \sigma_{2x} + \sigma_{1y} \sigma_{2y} \rangle_0 + \langle \sigma_{1z} \sigma_{2z} \rangle_0 \\
= s^2(1 - \langle \sigma_{1z} \sigma_{2z} \rangle_0) + \langle \sigma_{1z} \sigma_{2z} \rangle_0, \quad (66)
\]

\[
C_{zz}(p) = C_{zz}(0). \quad (67)
\]

Therefore, from the above properties, we obtain the evolution of the squeezing parameters,

\[
\xi_1^2 = 1 - s^2 C_r(0), \quad (68)
\]

\[
\xi_2^2 = \frac{\xi_3^2}{\langle \sigma_{1z} \rangle_0^2}, \quad (69)
\]

and the third parameter becomes

\[
\xi_3^2 = \frac{N \min \left[ \xi_1^2, 1 + C_{zz}(0) \right]}{(N - 1)[s^2 + (1 - s^2)\langle \sigma_{1z} \sigma_{2z} \rangle_0] + 1} \quad (70)
\]

\[
= \frac{N \xi_2^2}{(N - 1)[s^2 + (1 - s^2)\langle \sigma_{1z} \sigma_{2z} \rangle_0] + 1} \quad (71)
\]

where we have used Eqs. (66) and (67), and the property \( C_{zz}(0) \geq 0 \).

From Eq. (65) and the simplified form of the concurrence given by Eq. (59), the concurrence is found to be

\[
C_r = \max \left\{ 0, 2(N - 1) \times \left[ s^2 |a_0| - 4^{-1}(1 - \langle \sigma_{1z} \sigma_{2z} \rangle_0) \right] \right\}
\]

\[
= \max \left\{ 0, s^2 C_r(0) + \frac{a_0(s^2 - 1)}{2} \right\} \quad (72)
\]

where

\[
a_0 = (N - 1)(1 - \langle \sigma_{1z} \sigma_{2z} \rangle_0). \quad (73)
\]

Thus, we obtained all time evolutions of the spin-squeezing parameters and the concurrence. To study the phenomenon of SSSD, we below examine the vanishing times.

2. Decoherence strength \( p_c \) corresponding to the SSSD

The critical decoherence strengths \( p_c \) can be obtained from Eqs. (69), (70), and (72) as follows:

\[
p_c^{(k)} = 1 - \left[ \frac{a_0}{x_k C_r(0) + a_0} \right]^{\frac{k}{2}}, \quad (74)
\]

\[
p_c^{(2)} = 1 - \left[ \frac{1 - \langle \sigma_{1z} \rangle_0^2}{C_r(0)} \right]^{\frac{2}{3}}, \quad (75)
\]

where \( k = 1, 3 \) and \( x_1 = 2, x_3 = N \).

In Fig. 3, we plot the decoherence strength \( p_c \) versus the twist angle \( \theta_0 \) of the initial state for the PDC. For this decoherence channel, the critical value \( p_c^{(k)} \)s first decrease until they reach zero. Also, it is symmetric with respect to \( \theta_0 = \pi \), which is in contrast to the ADC. There are also intersections between the concurrence and parameter \( \xi_3^2 \), and the critical value \( p_c^{(3)} \) is always larger than \( p_c^{(1)} \) and \( p_c^{(2)} \).

D. Depolarizing channel

1. Squeezing parameters and concurrence

The decoherence of the squeezing parameter defined by Sørensen et al. [18] has been studied in Ref. [27] for the DPC. It is intimately related to the second squeezing parameter \( \xi_3^2 \). For the DPC, the evolution of correlations \( \langle \sigma_{1x} \sigma_{2x} \rangle \) and \( \langle \sigma_{1z} \sigma_{2z} \rangle \) are the same as those of the DPC given by Eq. (65), and the expectations \( \langle \sigma_{1z} \rangle \) and \( \langle \sigma_{1z} \sigma_{2z} \rangle \) change as (see Appendix D).

\[
\langle \sigma_{1z} \rangle = s \langle \sigma_{1z} \rangle_0, \quad (76)
\]

\[
\langle \sigma_{1z} \sigma_{2z} \rangle = s^2 \langle \sigma_{1z} \sigma_{2z} \rangle_0. \quad (77)
\]

