JOURNAL OF MODERN OPTICS, 1990, VOL. 37, NO. 12, 1935–1945

Quantum fluctuations in the Stokes parameters of light propagating in a Kerr medium

R. TANAŚ and S. KIELICH
Nonlinear Optics Division, Institute of Physics,
Adam Mickiewicz University, 60-780 Poznań, Poland

(*Received 11 August 1989; revision received 1 March 1990*)

Abstract. The quantum theory of light propagation in a nonlinear Kerr medium is applied to calculate the Stokes parameters and their variances in the process of light propagation. Exact quantum formulae are derived for the expectation values of the Stokes operators and thus for the azimuth θ and ellipticity η of the beam. The role of quantum fluctuations in light polarization characteristics is discussed. The periodic behaviour of quantum evolution of the light polarization is revealed explicitly. It is shown that the degree of polarization is diminished at early stages of each period of the evolution but then reverts to its initial state of complete polarization at the end of the period. The variances of the Stokes parameters are also periodic and intensity-dependent; however, they never fall below their coherent state values.

1. Introduction

It is a well known fact that an isotropic medium becomes birefringent when subjected to a strong optical field. The optical Kerr effect observed by Mayer and Gires [1] as well as the self-induced ellipse rotation observed by Maker *et al.* [2] belong to the earliest nonlinear optical phenomena observed experimentally. Nowadays, optically induced birefringence is a standard subject of textbooks on nonlinear optics [3, 4]. To understand phenomena like the optical Kerr effect and self-induced ellipse rotation there is no need for field quantization. On the other hand, it has been shown by us [5] that, if a strong optical field propagating through a nonlinear Kerr medium is treated as a quantum field, some new phenomena appear. For instance, the field propagating in such a medium can squeeze its own quantum fluctuations. We refer to this effect as self-squeezing. Such self-squeezed light can be to some extent controlled by means of a static magnetic field [6] and can serve, in other nonlinear processes, to produce for instance the third [7] or the second [8] optical harmonic. Recently, Kerr media have also been considered as suitable candidates for performing quantum non-demolition measurements [9, 10]. So there is growing interest in revealing those aspects of nonlinear propagation which are directly related to the quantum properties of the field.

Changes in the polarization state of light propagating through a nonlinear medium can be effectively described in terms of the Stokes parameters. The Stokes parameters, which are real numbers in the classical description of the field, become Hermitian operators in a quantum description. On having defined the Stokes operators, which are quantum mechanical observables, one is naturally led to address the problem of quantum fluctuations in these quantities as well as quantum field effects on the polarization state of fields propagating in a nonlinear Kerr medium. In this paper we discuss quantum fluctuations in the Stokes parameters of strong light propagating in an isotropic nonlinear medium. We show that the degree of light polarization can be degraded due to quantum fluctuations of the field. The results are particularly instructive because the problem of light propagation in an isotropic nonlinear medium admits exact operator solutions.

2. The Stokes operators and the polarization of the light beam

To describe quantized fields it is convenient to split the field into positive and negative frequency parts:

$$E_i(\mathbf{r}, t) = E_i^{(+)}(\mathbf{r}, t) + E_i^{(-)}(\mathbf{r}, t), \tag{1}$$

where i denotes a polarization component of the field. The next step is to perform a mode decomposition of this field. For plane-wave decomposition of the free field propagating in a medium with refractive index $n(\omega)$ one can write

$$E_i^{(+)}(\mathbf{r}, t) = \sum_{\mathbf{k},\lambda} \mathrm{i}\left(\frac{2\pi\hbar\omega_k}{n^2(\omega)V}\right)^{1/2} e_{ki}^{(\lambda)} a_{k\lambda} \exp\left[-\mathrm{i}(\omega_k t - \mathbf{k}\cdot\mathbf{r})\right], \tag{2}$$

where $e_{ki}^{(\lambda)}$ is the ith component of the polarization vector associated with the polarization state λ and the propagation vector $\mathbf{k}$, and V is the quantization volume. For the quantized field, $a_{k\lambda}$ is the annihilation operator of the photon with propagation vector $\mathbf{k}$ and polarization λ satisfying the commutation rules

$$[a_{k\lambda}, a_{k'\lambda'}^{+}] = \delta_{kk'}\delta_{\lambda\lambda'}. \tag{3}$$

The polarization vectors satisfy the orthogonality conditions

$$\sum_i e_{ki}^{(\lambda)*} e_{ki}^{(\lambda')} = \delta_{\lambda\lambda'}, \qquad \sum_i e_{ki}^{(\lambda)} k_i = 0 \tag{4}$$

For a monochromatic field of frequency ω propagating along the z axis of the laboratory reference frame, we can drop the index k in the above notation and write

$$E_i^{(+)}(z, t) = \mathrm{i}\left(\frac{2\pi\hbar\omega}{n^2(\omega)V}\right)^{1/2} \exp\left[-\mathrm{i}(\omega t - kz)\right] \sum_{\lambda\approx 1,2} e_i^{(\lambda)} a_\lambda, \tag{5}$$

with $k = n(\omega)\omega/c$. In equation (5) the sum over the two mutually orthogonal polarizations of the field still remains, so that we have a two-mode description of the field. If the field is a coherent superposition of these two modes however, the two-mode description can be replaced by one mode of a generally elliptically polarized field:

$$e_i a = e_i^{(1)} a_1 + e_i^{(2)} a_2, \tag{6}$$

where $e_i^{(1)}$ and $e_i^{(2)}$ are the ith components of the orthogonal unit polarization vectors $\hat{\mathbf{e}}^{(1)}$ and $\hat{\mathbf{e}}^{(2)}$ of the modes a_1 and a_2, and e_i is the ith component of the polarization vector $\hat{\mathbf{e}}$ of the mode a. The relation (6) can also be considered in the reverse sense as a decomposition of initially elliptically polarized light into two orthogonal modes. Applying the orthogonality condition (4) for the polarization vectors, we obtain the formula

$$a = e_1^* a_1 + e_2^* a_2, \tag{7}$$

where

$$e_1^* = \hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}^{(1)}, \qquad e_2^* = \hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}^{(2)}.$$

So far the decomposition (6) (or, equivalently, (7)) is quite general and can be further specified either for two modes with mutually perpendicular linear polarizations or for right- and left-circularly polarized modes.

In the case of a Cartesian basis, the unit polarization vectors are

$$\hat{\mathbf{e}}^{(1)}=\hat{\mathbf{x}}, \qquad \hat{\mathbf{e}}^{(2)}=\hat{\mathbf{y}},$$

whereas in that of a circular basis we have

$$\hat{\mathbf{e}}^{(1)}=\hat{\mathbf{e}}^{(+)}=(\hat{\mathbf{x}}+\mathrm{i}\hat{\mathbf{y}})/\sqrt{2}, \qquad \hat{\mathbf{e}}^{(2)}=\hat{\mathbf{e}}^{(-)}=(\hat{\mathbf{x}}-\mathrm{i}\hat{\mathbf{y}})/\sqrt{2}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit vectors in the x and y directions, respectively. The unit vector $\hat{\mathbf{e}}$ of the elliptically polarized light can be written in either a Cartesian or a circular basis as

$$\hat{\mathbf{e}}=e_x\hat{\mathbf{x}}+e_y\hat{\mathbf{y}}=e_+\hat{\mathbf{e}}^{(+)}+e_-\hat{\mathbf{e}}^{(-)}, \tag{8}$$

with [11]

$$e_x=\cos\eta\cos\theta-\mathrm{i}\sin\eta\sin\theta, \qquad e_y=\cos\eta\sin\theta+\mathrm{i}\sin\eta\cos\theta \tag{9}$$

and

$$e_\pm=(1/\sqrt{2})(e_x\mp\mathrm{i}e_y)=(1/\sqrt{2})(\cos\eta\pm\sin\eta)\exp(\mp\mathrm{i}\theta). \tag{10}$$

In equations (9) and (10) θ is the azimuth of the polarization ellipse denoting the angle between the major axis of the polarization ellipse and the x axis measured positive from the positive x axis towards the positive y axis, and η is the ellipticity parameter

$$-\frac{\pi}{4}\leqslant\eta\leqslant\frac{\pi}{4},$$

where $\tan\eta$ describes the ratio of the semi-minor axis and semi-major axis of the polarization ellipse and the sign defines its handedness (plus indicates right-handed polarization on the helicity convention).

According to (7) the annihilation operator of the elliptically polarized field can be written as

$$a=e_x^*a_x+e_y^*a_y=e_+^*a_++e_-^*a_-, \tag{11}$$

where e_x, e_y and $e_\pm$ are given by (9) and (10), and

$$a_\pm=(1/\sqrt{2})(a_x\mp\mathrm{i}a_y). \tag{12}$$

Hence, the annihilation operator a of the elliptically polarized light is a superposition of two orthogonal modes in either a Cartesian or a circular basis. Defining a coherent state of the field with respect to the operator a as follows:

$$a|\alpha\rangle=\alpha|\alpha\rangle, \tag{13}$$

we have simultaneously

$$|\alpha\rangle=|\alpha_x\rangle|\alpha_y\rangle=|\alpha_+\rangle|\alpha_-\rangle, \tag{14}$$

where $|\alpha_x\rangle$, $|\alpha_y\rangle$ and $|\alpha_+\rangle$, $|\alpha_-\rangle$ are the corresponding coherent states defined with respect to the operators a_x, a_y and a_+, a_-. By (11), (13) and (14) one can write