From these equations, we further have

\[
\langle \hat{\sigma}_1 \cdot \hat{\sigma}_2 \rangle = s^2 \langle \hat{\sigma}_1 \cdot \hat{\sigma}_2 \rangle_0 = s^2, \quad (78)
\]

\[
C_{zz} = s^2 (\langle \sigma_{1z} \sigma_{2z} \rangle - \langle \sigma_{1z} \rangle_0 \langle \sigma_{2z} \rangle_0) = s^2 C_{zz}(0). \quad (79)
\]
The squeezing parameter $\xi_2^2$ is given by Eq. (68), and the other two squeezing parameters are obtained as

\begin{align}
\xi_1^2 &= \frac{\xi_1^2}{s^2(\sigma_{1z})^2}, \\
\xi_2^2 &= \frac{N \min \left\{ \xi_1^2, 1 + s^2C_{zz}(0) \right\}}{(N-1)s^2 + 1} \\
&= \frac{N\xi_1^2}{(N-1)s^2 + 1}.
\end{align}

By making use of Eqs. (65) and (77) and starting from the simplified form of the concurrence (59), we obtain

\begin{align}
C_r &= \max \left\{ 0, 2(N-1) \left[ s^2|w_0| - \frac{1}{2}(1 - s^2<\sigma_{1z}\sigma_{2z}>_0) \right] \right\} \\
&= \max \left\{ 0, s^2C_r(0) + 2^{-1}(N-1)(s^2 - 1) \right\}.
\end{align}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
 & Amplitude-damping channel (ADC) & Phase-damping channel (PDC) & Depolarizing channel (DPC) \\
\hline
$\langle \sigma_{1z} \rangle$ & $s(\sigma_{1z})_0 - p$ & $\langle \sigma_{1z} \rangle_0$ & $s(\sigma_{1z})_0$ \\
\hline
$\langle \sigma_{1z}\sigma_{2z} \rangle$ & $s^2\langle \sigma_{1z}\sigma_{2z} \rangle_0 - 2sp\langle \sigma_{1z} \rangle_0 + p^2$ & $\langle \sigma_{1z}\sigma_{2z} \rangle_0$ & $s^2\langle \sigma_{1z}\sigma_{2z} \rangle_0$ \\
\hline
$\langle \sigma_{1+}\sigma_{2-} \rangle$ & $s\langle \sigma_{1+}\sigma_{2-} \rangle_0$ & $s^2\langle \sigma_{1+}\sigma_{2-} \rangle_0$ & $s^2\langle \sigma_{1+}\sigma_{2-} \rangle_0$ \\
\hline
$\langle \sigma_{1-}\sigma_{2-} \rangle$ & $s\langle \sigma_{1-}\sigma_{2-} \rangle_0$ & $s^2\langle \sigma_{1-}\sigma_{2-} \rangle_0$ & $s^2\langle \sigma_{1-}\sigma_{2-} \rangle_0$ \\
\hline
$\langle \sigma_1 \cdot \sigma_2 \rangle$ & $1 - sp x_0$ & $s^2(1 - \langle \sigma_{1z}\sigma_{2z} \rangle_0) + \langle \sigma_{1z}\sigma_{2z} \rangle_0$ & $s^2$ \\
\hline
$C_{xx}$ & $s^2C_{zz}(0)$ & $C_{zz}(0)$ & $s^2C_{zz}(0)$ \\
\hline
$\xi_1^2$ & $1 - sC_r(0)$ & $1 - s^2C_r(0)$ & $1 - s^2C_r(0)$ \\
\hline
$\xi_2^2$ & $\frac{1 - sC_r(0)}{(s\langle \sigma_{1z} \rangle_0 - p)^2}$ & $\frac{1 - s^2C_r(0)}{\langle \sigma_{1z} \rangle_0^2}$ & $\frac{1 - s^2C_r(0)}{s^2\langle \sigma_{1z} \rangle_0^2}$ \\
\hline
$\xi_3^2$ & $\frac{1 - sC_r(0)}{1 + (N-1)sp x_0}$ & $\frac{1 - s^2C_r(0)}{(1 - N^{-1})(s^2 + 1 - s^2\langle \sigma_{1z}\sigma_{2z} \rangle_0)} + N^{-1}$ & $\frac{1 - s^2C_r(0)}{(1 - N^{-1})s^2 + N^{-1}}$ \\
\hline
$C_r^{(s)}$ & $sC_r(0) - (N-1)sp x_0/2$ & $s^2C_r(0) + a_0(s^2 - 1)/2$ & $s^2C_r(0) + (N-1)(s^2 - 1)/2$ \\
\hline
$p_c^{(1)}$ & $\frac{2C_r(0)}{(N-1)x_0}$ & $1 - \left( \frac{a_0}{2C_r(0) + a_0} \right)^{\frac{1}{2}}$ & $1 - \left( \frac{N - 1}{2C_r(0) + N - 1} \right)^{\frac{1}{2}}$ \\
\hline
$p_c^{(2)}$ & $\frac{(\sigma_{1z})_0^2 + C_r(0) - 1}{1 + 2(\sigma_{1z})_0 + (\sigma_{1z})_0^2}$ & $1 - \left( \frac{1 - \langle \sigma_{1z} \rangle_0^2}{C_r(0)} \right)^{\frac{1}{2}}$ & $1 - \left( \frac{1}{C_r(0) + \langle \sigma_{1z} \rangle_0^2} \right)^{\frac{1}{2}}$ \\
\hline
$p_c^{(3)}$ & $\frac{NC_r(0)}{(N-1)x_0}$ & $1 - \left( \frac{a_0}{NC_r(0) + a_0} \right)^{\frac{1}{2}}$ & $1 - \left( \frac{N - 1}{NC_r(0) + N - 1} \right)^{\frac{1}{2}}$ \\
\hline
\end{tabular}
\end{table}

We observe that the concurrence is only dependent on the initial value itself, not other ones.

2. Decoherence strength $p_c$ corresponding to the SSSD

From Eqs. (82), (80), and (81), the vanishing times are analytically calculated as

\begin{align}
p_c^{(k)} &= 1 - \left[ \frac{N - 1}{x_kC_r(0) + N - 1} \right]^{\frac{1}{2}}, \\
p_c^{(2)} &= 1 - \left[ \frac{1}{C_r(0) + (\sigma_{1z})_0^2} \right]^{\frac{1}{2}}.
\end{align}

where $k = 1, 3$ and $x_1 = 2, x_3 = N$.

In Fig. 3, we plot critical values $p_c$ versus the initial
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APPENDIX A: SPIN-SQUEEZING PARAMETERS ξ FOR STATES WITH THE PARITY SYMMETRY

Here, we calculate the spin-squeezing parameter ξ for the collective states with either even or odd parity sym-
metry. For such states, we immediately have
\[ \langle J_x \rangle = \langle J_y \rangle = \langle J_x J_z \rangle = \langle J_y J_z \rangle = 0 \] (A1)
as the operators change the parity of the state. Then, the mean spin direction is along the z-direction and the correlation matrix given by Eq. (23) is simplified to
\[ C = \begin{pmatrix} \langle J_x^2 \rangle & C_{xy} & 0 \\ C_{xy} & \langle J_y^2 \rangle & 0 \\ 0 & 0 & \langle J_z^2 \rangle \end{pmatrix}, \] (A2)
where \( C_{xy} = \langle J_x, J_y \rangle / 2 \). From the correlation matrix \( C \) and the definition of covariance matrix \( \gamma \) given by Eq. (22), one finds
\[ \Gamma = \begin{pmatrix} N\langle J_x^2 \rangle & NC_{xy} & 0 \\ NC_{xy} & N\langle J_y^2 \rangle & 0 \\ 0 & 0 & N(\Delta J_z)^2 + \langle J_z^2 \rangle \end{pmatrix}. \] (A3)
This matrix has a block-diagonal form and the eigenvalues of the \( 2 \times 2 \) block are obtained as
\[ \lambda_{\pm} = \frac{N}{2} (\langle J_x^2 + J_y^2 \rangle \pm \langle J_z^2 \rangle). \] (A4)
In deriving the above equation, we have used the relation
\[ J_x^2 = J_x^2 - J_y^2 - i[J_x, J_y]. \] (A5)
Therefore, the smallest eigenvalue \( \lambda_{\min} \) of \( \Gamma \) is obtained as
\[ \lambda_{\min} = \min \left( \lambda_{-}, N(\Delta J_z)^2 + \langle J_z^2 \rangle \right), \] (A6)
where \( \lambda_{-} \) differs from the squeezing parameter \( \xi_1^2 \) given by Eq. (24) by only a multiplicative constant, as seen by comparing Eqs. (24) and (A6). From Eqs. (A6) and (20), one finds that the squeezing parameter \( \xi_2^2 \) is given by Eq. (26).