$$\alpha=e_x^*\alpha_x+e_y^*\alpha_y=e_+^*\alpha_++e_-^*\alpha_-, \tag{15}$$

and due to the orthogonality relations

$$e_x^* e_x + e_y^* e_y = 1, \qquad e_+^* e_+ + e_-^* e_- = 1$$

one obtains

$$\alpha_x = e_x \alpha, \qquad \alpha_y = e_y \alpha, \tag{16}$$

$$\alpha_\pm = e_\pm \alpha, \tag{17}$$

where e_x, e_y and $e_\pm$ are given by (9) and (10), and

$$|\alpha_x|^2 + |\alpha_y|^2 = |\alpha_+|^2 + |\alpha_-|^2 = |\alpha|^2.$$

A Cartesian or a circular basis can be used alternatively to describe the propagation of elliptically polarized light in a nonlinear Kerr medium. In isotropic media however, the circular basis is more advantageous as will become clear later on.

The two-mode description of the field allows us to introduce the following Hermitian Stokes operators [12]:

$$\left.\begin{aligned} S_0 &= a_x^+ a_x + a_y^+ a_y = a_+^+ a_+ + a_-^+ a_- \\ S_1 &= a_x^+ a_x - a_y^+ a_y = a_+^+ a_- + a_-^+ a_+ \\ S_2 &= a_x^+ a_y + a_y^+ a_x = -\mathrm{i}(a_+^+ a_- - a_-^+ a_+ \\ S_3 &= -\mathrm{i}(a_x^+ a_y - a_y^+ a_x) = a_+^+ a_+ - a_-^+ a_- \end{aligned}\right\} \tag{18}$$

where the annihilation and creation operators of the particular modes satisfy the commutation relations (3). It is easy to check that the Stokes operators themselves satisfy the commutation relations:

$$\begin{aligned} &[S_1, S_2] = 2\mathrm{i}S_3 \quad \text{(and cyclic interchange of indices)}, \\ &[S_i, S_0] = 0, \quad i = 1, 2, 3. \end{aligned} \tag{19}$$

Moreover, we have

$$S_1^2 + S_2^2 + S_3^2 = S_0(S_0 + 2). \tag{20}$$

The quantum mechanical expectation values of the Stokes operators (18) are the Stokes parameters describing the polarization of the light beam. The parameters of the polarization ellipse are given by

$$\left.\begin{aligned} \tan 2\theta &= \frac{\langle S_2 \rangle}{\langle S_1 \rangle} \\ \tan 2\eta &= \langle S_3 \rangle / (\langle S_1 \rangle^2 + \langle S_2 \rangle^2)^{1/2} \end{aligned}\right\} \tag{21}$$

One can also define the degree of polarization as

$$P = (\langle S_1 \rangle^2 + \langle S_2 \rangle^2 + \langle S_3 \rangle^2)^{1/2} / \langle S_0 \rangle. \tag{22}$$

When the state of the field is the coherent state given by (13) and (14), the quantum mechanical expectation values are easily calculated and, using equations (10) and (17), one easily recovers the relations (21) and obtains the value $P = 1$ for the degree of polarization. This means that the coherent state of the field corresponds to a classical, fully polarized field. However, the non-commutability of the Stokes operators puts well known limits on measurements of the physical quantities represented by these operators. For example, according to the commutation relations (19), we have the following Heisenberg uncertainty relation:

$$(\langle(\Delta S_1)^2\rangle\langle(\Delta S_2)^2\rangle)^{1/2} \geqslant |\langle S_3\rangle|. \tag{23}$$

It is also well known that the state of a field propagating in a nonlinear Kerr medium does not remain coherent; in fact, it can be self-squeezed [5]. This poses the question of quantum fluctuations of the Stokes parameters and the polarization characteristics of the light propagating in such a medium. Some of these problems are discussed in this paper.

3. Equations of motion for a light field propagating in a Kerr medium

In a classical description, the optical Kerr effect and self-induced ellipse rotation are related to the third-order nonlinear polarization of the medium. When a field at frequency ω propagates through a Kerr medium, the third-order nonlinear polarization in the electric-dipole approximation can be written as follows [3, 4]:

$$P_i^{(+)}(\omega) = 3\sum_{jkl} \chi_{ijkl}(-\omega, -\omega, \omega, \omega) E_j^{(-)}(\omega) E_k^{(+)}(\omega) E_l^{(+)}(\omega), \tag{24}$$

where $\chi_{ijkl}(-\omega, -\omega, \omega, \omega)$ is the third-order nonlinear susceptibility fourth-rank tensor of the medium and the electromagnetic field is decomposed into positive- and negative-frequency parts, as in (1); the field amplitudes $E_i^{(\pm)}(\omega)$ are however, classical quantities. Such a decomposition of the field gives the following expression for the intensity of the beam at the frequency ω:

$$I(\omega) = \frac{cn(\omega)}{2\pi} \sum_i E_i^{(-)}(\omega) E_i^{(+)}(\omega), \tag{25}$$

where $n(\omega)$ is the refractive index of the medium at frequency ω determined by its linear (first-order) polarization.