**APPENDIX B: SPIN-SQUEEZING PARAMETERS IN THE ONE-AXIS TWISTED STATE**

Here, we will use the Heisenberg picture to derive the relevant expectations and spin-squeezing parameters for the initial state [53, 54].

To determine the spin-squeezing parameter \( \xi_1^2 \) given by Eq. (30), one needs to know the expectation \( \langle \sigma_{1z} \rangle \), and correlations \( \langle \sigma_{1+} \sigma_{2-} \rangle \) and \( \langle \sigma_{1-} \sigma_{2+} \rangle \). We first consider the expectation \( \langle \sigma_{1z} \rangle \). For simplicity, we omit the subscript 0 in the following formulas.

1. **Expectation \( \langle \sigma_{1z} \rangle \)**

The evolution operator can be written as,
\[ U = \exp(-i\chi t J_x^2) = \exp \left( -i\theta \sum_{k \neq l} j_{kx} j_{lx} \right) \] (B1)
up to a trivial phase, where \( \theta = 2\chi t \) given by Eq. (4). From this form, the evolution of \( j_{1z} \) can be obtained as
\[ U^\dagger j_{1z} U = j_{1z} \cos[\theta j_{1x}^2] + j_{1y} \sin[\theta j_{1x}^2], \] (B2)
where
\[ j_{1x}^{(k)} = \sum_{i=0}^{N} j_{ix}. \] (B3)
Therefore, the expectations are
\[ \langle j_{1x} \rangle = -2^{-1} \langle 1^i | \cos[\theta j_{1x}^2] | 1^i \rangle \] (B4)
since \( \langle 1 | j_{iy} | 1 \rangle = 0 \). Here, \( 1^i = | 1 \rangle_2 \otimes \ldots \otimes | 1 \rangle_N \). So, one can find the following form for the expectation values
\[ \langle 1^i | \cos[\theta J_z] | 1 \rangle = \langle (1^i)e^{i\theta J_z}|1 \rangle + c.c. \rangle / 2 \]
\[ = \langle \Pi^{(N)}_{j=i=1} (| 1 \rangle e^{i\theta j_k} | 1 \rangle + c.c. \rangle / 2 \]
\[ = \cos(N(\theta')), \] (B5)
where \( \theta' = \theta / 2 \) and \( 1^i = | 1 \rangle \otimes N \).

By using Eqs. (B4) and (B5), one gets
\[ \langle \sigma_{z} \rangle = -\cos^{N-1}(\theta'). \] (B6)

2. **Correlation \( \langle \sigma_{1+} \sigma_{2-} \rangle \)**

Since the operator \( \sigma_{1z} \sigma_{2z} \) commutes with the unitary operator \( U \), we easily obtain
\[ \langle \sigma_{1z} \sigma_{2z} \rangle = 0. \] (B7)