For an isotropic medium with a centre of inversion the nonlinear susceptibility tensor $\chi_{ijkl}(\omega) = \chi_{ijkl}(-\omega, -\omega, \omega, \omega)$ takes the form [3]

$$\chi_{ijkl}(\omega) = \chi_{xxyy}(\omega)\delta_{ij}\delta_{kl} + \chi_{xyxy}(\omega)\delta_{ik}\delta_{jl} + \chi_{xyyx}(\omega)\delta_{il}\delta_{jk}, \tag{26}$$

with the additional relation

$$\chi_{xxxx}(\omega) = \chi_{yyyy}(\omega) = \chi_{xxyy}(\omega) + \chi_{xyxy}(\omega) + \chi_{xyyx}(\omega). \tag{27}$$

With regard to the permutation symmetry of the tensor χ_{ijkl} in its first and second pairs of indices we have, moreover, $\chi_{xyxy}(\omega) = \chi_{xyyx}(\omega)$. The light beam is assumed to propagate along the z axis of the laboratory reference frame.

On insertion of the polarization (24) into the Maxwell equations and applying the slowly varying amplitude approximation, one obtains the following equation for the amplitudes of the field [4]:

$$\frac{\mathrm{d}E_i^{(+)}(\omega)}{\mathrm{d}z} = \frac{\mathrm{i}2\pi\omega}{n(\omega)c} P_i^{(+)}(\omega), \tag{28}$$

where the slowly-varying amplitudes $E_i^{(+)}(\omega)$ are assumed to be dependent on z.

By (28) we have for example,

$$\frac{\mathrm{d}E_x^{(+)}(\omega)}{\mathrm{d}z}=\frac{\mathrm{i}2\pi\omega}{n(\omega)c}\{3\chi_{xxyy}(\omega)E_x^{(-)}(\omega)[E_x^{(+)}(\omega)^2+E_y^{(+)}(\omega)^2]$$
$$+6\chi_{xyxy}(\omega)[E_x^{(-)}(\omega)E_x^{(+)}(\omega)+E_y^{(-)}(\omega)E_y^{(+)}(\omega)]E_x^{(+)}(\omega)\} \qquad (29)$$

If the circular basis is introduced, which is the natural basis for isotropic media, with the circular components of the field

$$E_\pm^{(+)}(\omega)=\frac{1}{\sqrt{2}}[E_x^{(+)}(\omega)\mp\mathrm{i}E_y^{(+)}(\omega)], \qquad (30)$$

the nonlinear polarization becomes

$$P_\pm^{(+)}(\omega)=6\chi_{xyxy}(\omega)|E_\pm^{(+)}(\omega)|^2E_\pm^{(+)}(\omega)$$
$$+6[\chi_{xxyy}(\omega)+\chi_{xyxy}(\omega)]|E_\mp^{(+)}(\omega)|^2E_\pm^{(+)}(\omega), \qquad (31)$$

and the equations of motion for the field amplitudes are [8]

$$\frac{\mathrm{d}E_\pm^{(+)}(\omega)}{\mathrm{d}z}=\frac{\mathrm{i}2\pi\omega}{n(\omega)c}P_\pm^{(+)}(\omega)$$
$$=\frac{\mathrm{i}2\pi\omega}{n(\omega)c}6\{\chi_{xyxy}(\omega)|E_\pm^{(+)}(\omega)|^2$$
$$+[\chi_{xxyy}(\omega)+\chi_{xyxy}(\omega)]|E_\mp^{(+)}(\omega)|^2\}E_\pm^{(+)}(\omega). \qquad (32)$$

Equations (32) immediately visualize the advantage of the circular basis over the Cartesian basis used in equation (29). One easily checks that $\mathrm{d}/\mathrm{d}z\,|E_\pm^{(+)}(\omega)|^2=0$, that is the intensities $|E_\pm^{(+)}(\omega)|^2$ of both circular components are constants of motion. This is not the case for the Cartesian components, except for the case of linear polarization. Since the intensities $|E_\pm^{(+)}(\omega)|^2$ do not depend on z, equation (32) has the following simple exponential solution [13]:

$$E_\pm^{(+)}(\omega;z)=\exp(\mathrm{i}\Phi_\pm z)E_\pm^{(+)}(\omega;z=0), \qquad (33)$$