We now compute the correlations \( \langle \sigma_{1z} \sigma_{2z} \rangle \). From the unitary operator,
\[ U^\dagger j_{1z} j_{2z} U \]
\[ = \left[ j_{1z} \cos[\theta j_{1x}^2] + j_{1y} \sin[\theta j_{1x}^2] \right] \times \left[ j_{2z} \cos[\theta (j_{kx} + j_{kz})] + j_{2y} \sin[\theta (j_{kx} + j_{kz})] \right] \]
\[ = \left[ j_{1z} \cos[\theta j_{2x}] \cos[\theta j_{2x}] - j_{1z} \sin[\theta j_{2x}] \sin[\theta j_{2x}] \right] \]
\[ \times \left[ j_{2z} \cos[\theta j_{1x}] \cos[\theta j_{1x}] + j_{2y} \cos[\theta j_{1x}] \sin[\theta j_{1x}] \right] \]
\[ + j_{2y} \sin[\theta j_{1x}] \cos[\theta j_{1x}] + j_{2y} \cos[\theta j_{1x}] \sin[\theta j_{1x}] \right] \]
\[ + j_{2y} \sin[\theta j_{1x}] \cos[\theta j_{1x}] + j_{2y} \cos[\theta j_{1x}] \sin[\theta j_{1x}] \right] \]
\[ + j_{2y} \sin[\theta j_{1x}] \cos[\theta j_{1x}] + j_{2y} \cos[\theta j_{1x}] \sin[\theta j_{1x}] \right] \]
\[ + j_{2y} \sin[\theta j_{1x}] \cos[\theta j_{1x}] + j_{2y} \cos[\theta j_{1x}] \sin[\theta j_{1x}] \right] \]
\[ + j_{2y} \sin[\theta j_{1x}] \cos[\theta j_{1x}] + j_{2y} \cos[\theta j_{1x}] \sin[\theta j_{1x}] \right] \]
Although there are 16 terms after expanding the above equation, only four terms survive when calculating \( \langle \sigma_{1z} \sigma_{2z} \rangle \). We then have
\[ \langle j_{1z} j_{2z} \rangle = \langle 1^i | j_{1z} j_{2z} | \cos^2(\theta/2) \cos^2(\theta j_{1x}^2) \rangle \]
\[ - j_{1z} J_{2x} j_{2y} | \sin(\theta) \sin^2(\theta j_{1x}^2) \rangle \]
\[ + 4 j_{1y} j_{1x} j_{2x} j_{2y} | \sin^2(\theta/2) \cos^2(\theta j_{1x}^2) \rangle \]
\[ - j_{1y} j_{1x} J_{2x} | \sin(\theta) \sin^2(\theta j_{1x}^2) \rangle | 1 \rangle \]
\[ = 4^{-1} \langle 1 \rangle | \cos^2(\theta j_{1x}^2) \rangle | 1 \rangle \]
\[ = 8^{-1} \langle 1 \rangle \left[ 1 + \cos(2\theta j_{1x}^2) \right] | 1 \rangle \]
\[ = 8^{-1} \left[ 1 + \cos(2N(\theta)) \right] | 1 \rangle, \] (B8)
where $|1\rangle = |1\rangle_1 \otimes \ldots \otimes |1\rangle_N$. The second equality in Eq. (B8) is due to the property $j_xj_y = -j_yj_x = i j_z/2$, and the last equality from Eq. (B5). Finally, from the above equation, one finds

$$\langle \sigma_{1x}\sigma_{2x} \rangle = 2^{-1} (1 + \cos N^{-2} \theta). \quad (B9)$$

Due to the relation $\langle \sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z} \rangle = 1$ for the initial state, the correlation $\langle \sigma_{1y}\sigma_{2y} \rangle$ is obtained from Eqs. (B7) and (B9) as

$$\langle \sigma_{1y}\sigma_{2y} \rangle = 2^{-1} (1 - \cos N^{-2} \theta). \quad (B10)$$

Substituting Eqs. (B7) and (B10) into the following relations

$$\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} = 2 (\sigma_{1+}\sigma_{2+} + \sigma_{1-}\sigma_{2-})$$

leads to one element of the two-spin reduced density matrix,

$$y_0 = \langle \sigma_{1+}\sigma_{2-} \rangle = 8^{-1} (1 - \cos N^{-2} \theta), \quad (B11)$$

where the relation $\langle \sigma_{1+}\sigma_{2-} \rangle = \langle \sigma_{1-}\sigma_{2+} \rangle$ is used due to the exchange symmetry.