where

$$\Phi_\pm=\frac{2\pi\omega}{n(\omega)c}6\{\chi_{xyxy}(\omega)|E_\pm^{(+)}(\omega)|^2+[\chi_{xxyy}(\omega)+\chi_{xyxy}(\omega)]|E_\mp^{(+)}(\omega)|^2\}, \qquad (34)$$

determines the light-intensity-dependent phase of the field (self-phase-modulation or intensity-dependent refractive index). These classical nonlinear effects are well known [3, 4] and will not be discussed here. We are interested in effects related to the quantum properties of the light field. So we have to replace the classical field by the field operators which can be done according to (2). However, we shall also need quantum equations of motion for the field operators. Such equations (the Heisenberg equations of motion) can be obtained from the following effective interaction Hamiltonian [5]:

$$H_I=\frac{\hbar}{2}[\kappa_1(a_+^{+2}a_+^2+a_-^{+2}a_-^2)+2\kappa_2a_+^+a_-^+a_-a_+]$$
$$=\frac{\hbar}{2}\left[\kappa_1(a_x^{+2}a_x^2+a_y^{+2}a_y^2+2a_x^+a_y^+a_ya_x)+\frac{\kappa_2-\kappa_1}{2}(a_x^{+2}+a_y^{+2})(a_x^2+a_y^2)\right], \qquad (35)$$

where the nonlinear coupling constants κ_1 and κ_2 are real and are given by

$$\left.\begin{aligned}\kappa_1&=\frac{V}{\hbar}\left[\frac{2\pi\hbar\omega}{n^2(\omega)V}\right]^2 6\chi_{xyxy}(\omega),\\ \kappa_2&=\frac{V}{\hbar}\left[\frac{2\pi\hbar\omega}{n^2(\omega)V}\right]^2 6[\chi_{xxyy}(\omega)+\chi_{xyxy}(\omega)]\end{aligned}\right\} \tag{36}$$

The linear part of the interaction has already been incorporated into the refractive index of the medium. The interaction Hamiltonian (35) is written in two equivalent forms with the use of Cartesian as well as circular basis field operators. The annihilation operators in the two bases are related by equation (12) and satisfy the commutation relations (3).

Using the interaction Hamiltonian (35) and the commutation rules (3) one easily arrives at the Heisenberg equations of motion describing the time-evolution of the field operators. On replacing t by $-n(\omega)z/c$ (we are dealing with propagation rather than a field in a cavity) we obtain the following equations:

$$\frac{\mathrm{d}a_\pm(z)}{\mathrm{d}z}=\mathrm{i}\frac{n(\omega)}{c}[\kappa_1 a_\pm^+(z)a_\pm(z)+\kappa_2 a_\mp^+(z)a_\mp(z)]a_\pm(z). \tag{37}$$

The equations for the creation operators are Hermitian conjugates of (37). When the relation

$$E_\pm^{(+)}(\omega)=\mathrm{i}\left(\frac{2\pi\hbar\omega}{n^2(\omega)V}\right)^{1/2}a_\pm \tag{38}$$

between the annihilation operators and the corresponding field operators is applied, equation (37) reverts to the form of equation (32), but now the field is quantized. Quantum-classical correspondence is thus maintained.

An alternative version of the equations of motion, in a Cartesian basis, could be obtained from (35), but it has a much more complicated structure and would not be very useful here. Having solved the equations in a circular basis one can always apply the relation (12) to get the solutions in a Cartesian basis.

Since the numbers of photons in the two circular modes $a_+^+a_+$ and $a_-^+a_-$ are constants of motion (they commute with the Hamiltonian (35)), equation (37) has the simple exponential solution

$$a_\pm(z)=\exp\{\mathrm{i}z[\varepsilon a_\pm^+(0)a_\pm(0)+\delta a_\mp^+(0)a_\mp(0)]\}a_\pm(0), \tag{39}$$

where we have introduced the notation

$$\varepsilon=\frac{n(\omega)}{c}\kappa_1,\qquad \delta=\frac{n(\omega)}{c}\kappa_2. \tag{40}$$

The solutions (39) are exact operator solutions for the field operators of light propagating through an isotropic nonlinear medium which have been used for calculations of quantum effects such as photon anti-bunching [14] and squeezing [5]. In this paper we will use the solutions (39) to discuss the problem of quantum fluctuations in the Stokes parameters, as well as the polarization characteristics of light propagating in a Kerr medium.