3. Correlation $\langle \sigma_{1-}\sigma_{2-} \rangle$

To calculate the correlation $\langle \sigma_{1-}\sigma_{2-} \rangle$, we consider the following relations

$$\sigma_{1x}\sigma_{2x} - \sigma_{1y}\sigma_{2y} = 2 (\sigma_{1+}\sigma_{2-} - \sigma_{1-}\sigma_{2+}), \quad (B12)$$

$$i (\sigma_{1x}\sigma_{2y} + \sigma_{1y}\sigma_{2x}) = 2 (\sigma_{1+}\sigma_{2+} - \sigma_{1-}\sigma_{2-}), \quad (B13)$$

we need to know the expectations $\langle j_{1x}j_{2y} \rangle$. The evolution of $j_{1x}j_{2y}$ is given by

$$U^j_{1x}s_{1y}U = j_{1x} \left\{ \begin{array}{l} j_{2y} \cos \theta(j_{1x} + j_{1x}^{(3)}) \\ - j_{2x} \sin \theta(j_{1x} + j_{1x}^{(3)}) \end{array} \right\},$$

and the expectation is obtained as

$$\langle j_{1x}j_{2y} \rangle = 2^{-1} (1' | j_{1x} \sin \theta (j_{1x} + j_{1x}^{(3)}) | 1')$$

$$= (4i)^{-1} (1' | j_{1x} e^{i\theta j_{1x}} \Pi_{k=3} e^{i\theta j_{1x}} - j_{1x} e^{-i\theta j_{1x}} \Pi_{k=3} e^{-i\theta j_{1x}} | 1')$$

$$= (4i)^{-1} \cos N^{-2} \theta (\theta') (1 | j_{1x} e^{i\theta j_{1x}} - j_{1x} e^{-i\theta j_{1x}} | 1)$$

$$= 2^{-1} \cos N^{-2} \theta (\theta') | 1 | j_{1x} \sin \theta(j_{1x}^{(3)}) | 1)$$

$$= 4^{-1} \sin \theta' \cos N^{-2} \theta (\theta')$$

Here, $|1'\rangle = |1\rangle_1 \otimes |1\rangle_3 \otimes \ldots \otimes |1\rangle_N$, where $|1\rangle_N$ is absent. Moreover, $\langle j_{1y}j_{2x} \rangle = \langle j_{1x}j_{2y} \rangle$ due to the exchange symmetry, and thus,

$$\langle j_{1x}j_{2y} + j_{1y}j_{2x} \rangle = 2^{-1} \sin (\theta') \cos N^{-2} (\theta').$$

For the initial state (2), we obtain the following expectations [16, 47]

$$\langle \sigma_{1x}\sigma_{2y} + \sigma_{1y}\sigma_{2x} \rangle = 2 \sin (\theta') \cos N^{-2} (\theta'). \quad (B14)$$

The combination of Eqs. (B7), (B10), (B12), (B13), and (B14) leads to the correlation

$$u_0 = \langle \sigma_{1-}\sigma_{2-} \rangle = -8^{-1} (1 - \cos N^{-2} \theta)$$

$$-i 2^{-1} \sin (\theta') \cos N^{-2} (\theta'). \quad (B15)$$

Substituting Eqs. (B11) and (B15) to Eq. (30) leads to the expression of the squeezing parameter $\xi_{1z}$ given by Eq. (43).

APPENDIX C: PROOF OF $C_{zz}(0) \geq 0$

To prove this, we will not use this specific function of the initial twist angle $\theta$ as given by Eq. (46), but use the positivity of the reduced density matrix (37). We first notice an identity

$$C_{zz} = 4(v_+ v_- - w^2),$$

which results from Eqs. (38) and (39). This is a key step. Also there exists another identity

$$w_0 = y_0 \quad (C1)$$

as $\langle \sigma_1 \cdot \sigma_2 \rangle_0 = 1$. From the positivity of the reduced density matrix (37), one has

$$v_0 + v_0 - \geq |u_0|^2 \geq w_0^2,$$

where the second inequality follows from Eq. (39) and the last equality results from Eq. (C1). This completes the proof.