4. Quantum fluctuations in the Stokes parameters of light propagating in a Kerr medium

If the solutions (39) are inserted into equations (18) defining the Stokes operators, exact operator solutions are obtained for the evolution of the Stokes operators of light propagating in a Kerr medium. The Stokes parameters defining the polarization characteristics of the field are the expectation values of the Stokes operators taken in the initial state of the field. Such expectation values are easy to calculate and at the same time most interesting if the initial state is a coherent state of (generally) an elliptically polarized field defined by equations (13)–(17). The results are the following:

$$\begin{aligned}
\langle\alpha|S_0(z)|\alpha\rangle &= \langle\alpha|a_+^\dagger(z)a_+(z)|\alpha\rangle + \langle\alpha|a_-^\dagger(z)a_-(z)|\alpha\rangle = \langle\alpha|S_0(0)|\alpha\rangle = |\alpha|^2\\
\langle\alpha|S_1(z)|\alpha\rangle &= 2\,\mathrm{Re}\,\{\alpha_+^*\alpha_-\exp[(\exp -iz(\varepsilon-\delta)-1)|\alpha_+|^2 + (\exp iz(\varepsilon-\delta)-1)|\alpha_-|^2]\},\\
\langle\alpha|S_2(z)|\alpha\rangle &= 2\,\mathrm{Im}\,\{\alpha_+^*\alpha_-\exp[(\exp -iz(\varepsilon-\delta)-1)|\alpha_+|^2 + (\exp iz(\varepsilon-\delta)-1)|\alpha_-|^2]\},\\
\langle\alpha|S_3(z)|\alpha\rangle &= |\alpha_+|^2 - |\alpha_-|^2,
\end{aligned} \tag{41}$$

where the $\alpha_\pm$ are defined by (17). Since $a_+^\dagger a_+$ and $a_-^\dagger a_-$ are constants of motion, the Stokes parameters $\langle\alpha|S_0(z)|\alpha\rangle$ and $\langle\alpha|S_3(z)|\alpha\rangle$ do not change during the propagation process of light in isotropic media. So, only the parameters $\langle\alpha|S_1(z)|\alpha\rangle$ and $\langle\alpha|S_2(z)|\alpha\rangle$ will change as the light propagates. The internal exponentials that appear in the evolution (41) are the result of the quantum treatment of the field. Were the field classical, the exponentials would have never appeared [15]. According to (10) and (17), we have

$$\begin{aligned}
\langle\alpha|S_1(z)|\alpha\rangle &= |\alpha|^2\cos 2\eta\exp\{\{\cos[z(\varepsilon-\delta)]-1\}|\alpha|^2\}\cos\{2\theta-|\alpha|^2\sin 2\eta\sin[z(\varepsilon-\delta)]\},\\
\langle\alpha|S_2(z)|\alpha\rangle &= |\alpha|^2\cos 2\eta\exp\{\{\cos[z(\varepsilon-\delta)]-1\}|\alpha|^2\}\sin\{2\theta-|\alpha|^2\sin 2\eta\sin[z(\varepsilon-\delta)]\},
\end{aligned} \tag{42}$$

where $|\alpha|^2$ is the mean number of photons of the incoming beam whereas θ and η are the azimuth and ellipticity of the polarization ellipse. When the solutions (42) are inserted into equations (21) the parameters of the polarization ellipse of light that has traversed a path z in the Kerr medium are obtained as

$$\tan 2\theta(z) = \tan\{2\theta-|\alpha|^2\sin 2\eta\sin[z(\varepsilon-\delta)]\}, \tag{43}$$

$$\tan 2\eta(z) = \exp\{-\{\cos[z(\varepsilon-\delta)]-1\}|\alpha|^2\}\tan 2\eta. \tag{44}$$

These are exact solutions taking into account the quantum properties of the field. Classical solutions are obtained on replacing $\sin[z(\varepsilon-\delta)]$ by $z(\varepsilon-\delta)$ in (43) and dropping the exponential in (44). Thus, classical solutions are obtained when only linear terms in $z(\varepsilon-\delta)$ are retained in the series expansions of $\sin[z(\varepsilon-\delta)]$ and $\cos[z(\varepsilon-\delta)]$. This means that as long as $z(\varepsilon-\delta)$ is small the classical solutions are valid. The classical solutions for the Stokes parameters of light propagating in isotropic nonlinear media subjected to a d.c. electric field have been obtained by Sala [16]. It is seen from (44) that, classically, the ellipticity of the beam does not change whereas its azimuth does change whenever the initial ellipticity η differs from zero. This is the well known effect of self-ellipse rotation [2–4]. The quantum solutions are periodic in $z(\varepsilon-\delta)$, and for $z(\varepsilon-\delta)=2\pi$ the initial values of θ and η are recovered. If $z(\varepsilon-\delta)$ is not small the polarization of the field is essentially effected by the quantum fluctuations of the field. However, if the initial field is linearly polarized ($\eta=0$) both θ

and η remain unchanged even in the quantum case. This result can seem a bit surprising if we remember that in a Cartesian basis the number of photons $a_x^+ a_x$ and $a_y^+ a_y$ do not commute with the interaction Hamiltonian (35) meaning that linear polarization is not preserved in the course of propagation. To resolve this contradiction we have to calculate the degree of polarization defined by (22). The result is

$$P^2(z) = 1 - \cos^2 2\eta\{1 - \exp\{2\{\cos[z(\varepsilon - \delta)] - 1\}|\alpha|^2\}\}. \tag{45}$$

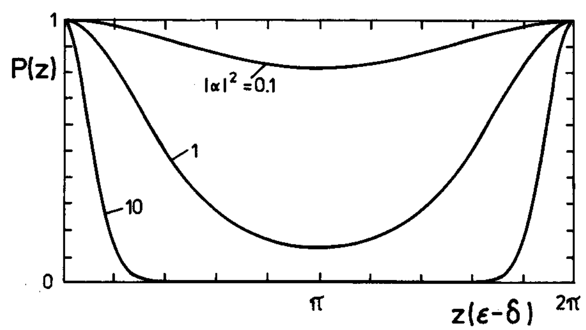
In the case of linear polarization ($\eta = 0$) we have (see also [17])

$$P(z) = \exp\{\{\cos[z(\varepsilon - \delta)] - 1\}|\alpha|^2\}. \tag{46}$$

It is now clear that the degree of polarization becomes smaller than unity due to quantum fluctuations of the field. Classically, this quantity remains unity all the time. We can say that the polarization of the initially completely polarized light is degraded as a result of the quantum fluctuations, and the initially linear polarization of the beam preserves its direction of polarization in that part of the field that is polarized whereas part of the incoming intensity becomes unpolarized (isotropic). An interesting feature of this quantum evolution, as is seen from (45) and (46), resides in the periodicity of the degree of polarization. This means that after initial degradation the polarization of the field returns to its initial state of complete polarization. This is illustrated in the figure, where the degree of polarization P is plotted against the dimensionless length of the medium $z(\varepsilon - \delta)$ for several values of the mean number of photons $|\alpha|^2$. Since, according to the operator relation (20), the square of the 'total spin' is conserved, one can easily calculate the variance of the 'total spin' as

$$\begin{aligned}\langle \mathbf{s}^2 \rangle - \langle \mathbf{s} \rangle^2 &= \langle S_1^2 \rangle + \langle S_2^2 \rangle + \langle S_3^2 \rangle - (\langle S_1 \rangle^2 + \langle S_2 \rangle^2 + \langle S_3 \rangle^2) \\ &= |\alpha|^2 \cos^2 2\eta\{1 - \exp\{2\{\cos[z(\varepsilon - \delta)] - 1\}|\alpha|^2\}\} + 3|\alpha|^2 \\ &= 3|\alpha|^2 + |\alpha|^2[1 - P^2(z)]. \end{aligned} \tag{47}$$

The lowest level of this variance is reached when the field becomes completely polarized. According to the uncertainty relation (23), the two Stokes parameters $\langle S_1 \rangle$ and $\langle S_2 \rangle$ cannot be measured simultaneously with high precision. Using the solutions (39) and assuming that the initial state of the field is the coherent state



The degree of polarization $P(z)$ plotted against $z(\varepsilon - \delta)$ for the mean number of photons $|\alpha|^2$ equal to 0·1, and 10.

defined by (13) and (14) the uncertainties $\langle\alpha|(\Delta S_1)^2|\alpha\rangle$ and $\langle\alpha|(\Delta S_2)^2|\alpha\rangle$ can be calculated explicitly, giving the results

$$\begin{aligned}\langle\alpha|(\Delta S_{1,2})^2|\alpha\rangle &= \langle\alpha|S_{1,2}^2|\alpha\rangle - \langle\alpha|S_{1,2}|\alpha\rangle^2 \\ &= \frac{|\alpha|^4}{2}\cos^2 2\eta\,\{\pm[\exp(\{\cos[2z(\varepsilon-\delta)]-1\}|\alpha|^2) \\ &\quad \times\cos\{4\theta-|\alpha|^2\sin 2\eta\sin[2z(\varepsilon-\delta)]\} \\ &\quad -\exp(2\{\cos[z(\varepsilon-\delta)]-1\}|\alpha|^2) \\ &\qquad \cos\{4\theta-2|\alpha|^2\sin 2\eta\sin[z(\varepsilon-\delta)]\}] \\ &\quad +1-\exp(2\{\cos[z(\varepsilon-\delta)]-1\}|\alpha|^2)\}+|\alpha|^2. \end{aligned} \tag{48}$$

Both variances are equal to $|\alpha|^2$ (the mean number of photons) when the light is circularly polarized ($\eta=\pm\pi/4$), and do not change during propagation. For linear polarization of the field ($\eta=0$), we have

$$\begin{aligned}\langle\alpha|(\Delta S_{1,2})^2|\alpha\rangle &= \frac{|\alpha|^4}{2}\{\pm[\exp(\{\cos[2z(\varepsilon-\delta)]-1\}|\alpha|^2) \\ &\quad -\exp(2\{\cos[z(\varepsilon-\delta)]-1\}|\alpha|^2)]\cos 4\theta \\ &\quad +1-\exp(2\{\cos[z(\varepsilon-\delta)]-1\}|\alpha|^2)\}+|\alpha|^2. \end{aligned} \tag{49}$$

The quantum fluctuations in the Stokes parameters are intensity-dependent and periodic in $z(\varepsilon-\delta)$. However, they never become lower than $|\alpha|^2$. This means that the quantum fluctuations in the Stokes parameter cannot be squeezed below the level for a coherent state, contrary to the field variances which can undergo squeezing in the propagation process [5].