APPENDIX D: DERIVATION OF THE EVOLUTION OF THE CORRELATIONS AND EXPECTATIONS UNDER DECOHERENCE

For an arbitrary matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

from the Kraus operators (12) for the ADC, it is straightforward to find

$$E(A) = \begin{pmatrix} sa & \sqrt{sb} \\ \sqrt{sc} & d + pa \end{pmatrix},$$

$$E^j(A) = \begin{pmatrix} sa + pd \sqrt{sb} \\ \sqrt{sc} & d \end{pmatrix}.$$

The above equations imply that

$$E^j(\sigma_\mu) = \sqrt{s} \sigma_\mu \quad \text{for } \mu = x, y,$$

$$E^j(\sigma_z) = s \sigma_z - p.$$

As we considered independent and identical decoherence channels acting separately on each spin, the evolution
correlations and expectations in Eqs. (51b), (51c), and (51d) are obtained directly from the above equations.

From the Kraus operators (14), the evolution of the matrix $A$ under the PDC is obtained as

$$
\mathcal{E}(A) = \mathcal{E}^\dagger(A) = \begin{pmatrix}
a & sb \\
sc & d
\end{pmatrix},
$$

from which one finds

$$
\mathcal{E}^\dagger(\sigma_\mu) = s\sigma_\mu \quad \text{for} \quad \mu = x, y
$$

$$
\mathcal{E}^\dagger(\sigma_z) = \sigma_z.
$$

So expectations $\langle \sigma_z^{\otimes n} \rangle$ are unchanged and Eq. (65) is obtained.

From the Kraus operators (16) of the DPC, the evolution of the matrix $A$ is given by

$$
\mathcal{E}(A) = \mathcal{E}^\dagger(A) = \begin{pmatrix}
as + \frac{\zeta}{2}(a + d) & sb \\
sc & ds + \frac{\zeta}{2}(a + d)
\end{pmatrix}
$$

from which one finds

$$
\mathcal{E}^\dagger(\sigma_\alpha) = s\sigma_\alpha \quad \text{for} \quad \alpha \in \{x, y, z\}.
$$

Then, Eq. (77) is obtained.

### APPENDIX E: SIMPLIFICATION OF THE CONCURRENCE

For our three kinds of decoherence channels, the concurrence (42) can be simplified, which is given by

$$
C = \max \left\{ 0, 2(\langle |u| - w \rangle \cdot 2(y - \sqrt{v_+v_-})) \right\}
$$

$$
= \max \left\{ 0, 2(\langle |u| - w \rangle) \right\}. \quad (E1)
$$

If one can prove

$$
|u| - y \geq 0, \quad (E2)
$$

$$
w - \sqrt{v_+v_-} \leq 0, \quad (E3)
$$

then we obtain the simplified form. The last inequality can be replaced by

$$
w^2 - v_+v_- \leq 0 \quad (E4)
$$
as $w$ and $v_+v_-$ are real.

We first consider the ADC channel. From Eqs. (51b), (51c), and (53), one obtains

$$
|u| - y = s(|u_0| - y_0) \geq 0, \quad (E5)
$$

$$
w^2 - v_+v_- = -\frac{1}{4}c_{zz} = -\frac{1}{4}s^2c_{zz}(0) \leq 0. \quad (E6)
$$

where the inequalities result from Eqs. (43) and (46), respectively. So, the inequality (E4) follows.

For the PDC, from Eq. (65) and fact that $\langle \sigma_z^{\otimes n} \rangle$ is unchanged under decoherence, the concurrence can also be simplified due to the following properties:

$$
|u| - y = s^2(|u_0| - y_0) \geq 0,
$$

$$
w^2 - v_+v_- = -\frac{1}{4}c_{zz}(0) \leq 0. \quad (E8)
$$

For the DPC, from Eqs. (65) and (77), one has

$$
|u| - y = s^2(|u_0| - y_0) \geq 0, \quad (E7)
$$

$$
w^2 - v_+v_- = -\frac{1}{4}s^2c_{zz}(0) \leq 0. \quad (E8)
$$

So, again, the concurrence can be simplified. This completes the proof.

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