5. Conclusions

In this paper we have considered the problem of quantum fluctuations in the Stokes parameters which determine the polarization of the field propagating in an isotropic Kerr medium. The two-mode description is needed to describe an elliptically polarized field propagating through such a medium. In the case of isotropic media, exact operator solutions of the Heisenberg equations of motion are allowed. We have used these solutions to calculate the Stokes parameters and the azimuth and ellipticity of the polarization ellipse of the field propagating in such a medium. The solutions are exact quantum solutions showing new features characteristic for quantum evolution, such as periodic behaviour. It is shown that, due to the quantum character of the field, the degree of polarization is lowered at the initial stage of evolution and then returns to its initial value after a period. Also, the azimuth θ and ellipticity η of the polarization ellipse exhibit periodic behaviour and their evolution in the quantum case essentially differs from the classical evolution. Since all the solutions are exact, they are highly instructive in revealing specific quantum features of the evolution. The variances of the Stokes parameters are also calculated, and although they are reminiscent of the field operator variances determining squeezing, it is shown that their values cannot fall below the level for a coherent state. Thus, there is no squeezing of quantum fluctuations in the Stokes parameters, at least when the initial state of the field is coherent.

When dealing with quantum fluctuations in the Stokes parameters we completely ignored the effect of dissipation. It has been shown by Milburn and Holmes [18] that in the case of an anharmonic oscillator, losses can degrade or even completely wash out the quantum effects. This should certainly hold as well in the case of the quantum effects considered here. So, to observe these effects, one would need a medium with minimal losses. Some approximate solutions of the problem of coupled nonlinear oscillators with losses have recently been obtained by Horák and Peřina [19].

Acknowledgment

This work was supported by the Polish Research Project CPBP 01.06.

References

[1] Mayer, G., and Gires, F., 1964, *C. r. hebd. Séanc. Acad. Sci.*, Paris, **258,** 2039.
[2] Maker, P. D., Terhune, R. W., and Savage, C. W., 1964, *Phys. Rev. Lett.*, **12,** 507.
[3] Kielich, S., 1981, *Nonlinear Molecular Optics* (Moscow: Nauka).
[4] Shen, Y. R., 1985, *The Principles of Nonlinear Optics* (New York: Wiley).
[5] Tanaś, R., and Kielich, S., 1983, *Optics Commun.*, **45,** 351; 1984, *Optica Acta*, **31,** 81.
[6] Kielich, S., Tanaś, R., and Zawodny, R., 1987, *Phys. Rev.* A, **36,** 5670.
[7] Kielich, S., Tanaś, R., and Zawodny, R., 1987, *J. opt. Soc. Am.* B, **4,** 1627.
[8] Kielich, S., Tanaś, R., and Zawodny, R., 1988, *Appl. Phys.* B, **45,** 249.
[9] Levenson, M. D., Shelby, R. M., Reid, M., and Walls, D. F., 1986, *Phys. Rev. Lett.*, **57,** 2473.
[10] Bachor, H. A., Levenson, M. D., Walls, D. F., Perlmutter, S. H., and Shelby, R. M., 1988, *Phys. Rev.* A, **38,** 180.
[11] Atkins, P. W., and Wilson, A. D., 1972, *Molec. Phys.*, **24,** 33.
[12] Jauch, J. M., and Rohrlich, F., 1959, *The Theory of Photons and Electrons* (London: Addison-Wesley).
[13] Kelley, P. L., 1965, *Phys. Rev. Lett.*, **15,** 1005.
[14] Ritze, H. H., 1980, *Z. Phys.* B, **39,** 353.
[15] Kielich, S., 1983, *Progress in Optics*, vol. 20 (Amsterdam: North Holland) p. 155.
[16] Sala, K. L., 1984, *Phys. Rev.* A, **29,** 1944.
[17] Horák, R., 1989, *Optics Commun.*, **72,** 239.
[18] Milburn, G. J., and Holmes, C. A., 1986, *Phys. Rev. Lett.*, **56,** 2237.
[19] Horák, R., and Peřina, J., 1989, *J. opt. Soc. Am.* B, **6,** 1239.